Cohomology of Frobenius–Lusztig Kernels of Quantized Enveloping Algebras

Christopher Martin Drupieski
Bel Air, Maryland

B.A., McDaniel College, 2004
M.S., University of Virginia, 2008

A Dissertation Presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia
May 2009

Abstract

Let k be a field of characteristic $p \neq 2$. Let Φ be a finite, indecomposable root system, G the simple, simply-connected algebraic group over k having root system Φ, and $\mathfrak{g} = \mathrm{Lie}(G)$. Let $U_\zeta(\mathfrak{g})$ denote the Lusztig (divided power) quantized enveloping algebra corresponding to $\mathfrak{g}$, with parameter q specialized to a primitive ℓ-th root of unity $\zeta \in k$. The Frobenius–Lusztig kernels $U_\zeta(G_r)$ of $U_\zeta(\mathfrak{g})$ are certain finite-dimensional Hopf subalgebras of $U_\zeta(\mathfrak{g})$ that play a role in the (integrable) representation theory of $U_\zeta(\mathfrak{g})$ analogous to the role played by the Frobenius kernels G_r of G in the rational representation theory of G. If $r = 0$, then $U_\zeta(G_r) = u_\zeta(\mathfrak{g})$, the "small" quantum algebra discovered by Lusztig [48, 49]. The higher Frobenius–Lusztig kernels of $U_\zeta(\mathfrak{g})$ (i.e., those parametrized by values $r \geq 1$) exist only if $p > 0$.

The goal of this dissertation is to study the cohomology of the Frobenius–Lusztig kernels of $U_\zeta(\mathfrak{g})$ when $p > 0$. Our strategy parallels the characteristic zero work of Ginzburg and Kumar [30] and of Bendel, Nakano, Parshall and Pillen [9], as well as the earlier work on Frobenius kernels of algebraic groups by Friedlander and Parshall [27] and Andersen and Jantzen [2]. For $r = 0$, we show (in most cases) that the cohomology $H^\bullet(u_\zeta(\mathfrak{g}), k)$ of $u_\zeta(\mathfrak{g})$ is isomorphic as a G-module to the induced module $\mathrm{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$ (for some subset J of simple roots depending on ℓ). If additionally $\ell \geq h$, h the Coxeter number of $\mathfrak{g}$, then $H^\bullet(u_\zeta(\mathfrak{g}), k)$ is isomorphic as an algebra to $k[\mathcal{N}]$, the coordinate ring of the variety of nilpotent elements in $\mathfrak{g}$. For $r = 1$, we show (under certain restrictions on ℓ, p, and Φ) that the cohomology ring $H^\bullet(U_\zeta(G_1), k)$ is Noetherian. For arbitrary $r \geq 1$, we show that the cohomology rings $H^\bullet(U_\zeta(U_r), k)$ and $H^\bullet(U_\zeta(B_r), k)$ for the "nilpotent" and Borel subalgebras $U_\zeta(U_r)$ and $U_\zeta(B_r)$ of $U_\zeta(G_r)$ are also Noetherian.

Contents

Acknowledgements **iv**

0 Introduction **1**
- 0.1 Summary of main results . . . 2
- 0.2 Notational conventions . . . 4

1 Quantized enveloping algebras **5**
- 1.1 Definitions . . . 5
- 1.2 Root vectors and distinguished subalgebras . . . 10
- 1.3 Representation theory . . . 12

2 Cohomology of augmented algebras **14**
- 2.1 Actions on cohomology groups . . . 14
- 2.2 Spectral sequences . . . 22

3 Integral results for quantized enveloping algebras **26**
- 3.1 An integral commutation formula . . . 26
- 3.2 The De Concini–Kac integral form . . . 28
- 3.3 Adjoint actions on integral forms . . . 31

4 Cohomology of the Frobenius–Lusztig kernel **40**
- 4.1 Cohomology of the De Concini–Kac quantum algebra . . . 40
- 4.2 Cohomology of the Frobenius–Lusztig kernel . . . 46

5 Cohomology of higher Frobenius–Lusztig kernels **58**
- 5.1 Cohomological finite generation for Borel subalgebras . . . 58
- 5.2 Restriction maps for Borel subalgebras . . . 67
- 5.3 Restriction maps for reductive subalgebras . . . 68
- 5.4 The second Frobenius–Lusztig kernel . . . 73

A The nilpotent variety **77**
- A.1 Coordinate ring of the nullcone . . . 77
- A.2 Characters of the nullcone . . . 79

B Representations of higher Frobenius–Lusztig kernels 82

Acknowledgements

First and most importantly, I thank my advisor Dr Brian Parshall, without whose time, energy, support and encouragement this thesis would not have been possible. Thanks also go to Terrell Hodge, for devoting considerable time and energy during her sabbatical at UVA to helping me learn algebraic geometry and algebraic group theory, and for providing much encouragement since then.

