COHOMOLOGY PRODUCTS

Let A be an associative algebra over the field k. Unless otherwise indicated, all tensor products in this note are taken over k, i.e., $\otimes = \otimes_k$. The *enveloping algebra* A^e of A is defined by $A^e := A \otimes A^{\mathrm{op}}$. We will frequently make use of the equivalence of categories between the category of A-bimodules and the category of left A^e-modules. Given an A-bimodule M, the left action of $\lambda \otimes \mu \in A \otimes A^{\mathrm{op}}$ on $m \in M$ is defined by $(\lambda \otimes \mu)m = \lambda m \mu$.

1. Cohomology Modules

1.1. **Definitions.** Let $\beta = \beta(A, A) = A \otimes A^{\otimes \bullet} \otimes A$ denote the (un-normalized) bimodule bar resolution of A. The differential $\partial_n : \beta_n \to \beta_{n-1}$ is defined by

$$\begin{aligned}\partial_n(a \otimes a_1 \otimes \cdots \otimes a_n \otimes a') = {} & aa_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes a' \\ & + \sum_{i=1}^{n-1} (-1)^i a \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes a' \\ & + (-1)^n a \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n a'.\end{aligned}$$

This makes $\beta(A, A)$ an A^e-projective resolution of A. The normalized bimodule bar resolution $\mathsf{B}(A, A)$ is defined by $\mathsf{B}_n(A, A) = A \otimes (A/k)^{\otimes n} \otimes A$, where $A/k := \operatorname{coker}(k \hookrightarrow A)$. The normalized bimodule bar resolution is also an A^e-projective resolution of A, with differential induced by that of $\beta(A, A)$. Given an element $a \otimes a_1 \otimes \cdots \otimes a_n \otimes a' \in \beta_n(A, A)$, it is customary to write the corresponding element of $\mathsf{B}_n(A, A)$ as $a[a_1|\ldots|a_n]a'$. It is clear that the constructions $A \mapsto \beta(A, A)$ and $A \mapsto \mathsf{B}(A, A)$ are functorial in A.

The Hochschild cohomology of A with coefficients in the A-bimodule M is defined by

$$\mathrm{HH}^\bullet(A, M) := \mathrm{Ext}^\bullet_{A^e}(A, M).$$

It may be computed as the homology of the complex $\mathrm{Hom}_{A^e}(B(A, A), M)$. Now assume that A is an augmented algebra over k with augmentation map $\varepsilon : A \to k$. The augmentation map defines the structure of a left A-module on k, called the *trivial module*. Then $\beta(A) := \beta(A, A) \otimes_A k$ and $B(A) := B(A, A) \otimes_A k$ are (left) A-projective resolutions of k. The cohomology of A with coefficients in the left A-module N is then defined by

$$\mathrm{H}^\bullet(A, N) := \mathrm{Ext}^\bullet_A(k, N).$$

It may be computed as the homology of the complex $\mathrm{Hom}_A(B(A), N)$. (If A is an augmented algebra, then we can replace A/k in the definition of $B(A, A)$ by $A_+ := \ker \varepsilon$, the augmentation ideal of A.)

Any left A-module N may be given the structure of an A-bimodule, denoted N_ε, by giving N the trivial right action, i.e., by having A act on the right via $\varepsilon : A \to k$. Then the map $N \mapsto N_\varepsilon$ yields a full embedding of the category of left A-modules into the category of left A^e-modules. Also, for any left A-module N and for any left A^e-module B, there exists a natural isomorphism $\mathrm{Hom}_{A^e}(B, N_\varepsilon) \cong \mathrm{Hom}_A(B \otimes_A k, N)$. In particular, taking $B = B(A, A)$, we get an isomorphism of complexes $\mathrm{Hom}_{A^e}(B(A, A), N_\varepsilon) \cong \mathrm{Hom}_A(B(A), N)$, and hence an isomorphism of graded spaces

$$\Phi : \mathrm{HH}^\bullet(A, N_\varepsilon) = \mathrm{Ext}^\bullet_{A^e}(A, N_\varepsilon) \xrightarrow{\sim} \mathrm{Ext}^\bullet_A(k, N) = \mathrm{H}^\bullet(A, N). \tag{1}$$

1.2. **Extensions.** Let V and W be left A-modules. An n-fold extension S of W by V is an exact sequence of A-modules

$$S : 0 \to W \to C_{n-1} \to C_{n-2} \to \cdots \to C_0 \to V \to 0.$$

There exists a congruence relation on the set of all n-extensions of W by V such that the set of all congruence classes $\overline{\operatorname{Ext}}^n_A(V, W)$ forms an abelian group (in fact, a k-vector space) under the operation of the Baer sum [7, III.5, VII.3]. In this note we follow the notation of [7] and write $[S]$ to denote the congruence class of S in $\overline{\operatorname{Ext}}^n_A(V, W)$. We also write $S \in\in \overline{\operatorname{Ext}}^n_A(V, W)$ to denote that S is a particular n-extension of W by V, and we write $S \in \sigma$ if $\sigma \in \overline{\operatorname{Ext}}^n_A(V, W)$ and $\sigma = [S]$.

Theorem 1. [7, Theorem III.6.4] *There exists a natural vector space isomorphism*

$$\zeta : \overline{\operatorname{Ext}}^n_A(V, W) \to \operatorname{Ext}^n_A(V, W). \tag{2}$$

Let $X \to V$ be an A-projective resolution of V, and let $S \in \sigma \in \overline{\operatorname{Ext}}^n_A(V, W)$. Lift the identity $1_V : V \to V$ to a chain map $f : X \to S$. Then $\zeta(\sigma) \in \operatorname{Ext}^n_A(V, W)$ is the cohomology class represented by the cocycle $f_n : X_n \to W$.

From now on we will not distinguish between the spaces $\overline{\operatorname{Ext}}^n_A(V, W)$ and $\operatorname{Ext}^n_A(V, W)$, but will just identify them via the isomorphism 2.

2. Cohomology Products

Let B, C be left A-modules. Depending on the structure on A, a number of different cohomology products can be defined on the spaces $\operatorname{Ext}^\bullet_A(B, C)$. We describe a few below, starting with those that require the least additional structure on A.

2.1. **The Yoneda product.** Let A be an arbitrary associative ring, and let B, C, D be left A-modules. The Yoneda product is a family of k-bilinear maps

$$\circ : \operatorname{Ext}^n_A(C, D) \otimes \operatorname{Ext}^m_A(B, C) \to \operatorname{Ext}^{n+m}_A(B, D).$$

If $m = n = 0$, then $\circ$ reduces to the composition of A-module homomorphisms $f \otimes g \mapsto f \circ g$. (This is the reason for using the symbol $\circ$ to denote the Yoneda product.)

