Definition 1 *A map* $f\colon Y \to X$ *between connected spaces is called a* homotopy monomorphism *at* p *if its homotopy fibre* F *is* $B\mathbf{Z}/p$*-local for every choice of basepoint.*

In the special case where $Y = BP$ *is the classifying space of a finite* p*-group, we say that* f *is a* p*-subgroup inclusion.*

Proposition 1 (Folklore) *Let* $f\colon Y \to X$ *be a map between two* p*-complete spaces, with Noetherian cohomology rings. Then* f *is a homotopy monomorphism if and only if the induced map in cohomology makes* $H^*(Y;\mathbf{F}_p)$ *a finitely generated* $H^*(X;\mathbf{F}_p)$*-module.*

Definition 2 *Let $f : Y \to X$ be a map of spaces. A* Frobenius transfer *of f is a stable map $t : \Sigma^{\infty}_{+} X \to \Sigma^{\infty}_{+} Y$ such that*

$$\Sigma^{\infty}_{+} f \circ t \simeq id_{\Sigma^{\infty}_{+} X}$$

and the following diagram commutes up to homotopy

$$\begin{array}{ccccc} \Sigma^{\infty}_{+} X & & \xrightarrow{\quad\quad\Delta\quad\quad} & & \Sigma^{\infty}_{+} X \wedge \Sigma^{\infty}_{+} X \\ \Big\downarrow t & & & & \Big\downarrow 1 \wedge t \\ \Sigma^{\infty}_{+} Y & \xrightarrow{\Delta} & \Sigma^{\infty}_{+} Y \wedge \Sigma^{\infty}_{+} Y & \xrightarrow{f \wedge 1} & \Sigma^{\infty}_{+} X \wedge \Sigma^{\infty}_{+} Y. \end{array} \tag{1}$$

Definition 3 *A* Frobenius transfer triple *over a finite p-group S is a triple (f, t, X), where*

X is a connected, p-complete space with finite fundamental group

f is a subgroup inclusion $BS \to X$

t is a Frobenius transfer for f.

In general, given a map $f\colon BS \to X$, we get a fusion system $\mathcal{F}_{S,f}(X)$ over S by putting

$$Hom_{\mathcal{F}_{S,f}}(P,Q) = \{\varphi \in Inj(P,Q) \mid f|_{BP} \simeq f|_{BQ} \circ B\varphi\}$$

for each $P, Q \leq S$.

For such a fusion system $\mathcal{F}_{S,f}$ we pose the following questions:

- Is $\mathcal{F}_{S,f}$ saturated?

- Does it have an associated centric linking system $\mathcal{L}$? If so, is it unique?

- What is the relationship between $BS \xrightarrow{f} X$ and $BS \xrightarrow{\theta} |\mathcal{L}|_p^\wedge$? Are they equivalent as objects under BS.

In these lectures, these questions will be answered in a special case.

Theorem 1 *Let S be a finite elementary abelian p-group. Let (f, t, X) be a Frobenius transfer triple over S and put $W := Aut_{\mathcal{F}_{S,f}(X)}(S)$. Then the following hold*

- *$\mathcal{F}_{S,f}(X)$ is equal to the saturated fusion system $\mathcal{F}_S(W \ltimes S)$*

- *$\mathcal{F}_{S,f}(X)$ has a unique associated centric linking system with classifying space $B(W \ltimes S)^{\wedge}_p$.*

- *There is a natural equivalence $B(W \ltimes S)^{\wedge}_p \xrightarrow{\simeq} X$ of objects under BS.*

Conversely, we will show the following for p-local finite groups over *any* finite p-group S:

Theorem 2 *Let* $(S, \mathcal{F}, \mathcal{L})$ *be a* p*-local finite group. Then the natural inclusion*

$$\theta\colon BS \longrightarrow |\mathcal{L}|_p^\wedge.$$

has a Frobenius transfer

$$t :: \Sigma_+^\infty |\mathcal{L}|_p^\wedge \to \Sigma_+^\infty BS$$

making (θ, t, X) *a Frobenius transfer triple over* S*.*

In the course of the proof of Theorem 2, we also obtain the following interesting results:

Theorem 3 (The next best thing) *There is a functorial assignment*

$$\Upsilon\colon (S,\mathcal{F}) \longmapsto \Sigma^{\infty}_{+}BS \xrightarrow{\sigma_{\mathcal{F}}} B\mathcal{F}$$

of a classifying spectrum *to every saturated fusion system, such that:*

$(S,\mathcal{F})$ *can be recovered from* $(\sigma_{\mathcal{F}}, B\mathcal{F})$*.*

$\sigma_{\mathcal{F}}$ *admits a "transfer"* $t_{\mathcal{F}}\colon B\mathcal{F} \xrightarrow{\sigma_{\mathcal{F}}} \Sigma^{\infty}_{+}BS$*, such that* $\sigma_{\mathcal{F}} \circ t_{\mathcal{F}} \simeq id_{B\mathcal{F}}$*.*

$(\sigma_{\mathcal{F}}, B\mathcal{F})$ *agrees with* $(\Sigma^{\infty}_{+}\theta, \Sigma^{\infty}_{+}|\mathcal{L}|^{\wedge}_{p})$ *in the case of* p*-local finite groups.*

There is a theory of transfers for "injective" morphisms between saturated fusion systems.