I thank my friends among the graduate students at UVA, without whom these past five years would have been much less interesting. Finally, I thank Jennie Callas, for (almost) always being willing to listen to me prattle on about mathematics, even though I'm sure it rarely ever made any sense to her.

Portions of the work for this thesis were conducted while the author was under the support of NSF grant OISE–0813052.

Chapter 0

Introduction

Quantized enveloping algebras (also called quantum algebras or quantum groups) were introduced independently by Drinfel'd [23] and Jimbo [38] around 1985 as a tool to find solutions to the quantum Yang-Baxter equation, an equation from the field of statistical mechanics. The quantum algebra $\mathbb{U}_k(\mathfrak{g})$ with parameter q (an indeterminate) associated to the simple Lie algebra $\mathfrak{g}$ is a $k(q)$-algebra, defined, loosely speaking, as a "q-deformation" of the universal enveloping algebra of $\mathfrak{g}$. Following Lusztig [49], one can specialize the parameter q to a unit ζ in the (arbitrary) field k of characteristic $p \geq 0$ by first defining an integral $\mathbb{Z}[q, q^{-1}]$-form in $\mathbb{U}_k(\mathfrak{g})$, and then extending scalars to k. The resulting Hopf algebra is denoted by $U_\zeta(\mathfrak{g})$.

The representation theory of $U_\zeta(\mathfrak{g})$ depends greatly on the chosen parameter $\zeta \in k$. We are interested in the case when $\zeta \in k$ is a primitive ℓ-th root of unity, with ℓ odd and coprime to 3 if the root system Φ of $\mathfrak{g}$ has type G_2. In this case, the (integrable) representation theory of $U_\zeta(\mathfrak{g})$ closely resembles the rational representation theory of the simple, simply-connected algebraic group G over k having root system Φ. Many authors have successfully exploited this strong connection in order to study the structure and representation theory of the quantum algebra $U_\zeta(\mathfrak{g})$; see, for example, [5, 6, 9, 30] and others. The goal of this dissertation is to exploit connections between algebraic groups and quantized enveloping algebras in order to study the cohomology of certain finite-dimensional Hopf-subalgebras of $U_\zeta(\mathfrak{g})$, called the Frobenius–Lusztig kernels of $U_\zeta(\mathfrak{g})$.

In 1993, Ginzburg and Kumar [30] computed the structure of the cohomology ring $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ for the Frobenius–Lusztig kernel $u_\zeta(\mathfrak{g})$ of $U_\zeta(\mathfrak{g})$, assuming $\zeta \in \mathbb{C}$ to be a primitive ℓ-th root of unity with $\ell > h$ (h the Coxeter number of Φ). In exact analogy to the earlier work of Friedlander and Parshall [28] and of Andersen and Jantzen [2] on the cohomology of the first Frobenius kernel G_1 of G, Ginzburg and Kumar proved the existence of a graded G-algebra isomorphism between $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ and the coordinate ring $\mathbb{C}[\mathcal{N}]$ of the nilpotent variety $\mathcal{N} \subset \mathfrak{g}$. In particular, $\mathcal{N}$ being an affine variety, Ginzburg and Kumar showed that $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ is a Noetherian ring. Kazhdan and Verbitsky [40] independently reached this same qualitative conclusion around the same time, albeit only for ℓ an odd prime, via a more direct comparison of $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ with $H^\bullet(G_1, \overline{\mathbb{F}}_\ell)$, which was known to be finitely-generated by the work of Friedlander and Parshall [27]. More recently, Bendel, Nakano, Parshall and Pillen [9] have extended the computation of $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ to small values of ℓ, proving in most cases the existence of a G-module isomorphism between $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ and the coordinate ring of a nilpotent subvariety of $\mathcal{N}$. A principal goal of this dissertation is to extend the above characteristic zero cohomology calculations for $u_\zeta(\mathfrak{g})$ to fields k of positive characteristic.