Let

$$S : 0 \to D \to C_{n-1} \to C_{n-2} \to \cdots \to C_0 \to C \to 0 \qquad \text{and}$$
$$S' : 0 \to C \to C'_{m-1} \to C'_{m-2} \to \cdots \to C'_0 \to B \to 0$$

represent elements $\sigma \in \operatorname{Ext}^n_A(C, D)$ and $\sigma' \in \operatorname{Ext}^m_A(B, C)$. The $(n+m)$-extension $S \circ S'$ of D by B is defined by splicing S and S' together along the composite map $C_0 \to C \to C'_{m-1}$. Then the Yoneda product $\sigma \circ \sigma'$ is defined by $\sigma \circ \sigma' = [S \circ S']$. (For this reason, the Yoneda product is also referred to as the *Yoneda composition* of extensions.)

The Yoneda product can be characterized axiomatically; see [8, §9.5].

Lemma 2. [7, Exercise III.6.2] *Let A be an arbitrary associative ring, and let B, C, D be left A-modules. Let $X \to B$ and $Y \to C$ be A-projective resolutions of B and C, respectively. Given $\alpha \in \operatorname{Ext}^m_A(C, D)$ and $\beta \in \operatorname{Ext}^n_A(B, C)$, choose cocycle representatives $h \in \operatorname{Hom}_A(Y_m, D)$ and $g \in \operatorname{Hom}_A(X_n, C)$, respectively. Let $\partial : X_{n+1} \to X_n$ denote the differential of the resolution X, and* *write g as the composition $g = g_0 \circ \partial'$, where $\partial' : X_n \to \operatorname{coker}(\partial) = X_n/\partial(X_{n+1})$ is the projection map. Lift g_0 to a map $f : X_{n+m} \to Y_m$ as in the diagram*

$$\begin{array}{ccccccccc} X_{m+n} & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & \operatorname{coker}(\partial) & \longrightarrow & 0 \\ \downarrow{\scriptstyle f} & & & & \downarrow & & \downarrow{\scriptstyle g_0} & & \\ Y_m & \longrightarrow & \cdots & \longrightarrow & Y_0 & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

Then $h \circ f \in \operatorname{Hom}_A(X_{n+m}, D)$ is a cocycle representative for the Yoneda composition $\alpha \circ \beta$.

2.2. **Composition product on Hochschild cohomology.** Let A be a (not necessarily augmented) algebra over the field k, and let M and N be A-bimodules. The product we describe in this section is a family of k-bilinear maps

$$\sqcup : \mathrm{HH}^n(A, M) \otimes \mathrm{HH}^m(A, N) \to \mathrm{HH}^{n+m}(A, M \otimes_A N). \tag{3}$$

This product structure on Hochschild cohomology was first studied by Eilenberg and Mac Lane [5] in the case $A = kG$ (the group algebra of a group G), and later by Gerstenhaber [6] for arbitrary associative algebras. We use the symbol $\sqcup$ for the composition product in order to distinguish it from the usual cup product $\cup$, which we will discuss later.

The product (3) is defined as follows: Given $\zeta \in \mathrm{HH}^n(A, M)$ and $\eta \in \mathrm{HH}^m(A, N)$, choose representative cocycles $f \in \operatorname{Hom}_{A^e}(B_n(A, A), M)$ and $g \in \operatorname{Hom}_{A^e}(B_m(A, A), N)$. Define $f \sqcup g \in \operatorname{Hom}_{A^e}(B_{n+m}(A, A), M \otimes_A N)$ by

$$(f \sqcup g)([a_1|\cdots|a_n|b_1|\cdots|b_m]) = f([a_1|\cdots|a_n]) \otimes_A g([b_1|\cdots|b_m]). \tag{4}$$

The function $f \sqcup g$ is a cocycle because

$$\delta(f \sqcup g) = \delta(f) \sqcup g + (-1)^n f \sqcup \delta(g). \tag{5}$$

Now $\zeta \sqcup \eta$ is defined to be the class of $f \sqcup g$ in $\mathrm{HH}^{n+m}(A, M \otimes_A N)$. It follows from (5) that the product $\zeta \sqcup \eta$ does not depend on the particular choice of representative cocycles f and g. If $\mu : M \otimes_A N \to P$ is an A^e-module homomorphism, then we can compose $\sqcup$ with the induced map $\mu_* : \mathrm{HH}^\bullet(A, M \otimes_A N) \to \mathrm{HH}^\bullet(A, P)$ in order to obtain

$$\sqcup : \mathrm{HH}^n(A, M) \otimes \mathrm{HH}^m(A, N) \to \mathrm{HH}^{n+m}(A, P),$$

the composition product with respect to the pairing $\mu : M \otimes_A N \to P$.

Gerstenhaber proved the following results for the composition product $\sqcup$:

Theorem 3. [6, Corollary 1] *The composition product $\sqcup : \mathrm{HH}^\bullet(A, A) \otimes \mathrm{HH}^\bullet(A, A) \to \mathrm{HH}^\bullet(A, A)$ makes $\mathrm{HH}^\bullet(A, A)$ a graded-commutative ring, with grading given by dimension.*

Theorem 4. [6, Corollary 2] *Let P be an A-bimodule. Let $\zeta \in \mathrm{HH}^n(A, A)$, and let $\eta \in \mathrm{HH}^m(A, P)$. Then*

$$\zeta \sqcup \eta = (-1)^{nm} \eta \sqcup \zeta.$$

The composition product $\sqcup$ can be characterized axiomatically; see [10, §1].

2.3. **Wedge product.** Let Λ and Λ' be algebras over k. Let V and W be left Λ-modules, and let V' and W' be left Λ'-modules. Set $\Omega = \Lambda \otimes \Lambda'$. The *external* or *wedge product* is a family of k-bilinear maps

$$\vee : \mathrm{Ext}^n_\Lambda(V, W) \otimes \mathrm{Ext}^m_{\Lambda'}(V', W') \to \mathrm{Ext}^{n+m}_\Omega(V \otimes V', W \otimes W'). \tag{6}$$

It is defined as follows: Take projective resolutions $X \to V$ and $X' \to V'$ by Λ- and Λ'-modules, respectively. Then, for each $n, m \in \mathbb{N}$, $X_n \otimes X'_m$ is projective for Ω, and by the Künneth Theorem, $X \otimes X'$ is an Ω-projective resolution of $V \otimes V'$. Now given $f \in \mathrm{Hom}_\Lambda(X, W)$ and $g \in \mathrm{Hom}_{\Lambda'}(X', W')$, define $f \vee g \in \mathrm{Hom}_\Omega(X \otimes X', W \otimes W')$ by $(f \vee g)(x \otimes x') = f(x) \otimes g(x')$. Then (6) is the map in cohomology induced by

$$\vee : \mathrm{Hom}_\Lambda(X, W) \otimes \mathrm{Hom}_{\Lambda'}(X', W') \to \mathrm{Hom}_\Omega(X \otimes X', W \otimes W'),$$

2.4. **Cup product for bialgebras.** Assume now that Λ is a bialgebra. Then Λ is equipped with an algebra homomorphism $\Delta : \Lambda \to \Lambda \otimes \Lambda$, called *comultiplication* or the *diagonal map*, as well as an augmentation map $\varepsilon : \Lambda \to k$ (called the *counit*). Identifying $\Lambda \otimes k = \Lambda = k \otimes \Lambda$, we have $(\mathrm{id}_\Lambda \otimes \varepsilon) \circ \Delta = \mathrm{id}_\Lambda = (\varepsilon \otimes \mathrm{id}_\Lambda) \circ \Delta$.