The functor Υ also agrees with the p-completed stable classifying spaces of finite groups. That is, the following diagram of functors commutes:

$$\begin{array}{ccc} \text{Groups} & \xrightarrow{B(-)^{\wedge}_p} & \text{Spaces} \\ \Big\downarrow \mathcal{F}(-) & & \Big\downarrow \Sigma^{\infty}_{+}(-) \\ \text{Fusion systems} & \xrightarrow{\Upsilon} & \text{Spectra.} \end{array}$$

In view of Oliver's solution of the Martino-Priddy conjecture, we get the following Corollary.

Corollary 1 *Let G and G' be finite groups with Sylow subgroups S and S', respectively. Then the following are equivalent*

(i) $BG^\wedge_p \simeq BG'^\wedge_p$

(ii) *There is an isomorphism $\varphi : S \to S'$ and a stable equivalence h: $\Sigma^\infty_+ BG^\wedge_p \to \Sigma^\infty_+ BG'^\wedge_p$ making the following diagram commute*

$$\begin{array}{ccc} \Sigma^\infty_+ BS & \longrightarrow & \Sigma^\infty_+ BG^\wedge_p \\ \Big\downarrow B\varphi & & \Big\downarrow h \\ \Sigma^\infty_+ BS' & \longrightarrow & \Sigma^\infty_+ BG'^\wedge_p. \end{array}$$

Using the transfer theory, we also obtain the following corollary

Corollary 2 *Let $\mathcal{F}$ and $\mathcal{F}'$ be saturated fusion systems over a finite p-group S. Then $\mathcal{F} \subset \mathcal{F}'$ if and only if $B\mathcal{F}'$ is a stable summand of $B\mathcal{F}$ (as objects under $\Sigma^{\infty}_{+} BS$).*

Let S be a finite abelian p-group and $\mathcal{F}$ be a fusion system over S. Then we get the following simplifications:

• Every $P \leq S$ is both fully centralized and fully normalized, since $C_S(P) = N_S(P) = S$.

• $Aut_S(P) = \{id\}$ for all $P \leq S$.

• For $P \leq S$ and $\varphi \in Hom_{\mathcal{F}}(P, S)$, we have

$$\begin{aligned} N_\varphi &= \{g \in N_S(P) \mid \varphi \circ c_g \circ \varphi^{-1} \in Aut_S(\varphi P)\} \\ &= \{g \in S \mid \varphi \circ id \circ \varphi^{-1} \in \{id\}\} \\ &= S. \end{aligned}$$

• Since $C_S(P) = S$ for every $P \leq S$, the only $\mathcal{F}$-centric subgroup is S itself.

Lemma 1 *Let $\mathcal{F}$ be a fusion system over a finite abelian p-group S. Then $\mathcal{F}$ is saturated if and only if the following two conditions are satisfied:*

($\mathbf{I}_{ab}$) $|Aut_{\mathcal{F}}(S)|$ *is prime to p.*

($\mathbf{II}_{ab}$) *Every $\varphi \in Hom_{\mathcal{F}}(P,Q)$ is the restriction of some $\tilde{\varphi} \in Aut_{\mathcal{F}}(S)$.*

Therefore, the only saturated fusion systems over S are the ones coming from semi-direct products $W \ltimes S$, where $W \leq Aut(S)$ has order prime to p.

These have a canonical classifying space $B(W \ltimes S)^{\wedge}_{p}$. The obstructions to existence and uniqueness of classifying spaces reduces to group cohomology

$$H^*(W; S),$$

which vanishes by a transfer argument. Hence the classifying space is unique.

Proposition 2 *If S is an abelian finite p-group, then the assignment*

$$W \mapsto (S, \mathcal{F}(W \ltimes S), \mathcal{L}_S^c(W \ltimes S))$$

gives a bijective correspondence between subgroups $W \leq Aut(S)$ of order prime to p and p-local finite groups over S. In particular, there are no exotic p-local finite groups over S.

Outline of proof of Theorem 1:

1. Use a theorem of Adams and Wilkerson to show that

$$H^*(X) = H^*(BS)^W = H^*(B(W \ltimes S)^\wedge_p),$$

for some $W \leq Aut(S)$ of order prime to p.

2. Use Lannes's theorem to deduce that

$$\mathcal{F}_{S,f}(X) = \mathcal{F}_S(W \ltimes S)$$

3. By classification of p-local finite groups, we have a unique classifying space $B(W \ltimes S)^\wedge_p$ for $\mathcal{F}_{S,f}(X)$. We use Wojtkowiak's obstruction theory to produce an equivalence

$$B(W \ltimes S)^\wedge_p \xrightarrow{\simeq} X$$

of objects under BS.