The case $\text{char}(k) > 0$ is of special interest in part because it has received less attention in the literature than the $\text{char}(k) = 0$ case, but also because of the appearance of certain finite-dimensional Hopf-subalgebras $U_\zeta(G_r)$ ($r \geq 1$) in $U_\zeta(\mathfrak{g})$, the higher Frobenius–Lusztig kernels of $U_\zeta(\mathfrak{g})$, which do not exist in characteristic zero. These finite-dimensional Hopf algebras play a role in the positive characteristic (integrable) representation theory of $U_\zeta(\mathfrak{g})$ analogous to the role played by the Frobenius kernels G_r of G in the rational representation theory of G. A second principal motivation for this dissertation is the desire to generalize to the $U_\zeta(G_r)$ certain classical cohomological finite-generation results (e.g., [27, Theorem 1.11] and [29, Theorem 1.1]) that are well-known for the G_r.

0.1 Summary of main results

Chapter 1 provides an overview of the theory of quantized enveloping algebras as needed for this dissertation, including basic definitions, PBW-bases, and a brief summary of the integrable representation theory of $U_\zeta(\mathfrak{g})$. The material here is standard, and most of it can be found in [5] and [36].

Chapters 2 and 3 establish the necessary preliminary results for the cohomology calculations of Chapters 4 and 5. We begin in Chapter 2 by discussing the cohomology of an augmented algebra A with fixed normal subalgebra B. If A is right B-flat and if V is an A-module, it is well-known that the cohomology groups $H^\bullet(B, V)$ admit natural A-module structures extending the natural action of A on V^B [30, Lemma 5.2.1]. More generally, we show in Theorem 2.11 that if H is a Hopf algebra acting compatibly on A, B, and V, then $H^\bullet(B, V)$ admits a natural H-module structure. Some care is required in defining the notion of a compatible H-module structure: we require that A be a right H-module algebra, that B be a right H-submodule of A, and that V be a left H-module satisfying a certain compatibility condition (which is automatically satisfied if $H = A$). As special cases of Theorem 2.11, we obtain a new description for the action of A on $H^\bullet(B, V)$ whenever A is itself a Hopf algebra (see Example 2.16), and an easy proof of [30, Lemma 5.2.2] (which describes the action of A on $H^\bullet(B, k)$ in terms of derivations).

Chapter 2 culminates with Theorem 2.24, which states that the Lyndon–Hochschild–Serre spectral sequence relating the cohomology of A and B admits the structure of an H-module spectral sequence whenever A, B, and the coefficient module V admit compatible H-module structures. Theorem 2.24 is used in Chapter 4 in order to inductively compute the module structure of certain cohomology groups.

Chapter 3 is devoted to showing that the algebras considered in Chapters 4 and 5 satisfy the flatness and compatible action hypotheses of Chapter 2. Some of the results in Chapter 3 are already well-known if $\mathrm{char}(k) = 0$; with some extra work, we are able to apply base-change arguments in order to show that they hold over fields of (almost) arbitrary characteristic.

We begin in §3.1 by stating an integral version of a commutation formula originally observed by Levendorskii and Soibelman (and note some incorrect statements of the formula appearing in the literature). Next, in §3.2 we define the De Concini–Kac integral form $\mathcal{U}_{\mathcal{A}}(\mathfrak{g}) \subset \mathbb{U}_k(\mathfrak{g})$, and the De Concini–Kac quantum algebra $\mathcal{U}_\zeta(\mathfrak{g})$. The most important content Chapter 3 is in §3.3, where we show that the (right) adjoint action of $\mathbb{U}_k(\mathfrak{g})$ on itself induces an action of $U_\zeta(\mathfrak{g})$ on the Frobenius–Lusztig kernel $u_\zeta(\mathfrak{g})$ of $U_\zeta(\mathfrak{g})$, as well as actions of the parabolic subalgebra $U_\zeta(\mathfrak{p}_J) \subset U_\zeta(\mathfrak{g})$ on the "nilpotent" subalgebra $\mathcal{U}_\zeta(\mathfrak{u}_J) \subset \mathcal{U}_\zeta(\mathfrak{g})$ and the central subalgebra $Z_J \subset \mathcal{U}_\zeta(\mathfrak{u}_J)$. This establishes the existence of compatible Hopf algebra actions for $U_\zeta(\mathfrak{g})$ and $U_\zeta(\mathfrak{p}_J)$ in the sense of Chapter 2. Most of the results in §3.3 are obtained via base-change from the "integral" adjoint action result of Proposition 3.12.