Take $\Lambda = \Lambda'$ in (6). Then the wedge product becomes a bilinear map

$$\vee : \mathrm{Ext}^n_\Lambda(V, W) \otimes \mathrm{Ext}^m_\Lambda(V', W') \to \mathrm{Ext}^{n+m}_{\Lambda\otimes\Lambda}(V \otimes V', W \otimes W').$$

Pulling back the $\Lambda \otimes \Lambda$-module structures of $V \otimes V'$ and $W \otimes W'$ along $\Delta : \Lambda \to \Lambda \otimes \Lambda$, we obtain Λ-module structures on $V \otimes V'$ and $W \otimes W'$. Thus, the change-of-rings map for Ext yields in this case a map

$$\Delta^* : \mathrm{Ext}^\bullet_{\Lambda\otimes\Lambda}(V \otimes V', W \otimes W') \to \mathrm{Ext}^\bullet_\Lambda(V \otimes V', W \otimes W'). \tag{7}$$

The cup product

$$\cup : \mathrm{Ext}^n_\Lambda(V, W) \otimes \mathrm{Ext}^m_\Lambda(V', W') \to \mathrm{Ext}^{n+m}_\Lambda(V \otimes V', W \otimes W') \tag{8}$$

is then defined as the composition $\Delta^* \circ \vee$. (Note that if S is an extension of $W \otimes W'$ by $V \otimes V'$ consisting of $\Lambda \otimes \Lambda$-modules, then $\Delta^*([S]) = [\Delta^*(S)]$, where $\Delta^*(S)$ is equal to S as an exact sequence of vector spaces, but the terms of S are considered instead as Λ-modules via Δ.)

The cup product can be described at the level of cochains as follows. Let $X \to V$ and $X' \to V'$, and $Z \to V \otimes V'$ be Λ-projective resolutions of V, V', and $V \otimes V'$, respectively. By the Künneth Formula, $X \otimes X'$ is an acyclic chain complex with homology equal to $V \otimes V'$ in degree 0. Choose a chain map $\varphi : Z \to X \otimes X'$ lifting the identity $\mathrm{id} : V \otimes V' \to V \otimes V'$. Given $\zeta \in \mathrm{Ext}^n_\Lambda(V, W)$ and $\eta \in \mathrm{Ext}^m_\lambda(V', W')$, choose cocycle representatives $f \in \mathrm{Hom}_\Lambda(X_n, W)$ and $g \in \mathrm{Hom}_\Lambda(X'_m, W')$. Then $\zeta \cup \eta$ is the cohomology class of the cocycle $(f \vee g) \circ \varphi \in \mathrm{Hom}_\Lambda(Z_{n+m}, W \otimes W')$.

The cup product $\cup$ can be characterized axiomatically; see [2, II.1, II.2]. Below are some important special cases of the cup product.

2.4.1. *Cup product for algebra cohomology.* Let A be a bialgebra. Take $\Lambda = A$, and take $V = V' = k$ in (8). Then the cup product is a family of maps

$$\cup : \mathrm{H}^n(A, W) \otimes \mathrm{H}^m(A, W') \to H^{n+m}(A, W \otimes W'). \tag{9}$$

If $m : W \otimes W' \to V$ is an A-module homomorphism, then we also have the cup product $m_* \circ \cup : \mathrm{H}^n(A, W) \otimes \mathrm{H}^m(A, W') \to \mathrm{H}^{n+m}(A, V)$ with respect to the pairing $m : W \otimes W' \to V$.

Working at the level of chain complexes, it is customary to take $X = X' = Z = B(A)$, the normalized bar complex of A. Then an explicit chain map $\varphi : Z \to X \otimes X'$ lifting the identity $\mathrm{id} : k \to k$ is determined by the formula

$$\varphi([a_1|\cdots|a_r]) = \sum_{p=0}^{r}[a_1^{(1)}|\cdots|a_p^{(1)}]\varepsilon(a_{p+1}^{(1)}\cdots a_r^{(1)}) \otimes a_1^{(2)}\cdots a_p^{(2)}[a_{p+1}^{(2)}|\cdots|a_r^{(2)}], \tag{10}$$

where we have written $\Delta(a_i) = a_i^{(1)} \otimes a_i^{(2)}$ (Einstein notation). The chain map φ described here is obtained by composing the chain map $\varphi_1 : B(A) \to B(A \otimes A)$ induced by Δ (which lifts the identity $\mathrm{id} : k \to k$) with the chain map $\varphi_2 : B(A \otimes A) \to B(A) \otimes B(A)$ defined in [4, XI.7(3)] (which also lifts the identity $\mathrm{id} : k \to k$). The map φ_1 is a homomorphism of A-modules (letting A act on $B(A \otimes A)$ via Δ), and the map φ_2 is a homomorphism of $A \otimes A$-modules.

2.4.2. *Cup product for Hochschild cohomology.* Let A be a bialgebra. The comultiplication $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to k$ induce in a natural way a comultiplication $\Delta^e : A^e \to A^e \otimes A^e$ and a counit $\varepsilon^e = \varepsilon \otimes \varepsilon : A \otimes A^{\mathrm{op}} \to k$ for A^e. Thus A^e is also a bialgebra. Now take $\Lambda = A^e$ and $V = V' = A$. Then (8) is a family of maps

$$\cup' : \mathrm{Ext}^n_{A^e}(A, W) \otimes \mathrm{Ext}^m_{A^e}(A, W') \to \mathrm{Ext}^{n+m}_{A^e}(A \otimes A, W \otimes W').$$

The diagonal map $\Delta : A \to A \otimes A$ is an algebra homomorphism, hence also an A-bimodule homomorphism (i.e., a homomorphism of A^e-modules). Thus, we obtain the morphism

$$\Delta^* : \mathrm{Ext}^\bullet_{A^e}(A \otimes A, W \otimes W') \to \mathrm{Ext}^\bullet_{A^e}(A, W \otimes W').$$

Now the cup product for Hochschild cohomology

$$\cup : \mathrm{HH}^n(A, W) \otimes \mathrm{HH}^m(A, W') \to \mathrm{HH}^{n+m}(A, W \otimes W') \tag{11}$$

is defined as the composite $\Delta^* \circ \cup'$.