In Chapter 4 we apply the results of Chapters 2 and 3 in order to compute the structure of the cohomology ring $H^\bullet(u_\zeta(\mathfrak{g}), k)$. Our strategy is essentially the same as that utilized in [9] for the characteristic zero case, though since we are interested in generalizing the results of [9] to positive characteristics, our arguments are occasionally different. For example, if $\mathrm{char}(k) = 0$, then [5, Theorem 9.12] and [4, Corollary 4.5] state that the Steinberg module St_ℓ is injective for $U_\zeta(\mathfrak{g})$, hence that the category of finite-dimensional $U_\zeta(\mathfrak{g})$-modules contains enough injectives, and every injective $u_\zeta(\mathfrak{g})$-module is the restriction of an injective $U_\zeta(\mathfrak{g})$-module. But if $\mathrm{char}(k) > 0$, then St_ℓ is not injective for $U_\zeta(\mathfrak{g})$, so one must invoke Proposition 4.12 and Lemma 2.1 in order to conclude that resolutions of finite-dimensional $U_\zeta(\mathfrak{g})$-modules by injective $U_\zeta(\mathfrak{g})$-modules are automatically resolutions by injective $u_\zeta(\mathfrak{g})$-modules. The results in Chapter 4 are also occasionally less sharp than those in [9]. For example, due to the failure of the Grauert–Riemenschneider vanishing theorem in positive characteristics, we must assume as hypotheses certain algebraic group cohomological vanishing conditions that are known if $\mathrm{char}(k) = 0$, but which are as yet unproved if $\mathrm{char}(k) > 0$.

The main results of Chapter 4 are Theorem 4.21 and Corollary 4.23, which state (under certain assumptions on ℓ and p) that the cohomology $H^\bullet(u_\zeta(\mathfrak{g}), k)$ vanishes in odd degree, and in even degree there exists a graded G-module isomorphism $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong \mathrm{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$. Moreover, if $\ell \geq h$, then $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong \mathrm{ind}_B^G S^\bullet(\mathfrak{u}^*)$ as algebras, and if k is algebraically closed, then $H^{2\bullet}(u_\zeta(\mathfrak{g}), k)$ can be identified with $k[\mathcal{N}]$, the coordinate ring of the variety of nilpotent elements in $\mathfrak{g}$. This generalizes the characteristic zero calculations of Ginzburg and Kumar [30] and of Bendel et al. [9], as well as the classical results of Friedlander and Parshall [28] and of Andersen and

Jantzen [2] on the cohomology of Frobenius kernels of algebraic groups.

Chapter 5 generalizes the results of Chapter 4 in two directions. First, in §5.1 we study the cohomology of the higher Frobenius–Lusztig kernels $U_\zeta(U_r)$ of the "nilpotent" subalgebra $U_\zeta(\mathfrak{u}) \subset U_\zeta(\mathfrak{g})$. After carefully analyzing the cohomology of an associated graded algebra in Propositions 5.3 and 5.4, we obtain in Theorem 5.6, for any finite-dimensional $U_\zeta(U_r)$-module M, that the cohomology group $H^\bullet(U_\zeta(U_r), M)$ is finitely-generated over the Noetherian ring $H^\bullet(U_\zeta(U_r), k)$. Analogous results are obtained simultaneously for the higher Frobenius–Lusztig kernels $U_\zeta(B_r)$ of the Borel subalgebra $U_\zeta(\mathfrak{b}) \subset U_\zeta(\mathfrak{g})$. This generalizes the classical result of Friedlander and Parshall on cohomology for Frobenius kernels of unipotent and solvable algebraic groups [27, Proposition 1.12].

Next, in §§5.2–5.3 we study the restriction maps $H^\bullet(u_\zeta(\mathfrak{b}'), k) \to H^\bullet(u_\zeta(\mathfrak{b}), k)$ and $H^\bullet(u_\zeta(\mathfrak{g}'), k) \to H^\bullet(u_\zeta(\mathfrak{g}), k)$ corresponding to an inclusion of root systems $\Phi' \subset \Phi$. If $\ell \geq h$, then under the identifications $H^\bullet(u_\zeta(\mathfrak{b}), k) \cong S^\bullet(\mathfrak{u}^*)$ and $H^\bullet(u_\zeta(\mathfrak{g}), k) \cong k[\mathcal{N}]$ of Corollary 4.20 and Corollary 4.23, the restriction maps are simply the restriction of functions. This realization is put to good effect in §5.4, where we show, under certain restrictions on p, ℓ, and Φ, that Theorems 4.24 and 5.6 generalize to the higher Frobenius–Lusztig kernel $U_\zeta(G_1)$ of $U_\zeta(\mathfrak{g})$, that is, given a finite-dimensional $U_\zeta(G_1)$-module M, the cohomology group $H^\bullet(U_\zeta(G_1), M)$ is finitely-generated over the Noetherian ring $H^\bullet(U_\zeta(G_1), k)$.

Appendix A provides a brief summary of results concerning the coordinate ring $k[\mathcal{N}]$ of the nilpotent variety $\mathcal{N} \subset \mathfrak{g}$ when $\mathrm{char}(k)$ is good for G, while Appendix B summarizes certain fundamental results on the irreducible representations of the higher Frobenius–Lusztig kernels $U_\zeta(G_r)$ of $U_\zeta(\mathfrak{g})$.

0.2 Notational conventions

In this paper we assume all rings to be associative and with unit. By an augmented algebra over k we mean a k-algebra A equipped with a k-algebra homomorphism $\varepsilon : A \to k$, called the *counit* or *augmentation map*. Denote the kernel of ε by A_ε. We say that a subalgebra B of A is a normal subalgebra of A if the left ideal in A generated by $B_\varepsilon := \ker \varepsilon|_B$ is also a right ideal in A. (Many authors define B to be normal in A if $AB_\varepsilon = B_\varepsilon A$; we have followed the convention of Cartan and Eilenberg [14, XVI.6] in adopting the less restrictive definition. If A and B are Hopf algebras with bijective antipode, then our definition implies the more restrictive equality.) If B is a normal subalgebra of A, write $A//B$ to denote the quotient $A/(AB_\varepsilon)$. Given a left A-module V, define the cohomology group $H^\bullet(A, V)$ by $H^\bullet(A, V) := \mathrm{Ext}^\bullet_A(k, V)$. (The A-module structure on k is defined by $\varepsilon : A \to k$.)

In this paper we use the symbol $\mathbb{N}$ to denote the set of all non-negative integers.

Chapter 1

Quantized enveloping algebras

In this chapter we recall the definition and basic properties of the quantized enveloping algebra $\mathbb{U}_k(\mathfrak{g})$ and its specialization $U_\zeta(\mathfrak{g})$. Our primary references are [3, 5, 36, 49].

1.1 Definitions

Let Φ be a finite, irreducible root system. Fix a set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots in Φ, and let Φ^+ (resp. Φ^-) denote the corresponding set of positive (resp. negative) roots in Φ. Let $\mathbb{Z}\Phi$ denote the root lattice of Φ, X the weight lattice of Φ, and $X^+ \subset X$ the subset of dominant weights. Let W denote the Weyl group of Φ. It is generated by the simple reflections $\{s_\alpha : \alpha \in \Pi\}$. The root system Φ spans a real vector space $\mathbb{E}$, possessing a positive definite, W-invariant inner product $(\cdot, \cdot)$, normalized so that $(\alpha, \alpha) = 2$ if $\alpha \in \Phi$ is a short root.

Let v be an indeterminate over $\mathbb{Z}$. For $a \in \mathbb{Z}$, put

$$[a] = \frac{v^a - v^{-a}}{v - v^{-1}} \in \mathbb{Z}[v, v^{-1}].$$

Recall that the Gaussian binomial coefficients are defined for $a \in \mathbb{Z}$ and $n \in \mathbb{N}$ by

$$\begin{bmatrix} a \\ n \end{bmatrix} = \frac{[a][a-1]\cdots[a-n+1]}{[1][2]\cdots[n]}$$

if $n > 0$, and by

$$\begin{bmatrix} a \\ 0 \end{bmatrix} = 1.$$

Define $[a]^!$ for $a \in \mathbb{N}$ by setting $[a]^! = [a][a-1]\cdots[2][1]$ if $a > 0$, and by setting $[0]^! = 1$. Then for all $a \geq n \geq 0$, we have

$$\begin{bmatrix} a \\ n \end{bmatrix} = \frac{[a]^!}{[n]^![a-n]^!}.$$

There exists an involutory automorphism of $\mathbb{Z}[v, v^{-1}]$ mapping $v \mapsto v^{-1}$. Denote it by $x \mapsto \overline{x}$. Then clearly $\overline{[a]} = [a]$ for all $a \in \mathbb{Z}$. One can show that the Gaussian binomial coefficients are all elements of $\mathbb{Z}[v, v^{-1}]$. It follows then that they too are invariant under the map $x \mapsto \overline{x}$.