At the level of chain complexes, the cup product (11) admits the following description. Given $\zeta \in \mathrm{HH}^n(A, W)$ and $\eta \in \mathrm{HH}^m(A, W')$, choose representative cocycles $f \in \mathrm{Hom}_{A^e}(B_n(A, A), W)$ and $g \in \mathrm{Hom}_{A^e}(B_m(A, A), W')$. Set $\Omega = A \otimes A$. (We identify Ω^e with $A^e \otimes A^e$.) Our first step is to determine a cocycle representative in $\mathrm{Hom}_{\Omega^e}(B_{n+m}(\Omega, \Omega), W \otimes W')$ for $\zeta \vee \eta$. To get this, we precompose $f \vee g$ with a chain map $\varphi : B(\Omega, \Omega) \to B(A, A) \otimes B(A, A)$ lifting the identity $A \otimes A \to A \otimes A$. Such a map is given by the formula

$$\varphi([\lambda_1 \otimes \gamma_1|\cdots|\lambda_r \otimes \gamma_r]) = \sum_{p=0}^{r}[\lambda_1|\cdots|\lambda_p]\lambda_{p+1}\cdots\lambda_r \otimes \gamma_1\cdots\gamma_p[\gamma_{p+1}|\cdots|\gamma_r];$$

see [4, XI.6(3)]. Next, pulling back the Ω^e-module structures of $B(\Omega, \Omega)$ and $W \otimes W'$ along $\Delta^e : A^e \to A^e \otimes A^e$, we have $\mathrm{Hom}_{\Omega^e}(B(\Omega, \Omega), W \otimes W') \subseteq \mathrm{Hom}_{A^e}(B(\Omega, \Omega), W \otimes W')$, so we consider $(f \vee g) \circ \varphi$ as an element of $\mathrm{Hom}_{A^e}(B(\Omega, \Omega), W \otimes W')$. Finally, by the functoriality of the bimodule bar resolution, the homomorphism $\Delta : A \to A \otimes A$ induces an A^e-module homomorphism of chain complexes $\psi : B(A, A) \to B(\Omega, \Omega)$ lifting Δ. Then $\zeta \cup \eta$ is the cohomology class of the cocycle $(f \vee g) \circ \varphi \circ \psi \in \mathrm{Hom}_{A^e}(B(A, A), W \otimes W')$. Explicitly,

$$(f \vee g) \circ \varphi \circ \psi([a_1|\cdots|a_{n+m}]) = \sum_{p=0}^{r} f([a_1^{(1)}|\cdots|a_p^{(1)}])a_{p+1}^{(1)}\cdots a_{n+m}^{(1)} \otimes a_1^{(2)}\cdots a_p^{(2)} g([a_{p+1}^{(2)}|\cdots|a_{n+m}^{(2)}]).$$

3. Comparing Cohomology Products

In this section we investigate certain relations between the cohomology products defined above.

3.1. **Comparison with Yoneda composition.** The wedge product may be expressed in terms of the Yoneda composition of extensions.

Theorem 5. [12] *Let Λ and Λ' be algebras over k. Let V and W be left Λ-modules, and let V' and W' be left Λ'-modules. Then for $\zeta \in \operatorname{Ext}^n_\Lambda(V, W)$ and $\eta \in \operatorname{Ext}^m_{\Lambda'}(V', W')$, we have*

$$\zeta \vee \eta = (\zeta \otimes W') \circ (V \otimes \eta) = (-1)^{nm}(W \otimes \eta) \circ (\zeta \otimes V'). \tag{12}$$

The expressions $\zeta \otimes W'$ and $V \otimes \eta$ in (12) have the following meaning: Choose n- and m-fold extensions $S \in\in \operatorname{Ext}^n_A(V, W)$ and $S' \in\in \operatorname{Ext}^m_{A'}(V', W')$ representing ζ and η, respectively. Then $\zeta \otimes W' = [S \otimes W']$, and $V \otimes \eta = [V \otimes S']$. If $n = 0$, then $\zeta \in \operatorname{Ext}^0_\Lambda(V, W) = \operatorname{Hom}_\Lambda(V, W)$, in which case $\zeta \otimes W'$ represents the homomorphism $V \otimes W' \to W \otimes W'$, and a similar interpretation holds for $V \otimes \eta$ if $m = 0$. If either of n or m is zero, then $(\zeta \otimes W') \circ (V \otimes \eta)$ is the usual composite of a homomorphism with an exact sequence, cf. [7, III.1, III.3].

Now let A be a bialgebra, and let W be a left A-module. Let $\zeta \in \mathrm{H}^n(A, k)$ and $\eta \in \mathrm{H}^m(A, W)$. Recall that $\Delta^* : \operatorname{Ext}^\bullet_{A\otimes A}(-,-) \to \operatorname{Ext}^\bullet_A(-,-)$ is the change-of-rings map induced by the comultiplication $\Delta : A \to A \otimes A$. Then, in the notation of Theorem 5, $\Delta^*(\zeta \otimes k) = \zeta = \Delta^*(k \otimes \zeta)$, and similarly for η. Theorem 5 then implies

$$\begin{aligned} \zeta \cup \eta = \Delta^*(\zeta \vee \eta) &= \Delta^*((-1)^{mn}(k \otimes \eta) \circ (\zeta \otimes k)) \\ &= (-1)^{mn}\eta \circ \zeta, \\ &= (-1)^{mn}\Delta^*((\eta \otimes k) \circ (k \otimes \zeta)) \\ &= (-1)^{mn}\Delta^*(\eta \vee \zeta) \\ &= (-1)^{mn}\eta \cup \zeta \end{aligned} \tag{13}$$

In particular,

$$\eta \cup \zeta = \eta \circ \zeta, \tag{14}$$

so the right cup product action of $\mathrm{H}^\bullet(A, k)$ on $\mathrm{H}^\bullet(A, W)$ coincides with the Yoneda composition product $\operatorname{Ext}^\bullet_A(k, W) \otimes \operatorname{Ext}^\bullet_A(k, k) \to \operatorname{Ext}^\bullet_A(k, W)$. In particular, the cup and Yoneda composition products on $\mathrm{H}^\bullet(A, k)$ coincide, and under either operation, $\mathrm{H}^\bullet(A, k)$ is a graded-commutative ring.

View W as an A-bimodule with trivial right action. Combining the observation of (14) with Lemma 7, we get $\eta \sqcup \zeta = \eta \cup \zeta = \eta \circ \zeta$ for all $\eta \in \mathrm{H}^n(A, W_\varepsilon)$ and $\zeta \in \mathrm{H}^m(A, k)$. In fact, the equality $\eta \sqcup \zeta = \eta \circ \zeta$ holds generally, as we show below.