Let k be a field of characteristic $p \neq 2$. Assume moreover that $p \neq 3$ if Φ is of type G_2. Let q be an indeterminate over k. For $\alpha \in \Pi$, set $q_\alpha = q^{d_\alpha}$, where $d_\alpha = (\alpha, \alpha)/2$. Then $d_\alpha \in \{1, 2, 3\}$. For $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, define

$$[a]_\alpha \quad \text{and} \quad \begin{bmatrix} a \\ n \end{bmatrix}_\alpha$$

to be the images in $k[q, q^{-1}]$ of the corresponding elements in $\mathbb{Z}[v, v^{-1}]$ under the unique ring homomorphism $\mathbb{Z}[v, v^{-1}] \to k[q, q^{-1}]$ with $v \mapsto q_\alpha$.

Let $\mathfrak{g}_\mathbb{C}$ denote the simple, complex Lie algebra with root system Φ. Then $\mathfrak{g}_\mathbb{C}$ admits a Chevalley basis (as defined, e.g., in [35, §25.2]). Let $\mathfrak{g} = \mathfrak{g}_k$ denote the Lie algebra over k obtained via base change from a Chevalley basis for $\mathfrak{g}_\mathbb{C}$. Then $\mathfrak{g} \cong \mathrm{Lie}(G)$, the Lie algebra of the simple, simply-connected algebraic group G over k of the same Lie type as Φ (cf. [10, §3.3] or [59, Remark, p.50]).

Definition 1.1. The quantized enveloping algebra $\mathbb{U}_k = \mathbb{U}_k(\mathfrak{g})$ is the $k(q)$-algebra with generators $\{E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1} : \alpha \in \Pi\}$ and relations (for all $\alpha, \beta \in \Pi$)

$$K_\alpha K_\alpha^{-1} = 1 = K_\alpha^{-1} K_\alpha, \qquad K_\alpha K_\beta = K_\beta K_\alpha \tag{1.1.1}$$

$$K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta, \tag{1.1.2}$$

$$K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta, \tag{1.1.3}$$

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}, \tag{1.1.4}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, and (for $\alpha \neq \beta$)

$$\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \begin{bmatrix} 1 - a_{\alpha\beta} \\ s \end{bmatrix}_\alpha E_\alpha^{1-a_{\alpha\beta}-s} E_\beta E_\alpha^s = 0, \tag{1.1.5}$$

$$\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \begin{bmatrix} 1 - a_{\alpha\beta} \\ s \end{bmatrix}_\alpha F_\alpha^{1-a_{\alpha\beta}-s} F_\beta F_\alpha^s = 0, \tag{1.1.6}$$

where $a_{\alpha\beta} \in \mathbb{Z}$ is defined by $a_{\alpha\beta} = (\beta, \alpha^\vee) := 2(\beta, \alpha)/(\alpha, \alpha)$.

Remark 1.2.

(1) In the literature, the algebra $\mathbb{U}_k(\mathfrak{g})$ is typically only defined for $\mathrm{char}(k) = 0$; we have followed the convention of [36] in permitting (almost) arbitrary fields.

(2) The algebra $\mathbb{U}_k(\mathfrak{g})$ is naturally $\mathbb{Z}\Phi$-graded if we assign E_α to have degree α, K_α and K_α^{-1} to each have degree zero, and F_α to have degree $-\alpha$. Given $\mu \in \mathbb{Z}\Phi$, the μ-graded component $\mathbb{U}_{k,\mu}$ of $\mathbb{U}_k$ is precisely the μ-weight space in $\mathbb{U}_k$ for the (left) adjoint action of $\mathbb{U}_k^0$ on $\mathbb{U}_k$; see Example 2.6 and §3.3 for details.