Lemma 6. *Let A be an arbitrary associative ring over the field k. Let W be an A-bimodule.*

(a) *Let $\alpha \in \mathrm{HH}^m(A, W)$, and let $\beta \in \mathrm{HH}^n(A, A)$. Then $\alpha \sqcup \beta = \alpha \circ \beta$.*
(b) *Assume that A is an augmented algebra over k, and that W has trivial right action. Let $\eta \in \mathrm{H}^m(A, W)$, and let $\zeta \in \mathrm{H}^n(A, k)$. Then $\eta \sqcup \zeta = \eta \circ \zeta$.*

Proof. We prove part (b) only, the proof of part (a) being similar. Choose cocycle representatives $g \in \operatorname{Hom}_A(B_n(A), k)$ and $h \in \operatorname{Hom}_A(B_m(A), W)$ for ζ and η, respectively. Write $\partial : B_{n+1}(A) \to B_n(A)$ for the differential, and write $g = g_0 \circ \partial'$, where $\partial' : B_n(A) \to \operatorname{coker}(\partial) = B_n(A)/\operatorname{im}(\partial)$ is the projection map. For $i \geq 0$, define $f_i : B_{i+n}(A) \to B_i(A)$ by

$$f_i(a[a_1|\cdots|a_{i+n}]) = a[a_1|\cdots|a_i] \cdot g([a_{i+1}|\cdots|a_{i+n}]).$$

Since g is a cocycle, the f_i form a chain map $B_{\bullet+n}(A) \to B_\bullet(A)$ lifting g_0, i.e., the f_i form a commutative diagram

$$\begin{array}{ccccccccc} B_{m+n}(A) & \longrightarrow & \cdots & \longrightarrow & B_n(A) & \longrightarrow & \operatorname{coker}(\partial) & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_m} & & & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle g_0} & & \\ B_m(A) & \longrightarrow & \cdots & \longrightarrow & B_0(A) & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

According to Lemma 2, the composite map $h \circ f_m : B_{n+m}(A) \to W$ is a cocycle representative for $\eta \circ \zeta$. But is is plain that $h \circ f_m([a_1|\cdots|a_{n+m}]) = (h \sqcup g)([a_1|\cdots|a_{n+m}])$. Thus $\eta \sqcup \zeta = \eta \circ \zeta$. □

In Lemma 6 we have implicitly made use of the graded space isomorphism (1). The compatibility of (1) with the cup products $\sqcup$ and $\cup$ is investigated below in Lemma 7. In general, if W is an A-bimodule with trivial right action, the right composition product action of $\mathrm{HH}^\bullet(A, A)$ on $\mathrm{HH}^\bullet(A, W_\varepsilon)$ factors through the right action of $\mathrm{HH}^\bullet(A, k)$ on $\mathrm{HH}^\bullet(A, W_\varepsilon)$. The induced map $\mathrm{HH}^\bullet(A, A) \to \mathrm{HH}^\bullet(A, k)$ is simply $\varepsilon_* : \mathrm{HH}^\bullet(A, A) \to \mathrm{HH}^\bullet(A, k)$, the map induced by the counit $\varepsilon : A \to k$.

3.2. **Products on $\mathrm{H}^\bullet(A, k)$.** Let A be a bialgebra. Taking into account the graded space isomorphism $\mathrm{HH}^\bullet(A, k) \cong \mathrm{H}^\bullet(A, k)$, we have three cup products on $\mathrm{H}^\bullet(A, k)$: the composition product $\sqcup$ defined in §2.2, the cup product $\cup$ on $\mathrm{H}^\bullet(A, k)$ defined in §2.4.1, and the Hochschild cup product $\cup$ defined in §2.4.2.

Lemma 7. *Let A be a bialgebra, and let M be a left A-module, viewed also as an A-bimodule with trivial right action. Then there exists a commutative square*

$$\begin{array}{ccc} \mathrm{H}^n(A, M) \otimes \mathrm{H}^m(A, k) & \xrightarrow{\ \cup\ } & \mathrm{H}^{n+m}(A, M) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ \mathrm{HH}^n(A, M_\varepsilon) \otimes \mathrm{HH}^m(A, k) & \xrightarrow{\ \sqcup\ } & \mathrm{HH}^{n+m}(A, M_\varepsilon), \end{array}$$

where the vertical maps are the graded space isomorphisms of (1).

Proof. Fix elements $\eta \in \mathrm{H}^n(A, M)$ and $\zeta \in \mathrm{H}^m(A, k)$, represented by cocycles $g \in \mathrm{Hom}_A(B_n(A), M)$ and $f \in \mathrm{Hom}_A(B_m(A), k)$, respectively. Then $\eta \cup \zeta$ is the cohomology class of the cocycle $\mu \circ (g \vee f) \circ \varphi$. Here $\mu : k \otimes k \to k$ is the multiplication map, and $\varphi : B(A) \to B(A) \otimes B(A)$ is the chain map (10). Let $[a_1|\cdots|a_{n+m}] \in B_{n+m}(A)$. Given $a_i \in A$, write $\Delta(a_i) = a_i^{(1)} \otimes a_i^{(2)}$ (Einstein notation). Then

$$\varphi([a_1|\cdots|a_{n+m}]) = \sum_{p=0}^{n+m} [a_1^{(1)}|\cdots|a_p^{(1)}]\varepsilon(a_{p+1}^{(1)} \cdots a_{n+m}^{(1)}) \otimes a_1^{(2)} \cdots a_p^{(2)}[a_{p+1}^{(2)}|\cdots|a_{n+m}^{(2)}],$$

and

$$\begin{aligned} &(g \vee f) \circ \varphi([a_1|\cdots|a_{n+m}]) \\ &\qquad = \sum_{p=0}^{n+m} g([a_1^{(1)}|\cdots|a_p^{(1)}])\varepsilon(a_{p+1}^{(1)} \cdots a_{n+m}^{(1)}) \otimes \varepsilon(a_1^{(2)} \cdots a_p^{(2)}) f([a_{p+1}^{(2)}|\cdots|a_{n+m}^{(2)}]) \\ &\qquad = \sum_{p=0}^{n+m} g([a_1^{(1)}|\cdots|a_p^{(1)}])\varepsilon(a_1^{(2)} \cdots a_p^{(2)}) \otimes \varepsilon(a_{p+1}^{(1)} \cdots a_{n+m}^{(1)}) f([a_{p+1}^{(2)}|\cdots|a_{n+m}^{(2)}]). \end{aligned}$$