As usual, we let $\mathbb{U}_k^+$ denote the subalgebra of $\mathbb{U}_k$ generated by $\{E_\alpha : \alpha \in \Pi\}$, we let $\mathbb{U}_k^-$ denote the subalgebra of $\mathbb{U}_k$ generated by $\{F_\alpha : \alpha \in \Pi\}$, and we let $\mathbb{U}_k^0$ denote the subalgebra of $\mathbb{U}_k$ generated by $\{K_\alpha, K_\alpha^{-1} : \alpha \in \Pi\}$. Then the multiplication in $\mathbb{U}_k$ induces vector space isomorphisms $\mathbb{U}_k^+ \otimes \mathbb{U}_k^0 \otimes \mathbb{U}_k^- \xrightarrow{\sim} \mathbb{U}_k \xleftarrow{\sim} \mathbb{U}_k^- \otimes \mathbb{U}_k^0 \otimes \mathbb{U}_k^+$, and $\mathbb{U}_k^0$ admits a basis consisting of all monomials $\{K_\mu : \mu \in \mathbb{Z}\Phi\}$, where $K_\mu = \prod_{\alpha\in\Pi} K_\alpha^{n_\alpha}$ if $\mu = \sum_{\alpha\in\Pi} n_\alpha \alpha$ [36, Theorem 4.21]. Bases for $\mathbb{U}_k^+$ and $\mathbb{U}_k^-$ are described in §1.2.

The quantized enveloping algebra $\mathbb{U}_k(\mathfrak{g})$ admits the structure of a Hopf algebra, with comultiplication Δ, counit ε, and antipode S satisfying (for each $\alpha \in \Pi$)

$$\begin{aligned} \Delta(E_\alpha) &= E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, & \varepsilon(E_\alpha) &= 0, & S(E_\alpha) &= -K_\alpha^{-1}E_\alpha, \\ \Delta(F_\alpha) &= F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, & \varepsilon(F_\alpha) &= 0, & S(F_\alpha) &= -F_\alpha K_\alpha, \\ \Delta(K_\alpha) &= K_\alpha \otimes K_\alpha, & \varepsilon(K_\alpha) &= 1, & S(K_\alpha) &= K_\alpha^{-1}. \end{aligned} \tag{1.1.7}$$

Evidently, $\mathbb{U}_k(\mathfrak{g})$ is neither commutative nor cocommutative as a Hopf algebra.

Lemma 1.3.

(1) There exists a unique $k(q)$-algebra automorphism ω of $\mathbb{U}_k(\mathfrak{g})$ with $\omega(E_\alpha) = F_\alpha$, $\omega(F_\alpha) = E_\alpha$, and $\omega(K_\alpha) = K_\alpha^{-1}$. One has $\omega^2 = \mathrm{id}$.

(2) There exists a unique k-algebra anti-automorphism κ of $\mathbb{U}_k(\mathfrak{g})$ with $\kappa(E_\alpha) = F_\alpha$, $\kappa(F_\alpha) = E_\alpha$, $\kappa(K_\alpha) = K_\alpha^{-1}$, and $k(q) = q^{-1}$. One has $\kappa^2 = \mathrm{id}$.

Proof. One must check that the images of the generators satisfy the defining relations for $\mathbb{U}_k(\mathfrak{g})$. This is clear from Definition 1.1. □

We can twist the Hopf algebra structure maps (Δ, ε, S) for $\mathbb{U}_k(\mathfrak{g})$ by the automorphism ω, in order to obtain a new set of Hopf algebra structure maps for $\mathbb{U}_k(\mathfrak{g})$,

$$(\overline{\Delta}, \overline{\varepsilon}, \overline{S}) = (\omega \otimes \omega \circ \Delta \circ \omega, \varepsilon, \omega \circ S \circ \omega),$$

satisfying

$$\begin{aligned} \overline{\Delta}(E_\alpha) &= 1 \otimes E_\alpha + E_\alpha \otimes K_\alpha, & \overline{\varepsilon}(E_\alpha) &= 0, & \overline{S}(E_\alpha) &= -E_\alpha K_\alpha^{-1}, \\ \overline{\Delta}(F_\alpha) &= F_\alpha \otimes 1 + K_\alpha^{-1} \otimes F_\alpha, & \overline{\varepsilon}(F_\alpha) &= 0, & \overline{S}(F_\alpha) &= -K_\alpha F_\alpha, \\ \overline{\Delta}(K_\alpha) &= K_\alpha \otimes K_\alpha, & \overline{\varepsilon}(K_\alpha) &= 1, & \overline{S}(K_\alpha) &= K_\alpha^{-1}. \end{aligned} \tag{1.1.8}$$

We will investigate the twisted Hopf algebra structure for $\mathbb{U}_k(\mathfrak{g})$ in §3.3.