Since $(\mathrm{id}_A \otimes \varepsilon) \circ \Delta = \mathrm{id}_A = (\varepsilon \otimes \mathrm{id}_A) \circ \Delta$, and since ε is an algebra homomorphism, it follows from the definition of $g \vee f$ that

$$\sum_{p=0}^{n+m} g([a_1^{(1)}|\cdots|a_p^{(1)}])\varepsilon(a_1^{(2)}\cdots a_p^{(2)}) \otimes \varepsilon(a_{p+1}^{(1)}\cdots a_{n+m}^{(1)}) f([a_{p+1}^{(2)}|\cdots|a_{n+m}^{(2)}])$$

$$= \sum_{p=0}^{n+m} g([a_1|\cdots|a_p]) \otimes f([a_{p+1}|\cdots|a_{n+m}]) = g([a_1|\cdots|a_n]) \otimes f([a_{n+1}|\cdots|a_{n+m}]).$$

Composing with μ, and making the identification $k \otimes k = k \otimes_A k$, the last term is precisely $(g \sqcup f)([a_1|\cdots|a_{n+m}])$. Thus, $\zeta \cup \eta = \zeta \sqcup \eta$. □

Next we compare the cup products discussed in §2.4. Note that if W and W' are left A-modules, viewed as A-bimodules with trivial right action, then $W_\varepsilon \otimes W'_\varepsilon = (W \otimes W')_\varepsilon$ as A-bimodules.

Lemma 8. *Let A be a bialgebra, and let W, W' be left A-modules, viewed also as A-bimodules with trivial right action. Then there exists a commutative square*

$$\begin{array}{ccc} \mathrm{H}^n(A,W) \otimes \mathrm{H}^m(A,W') & \xrightarrow{\ \cup\ } & \mathrm{H}^{n+m}(A, W \otimes W') \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{HH}^n(A,W_\varepsilon) \otimes \mathrm{HH}^m(A,W'_\varepsilon) & \xrightarrow{\ \cup\ } & \mathrm{HH}^{n+m}(A, W_\varepsilon \otimes W'_\varepsilon), \end{array}$$

where the horizontal maps are the cup products defined in §2.4.1 and §2.4.2, and the vertical maps are the graded space isomorphisms of (1).

Proof. As for Lemma 7, the proof follows from an explicit computation at the level of chain complexes using the explicit descriptions of the cup products provided in §2.4.1 and §2.4.2. □

Corollary 9. *Under the graded space isomorphism $\mathrm{HH}^\bullet(A,k) \cong \mathrm{H}^\bullet(A,k)$ of* (1)*, the three cup products on $\mathrm{H}^\bullet(A,k)$ defined in §2.2, §2.4.1 and §2.4.2 coincide.*

3.3. **Further comparison with the composition product.** Now assume that A is a Hopf algebra with bijective antipode S. Then the map $\delta := (1 \otimes S) \circ \Delta$ defines an embedding of A into $A^e = A \otimes A^{\mathrm{op}}$. Considering A^e as a right A-module via δ, Pevtsova and Witherspoon show that there exists an A^e-module isomorphism $A \cong k \uparrow_A^{A^e} = (A^e) \otimes_A k$ [9, Lemma 7.1]. (Here we use the notation $W \uparrow_H^K$ to denote the tensor induction functor $K \otimes_H W$.) The space A^e is projective as a right A-module via δ (see the proof of [9, Lemma 7.2]), hence the Eckmann–Shapiro Lemma [3, Corollary 2.8.4] implies for any A^e-module M the existence of a natural isomorphism

(15) $$\Psi : \mathrm{HH}^n(A,M) = \mathrm{Ext}^n_{A^e}(k \uparrow_A^{A^e}, M) \xrightarrow{\sim} \mathrm{Ext}^n_{\delta A}(k, M \downarrow_{\delta A}^{A^e}) = \mathrm{Ext}^n_A(k, M^{\mathrm{ad}}).$$

Here M^{ad} denotes the vector space M considered as a left A-module via the "adjoint" action $a \cdot m = \sum a_{(1)} m S(a_{(2)})$. Note that if M has trivial right A-action, then $M^{\mathrm{ad}} \cong M$ as left A-modules.

Theorem 10. *Let A be a Hopf algebra with bijective antipode, and let M be an A-bimodule. Then there exists a commutative square*

(16) $$\begin{array}{ccc} \mathrm{HH}^n(A,A) \otimes \mathrm{HH}^m(A,M) & \xrightarrow{\ \sqcup\ } & \mathrm{HH}^{n+m}(A,M) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{H}^n(A,A^{\mathrm{ad}}) \otimes \mathrm{H}^m(A,M^{\mathrm{ad}}) & \xrightarrow{\ \cup\ } & \mathrm{H}^{n+m}(A,M^{\mathrm{ad}}), \end{array}$$

where the vertical maps are the isomorphisms of (15)*, the top map is the composition product with respect to the pairing* $A \otimes_A M \xrightarrow{\sim} M$*, and the bottom map is the (usual) cup product with respect to the pairing* $\mu : A^{\mathrm{ad}} \otimes M^{\mathrm{ad}} \to M^{\mathrm{ad}}$, $a \otimes m \mapsto am$*. The theorem also holds if the order of the factors in the left-hand column is interchanged.*

Proof. We prove the commutativity of (16) by the strategy indicated in the proof of [9, Lemma 7.2]; it is a direct generalization of the proof of [11, Proposition 3.1].

Let $P \to k$ be an A-projective resolution of k. Since A^e is projective as a right (and left) A-module via δ, the induction functor $(-)\uparrow_A^{A^e} = A^e \otimes_A -$ is exact and takes projectives to projectives, hence $X := A^e \otimes_A P$ is an A^e-projective resolution of $A \cong A^e \otimes_A k$.

The map $\iota : P \hookrightarrow X = A^e \otimes_A P$ defined by $\iota(x) = (1,1) \otimes_A x$ is an A-module chain map (where the left action of A on $A^e \otimes_A P$ is via δ). If $\zeta \in \mathrm{HH}^n(A, M)$ is represented by the cocycle $f : X \to M$, then the corresponding element of $\mathrm{H}^n(A, M^{\mathrm{ad}})$ under the isomorphism (15) is the cohomology class represented by the cocycle $f \circ \iota : P \to M^{\mathrm{ad}}$.

The complex $P \otimes P$ is an A-projective resolution of $k \otimes k = k$ (the action of A on $P \otimes P$ is the diagonal action via Δ), and, as argued in [11, §2], the complex $X \otimes_A X$ is an A^e-projective resolution of A. Let $D : P \to P \otimes P$ be an A-module chain map lifting the identity $\mathrm{id} : k \to k$. Define $\theta : A^e \otimes_A (P \otimes P) \to X \otimes_A X$ by

$$(a, b) \otimes_A (x \otimes y) \overset{\theta}{\mapsto} ((a, 1) \otimes_A x) \otimes_A ((1, b) \otimes_A y).$$

Then θ is a (well-defined) A^e-module chain map lifting $\mathrm{id} : A \to A$, hence $D' := \theta \circ (D \uparrow_A^{A^e})$ is an A^e-module chain map $X \to X \otimes_A X$ lifting the identity $\mathrm{id} : A \to A$.

Let $\zeta \in \mathrm{HH}^n(A, A)$ and $\eta \in \mathrm{HH}^m(A, M)$ be represented by cocycles $f : X \to A$ and $f' : X \to M$, respectively. Then we have the commutative diagram of maps

$$\begin{array}{ccccccc} X & \xrightarrow{D'} & X \otimes_A X & \xrightarrow{f \otimes f'} & A \otimes_A M & \xrightarrow{\sim} & M \\ \iota \uparrow & & & & & & \| \\ P & \xrightarrow{D} & P \otimes P & \xrightarrow{(f\iota) \otimes (f'\iota)} & A^{\mathrm{ad}} \otimes M^{\mathrm{ad}} & \xrightarrow{\mu} & M. \end{array} \tag{17}$$

Let $\Psi : \mathrm{HH}^\bullet(A, -) \to \mathrm{H}^\bullet(A, -^{\mathrm{ad}})$ generically denote the isomorphism of (15). Then composition along the top row of (17) yields a cocycle representative for $\zeta \sqcup \eta$ (cf. [10, §1.2]), while composition along the bottom row yields a cocycle representative for $\Psi(\zeta) \cup \Psi(\eta)$. Composing from P to M along the top row yields a cocycle representative for $\Psi(\zeta \sqcup \eta)$, hence $\Psi(\zeta) \cup \Psi(\eta) = \Psi(\zeta \sqcup \eta)$, as desired. Similarly, $\Psi(\eta \sqcup \zeta) = \Psi(\eta) \cup \Psi(\zeta)$. This proves the theorem. □

Corollary 11. [9, Lemma 7.2] *Let A be a Hopf algebra with bijective antipode. Then there exists a ring isomorphism* $(\mathrm{HH}^\bullet(A, A), \sqcup) \cong (\mathrm{H}^\bullet(A, A^{\mathrm{ad}}), \cup)$*, which induces an embedding of* $\mathrm{H}^\bullet(A, k)$ *into* $\mathrm{HH}^\bullet(A, A)$*. In particular, the ring* $\mathrm{H}^\bullet(A, k)$ *is graded-commutative.*

Proof. The counit $\varepsilon : A \to k$ provides an A-module splitting to the embedding $k \hookrightarrow A^{\mathrm{ad}}$, so $\mathrm{H}^\bullet(A, k)$ embeds as a subalgebra of $\mathrm{H}^\bullet(A, A^{\mathrm{ad}})$. Now apply Theorem 10. The last statement follows from Theorem 3. □

We can use Theorem 10 to give a new proof of the graded-commutativity relation in (13).

Corollary 12. *Let A be a Hopf algebra with bijective antipode, and let M be an A-bimodule. Let* $\zeta \in \mathrm{H}^n(A, k)$*, and let* $\eta \in \mathrm{H}^m(A, M^{\mathrm{ad}})$*. Then* $\zeta \cup \eta = (-1)^{nm} \eta \cup \zeta$*.*

Proof. Let $\Upsilon : \mathrm{H}^\bullet(A, k) \to \mathrm{H}^\bullet(A, A^{\mathrm{ad}})$ denote the embedding of Corollary 11. Then $\Upsilon(\zeta) \cup \eta = \zeta \cup \eta$ and $\eta \cup \Upsilon(\zeta) = \eta \cup \zeta$. Now apply Theorems 10 and 4. □

3.4. **Adjoint associativity and cup products.** Consider the *adjoint associativity* isomorphism $\mathrm{Hom}_{A^e}(B, N_\varepsilon) \xrightarrow{\sim} \mathrm{Hom}_A(B \otimes_A k, N)$. In §1.1 we obtained the graded space isomorphism (1) by taking $B = B(A, A)$, but we could just as well have taken B to be any A^e-projective resolution of A. Indeed, suppose B is an A^e-projective resolution of A, and consider B as a complex of A-bimodules. Since $A \otimes A$ is free as a right A-module, $B \to A$ is a resolution of A by projective right A-modules. Then B splits as a complex of right A-modules. It follows that $B \otimes_A k$ is exact, hence that $B \otimes_A k$ is an A-projective resolution of $A \otimes_A k = k$.

Now take $B = B(A) \uparrow_A^{A^e}$. The reader can easily check that $(B(A) \uparrow_A^{A^e}) \otimes_A k \cong B(A)$ as complexes of left A-modules. With this choice of resolution for B, it is easy to see that the isomorphism

$$\Phi : \mathrm{HH}^\bullet(A, N_\varepsilon) \xrightarrow{\sim} \mathrm{H}^\bullet(A, N)$$

induced by adjoint associativity coincides with the isomorphism

$$\Psi : \mathrm{HH}^\bullet(A, N_\varepsilon) \xrightarrow{\sim} \mathrm{H}^\bullet(A, (N_\varepsilon)^{\mathrm{ad}}) = \mathrm{H}^\bullet(A, N)$$

from the Eckmann–Shapiro Lemma.

We turn our attention to another form of adjoint associativity. Continue to assume that A is a Hopf algebra with bijective antipode. Assume that A acts on all Hom-spaces by the usual diagonal action, i.e, given left A-modules M and N, and writing $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ for $a \in A$, then a acts on $f \in \mathrm{Hom}_k(M, N)$ by $(a.f)(m) = \sum a_{(1)} f(S(a_{(2)})m)$. Since the antipode S of A is bijective, we have $\mathrm{Hom}_k(M, N)^A = \mathrm{Hom}_A(M, N)$ by [1, Proposition 2.9].

Let B, M, N be left A-modules. There is a vector space isomorphism

$$\Theta : \mathrm{Hom}_k(B, \mathrm{Hom}_k(M, N) \xrightarrow{\sim} \mathrm{Hom}_k(B \otimes M, N), \tag{18}$$

which takes the linear map $\psi : B \to \mathrm{Hom}_k(M, N)$ to the linear map $\Theta(\psi) : B \otimes M \to N$ with $\Theta(\psi)(b \otimes m) = \psi(b)(m)$. The vector space isomorphism is a homomorphism of A-modules, hence it induces an isomorphism of the spaces of A-invariants, i.e., an isomorphism

$$\Theta : \mathrm{Hom}_A(B, \mathrm{Hom}_k(M, N)) \xrightarrow{\sim} \mathrm{Hom}_A(B \otimes M, N). \tag{19}$$

The isomorphism is natural in B.

Given a left A-module M, let M_{tr} denote M considered as a trivial A-module. There there is an isomorphism of left A-modules $A \otimes M_{tr} \xrightarrow{\sim} A \otimes M$, defined by $a \otimes m \mapsto \sum a_{(1)} \otimes a_{(2)} m$, with inverse map given by $a \otimes m \mapsto \sum a_{(1)} \otimes S(a_{(2)}) m$. In particular, this shows that if X is a projective left A-module, then $X \otimes M$ is also a projective left A-module. Furthermore, if $X \to k$ is an A-projective resolution of k, then the Künneth Theorem implies that $X \otimes M$ is an A-projective resolution of M.

Now take $B = B(A)$, the bar resolution of A. The preceding comments imply that (19) induces an isomorphism of cohomology groups

$$\Theta : \mathrm{H}^\bullet(A, \mathrm{Hom}_k(M, N)) = \mathrm{Ext}_A^\bullet(k, \mathrm{Hom}_k(M, N)) \xrightarrow{\sim} \mathrm{Ext}_A^\bullet(M, N). \tag{20}$$

Lemma 13. *Let M be a left A-module, and let $\iota : k \to \mathrm{Hom}_k(M, M)$ denote the map defined by $1 \mapsto \mathrm{id}_M$. Then there exists a commutative triangle*

$$\begin{array}{ccc} \mathrm{H}^\bullet(A, k) & \xrightarrow{\iota_*} & \mathrm{H}^\bullet(A, \mathrm{Hom}_k(M, M)) \\ & \searrow^{\Phi_M} & \downarrow \Theta \\ & & \mathrm{Ext}_A^\bullet(M, M), \end{array}$$

where the vertical map is the isomorphism of (20)*, and the diagonal map* Φ_M *is the algebra homomorphism that takes* $[S] \in H^n(A,k)$ *to* $[S \otimes M] \in \mathrm{Ext}_A^n(M,M)$*.*

Proof. Let $S \in\in H^n(A,k)$. Lift the identity $\mathrm{id}_k : k \to k$ to a chain map $f : B(A) \to S$. Then $[S]$ is represented by the cocycle $f_n : B_n(A) \to k$, and $\iota_*([S])$ is represented by the cocycle $\iota \circ f_n : B_n(A) \to \mathrm{Hom}_k(M,M)$. Now $\Theta \circ \iota_*([S])$ is represented by the function $\Theta(\iota \circ f_n) : B_n(A) \otimes M \to M$, which is just $f_n \otimes \mathrm{id}_M : B_n(A) \otimes M \to k \otimes M = M$.

Conversely, to obtain a cocycle representative for $[S \otimes M] \in \mathrm{Ext}_A^n(M,M)$, we follow the procedure of Theorem 1. Take $X = B(A) \otimes M$; it is an A-projective resolution of M. If ∂ denotes the differential of $B(A)$, then the differential of X is $\partial \otimes \mathrm{id}_M$. Now $f \otimes \mathrm{id}_M : X \to S \otimes M$ is a chain map lifting the identity $\mathrm{id}_M : M \to M$. Then $(f \otimes \mathrm{id}_M)_n = f_n \otimes \mathrm{id}_M$ is a cocycle representative for $[S \otimes M] \in \mathrm{Ext}_A^n(M,M)$. This shows that $[S \otimes M]$ and $\Theta \circ \iota_*([S])$ are represented by the same cocycle $B_n(A) \to M$, hence that $- \otimes M = \Theta \circ \iota_*$. □

The following lemma is an immediate consequence of Theorem 5.

Lemma 14. *Let* $\Phi_M : \mathrm{H}^\bullet(A,k) \to \mathrm{Ext}_A^\bullet(M,M)$ *be the algebra homomorphism defined in Lemma 13. Fix* $\zeta \in \mathrm{H}^n(A,k)$*, and* $\eta \in \mathrm{Ext}_A^m(M,M)$*. Let* $\mathrm{id}_M \in \mathrm{Hom}_A(M,M)$ *denote the identity map, considered also as an element of* $\mathrm{Ext}_A^0(M,M)$*. Let* $\circ$ *denote the Yoneda composition product. Then* $\eta = \mathrm{id}_M \circ \eta = \eta \circ \mathrm{id}_M$*,*

$$\zeta \cup \eta = \zeta \cup (\mathrm{id}_M \circ \eta) = (\zeta \cup \mathrm{id}_M) \circ \eta = \Phi_M(\zeta) \circ \eta, \quad \textit{and}$$
$$\eta \cup \zeta = (\eta \circ \mathrm{id}_M) \cup \zeta = \eta \circ (\mathrm{id}_M \cup \zeta) = \eta \circ \Phi_M(\zeta).$$

Theorem 15. *Let* M, N *be left* A*-modules. Then there exists a commutative square*

$$\begin{array}{ccc} \mathrm{H}^n(A,k) \otimes \mathrm{H}^m(A, \mathrm{Hom}_k(M,N)) & \longrightarrow & \mathrm{H}^{n+m}(A, \mathrm{Hom}_k(M,N)) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{H}^n(A,k) \otimes \mathrm{Ext}_A^m(M,N) & \longrightarrow & \mathrm{Ext}_A^{n+m}(M,N), \end{array}$$

where the vertical maps are induced by the isomorphism (20)*, and horizontal maps are the corresponding cup products*

Proof. Let Θ denote the natural isomorphism $\mathrm{Hom}_A(- \otimes M, N) \xrightarrow{\sim} \mathrm{Hom}_A(-, \mathrm{Hom}_k(M,N))$. Fix $\zeta \in \mathrm{H}^n(A,k)$ and $\eta \in \mathrm{Ext}_A^m(M,N)$, and choose cocycle representatives $f \in \mathrm{Hom}_A(B_n(A),k)$ and $g \in \mathrm{Hom}_A(B_m(A) \otimes M, N)$ for ζ and η, respectively. Then $X := B(A) \otimes (B(A) \otimes M)$ is an A-projective resolution of M, and $\zeta \cup \eta \in \mathrm{Ext}_A^{n+m}(M,N)$ is represented by the cocycle $f \vee g \in \mathrm{Hom}_A(X,N)$. Conversely, the image $\Theta(\eta)$ of η in $\mathrm{H}^m(A, \mathrm{Hom}_k(M,N))$ is represented by the cocycle $\Theta(g) \in \mathrm{Hom}_A(B_m(A), \mathrm{Hom}_k(M,N))$. The complex $Y := B(A) \otimes B(A)$ is an A-projective resolution of k, and the cup product $\zeta \cup \Theta(\eta)$ is represented by the cocycle $f \vee \Theta(g) \in \mathrm{Hom}_A(Y, \mathrm{Hom}_k(M,N))$. The theorem now follows, because $\Theta^{-1}(f \vee \Theta(g)) = f \vee g$. □

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