Now set $\mathsf{A} = \mathbb{Z}[q, q^{-1}]$ to be the ring of Laurent polynomials over $\mathbb{Z}$ in the indeterminate q. Let U_{A} denote the A-subalgebra of $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ generated by

$$\left\{E_\alpha^{(m)}, F_\alpha^{(m)}, K_\alpha, K_\alpha^{-1} : m \geq 0, \alpha \in \Pi\right\},$$

where the divided powers $E_\alpha^{(m)}, F_\alpha^{(m)} \in \mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ are defined by $E_\alpha^{(m)} = E_\alpha^m/[m]_\alpha^!$, $F_\alpha^{(m)} = F_\alpha^m/[m]_\alpha^!$. Then U_{A} is a Hopf-subalgebra of $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$. It is an A-form for $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ [49, Theorem 6.7(d)]. If $\mathscr{A}$ is a commutative algebra admitting the structure of a left A-module, then we set $U_{\mathscr{A}} = U_{\mathsf{A}} \otimes_{\mathsf{A}} \mathscr{A}$.

Remark 1.4.

(1) In this paper, the symbol $\mathbb{U}$ will always denote a quantized enveloping algebra defined over a rational function field, while the symbol U will always denote an algebra (or the quotient of such an algebra) obtained via base change from the divided power integral form U_{A}.

(2) We use the symbols $E_\alpha^{(m)}, F_\alpha^{(m)}, K_\alpha, K_\alpha^{-1}$, etc., to denote the corresponding elements of U_{A} as well as their images in the various specializations $U_{\mathscr{A}}$. We trust that our meaning will be clear from the context.

It is easy to see that $\mathbb{U}_k(\mathfrak{g}) \cong U_{k(q)}$. Indeed, there exists a canonical surjective algebra homomorphism $\mathbb{U}_k(\mathfrak{g}) \twoheadrightarrow U_{k(q)}$ mapping the generators for $\mathbb{U}_k(\mathfrak{g})$ to the corresponding elements in $U_{k(q)}$. The algebras $\mathbb{U}_k(\mathfrak{g})$ and $U_{k(q)}$ each admit PBW-type bases as in §1.2 (cf. [36, §8.24] and [49, Theorem 6.7]), and the canonical map $\mathbb{U}_k(\mathfrak{g}) \twoheadrightarrow U_{k(q)}$ takes the PBW-basis for $\mathbb{U}_k(\mathfrak{g})$ to the PBW-basis for $U_{k(q)}$, hence is an isomorphism.

Now set $\mathsf{B} = k[q, q^{-1}]$. The ring B admits an obvious left A-module structure. Under the identification $\mathbb{U}_k(\mathfrak{g}) \cong U_{k(q)}$, the subalgebra $U_{\mathsf{B}} = U_{\mathsf{A}} \otimes_{\mathsf{A}} k[q, q^{-1}]$ of $U_{k(q)}$ is the B-subalgebra of $\mathbb{U}_k(\mathfrak{g})$ generated by the subset

$$\left\{E_\alpha^{(m)}, F_\alpha^{(m)}, K_\alpha, K_\alpha^{-1} : m \geq 0, \alpha \in \Pi\right\} \subset \mathbb{U}_k(\mathfrak{g}).$$

Given a unit $\zeta \in k^\times$, we get an A-module structure on k by specializing $q \mapsto \zeta$. In this paper we typically assume ζ to be a primitive ℓ-th root of unity in k, with the following assumption on ℓ:

Assumption 1.5. Assume ℓ be be an odd positive integer that is coprime to 3 if Φ has type G_2. Assume moreover that ℓ is not a bad prime for Φ. (See §A.1 for a list of the bad primes corresponding to each irreducible root system.)

Definition 1.6. Let $\zeta \in k^\times$ be a primitive ℓ-th root of unity. Assume that ℓ satisfies the conditions of Assumption 1.5. Define $U_k = U_{\mathsf{A}} \otimes_{\mathsf{A}} k$, where the A-module structure on A is obtained via the ring homomorphism $\mathbb{Z}[q, q^{-1}] \to k$ with $q \mapsto \zeta$, and define $U_\zeta = U_\zeta(\mathfrak{g})$ to be the quotient of U_k by the two-sided ideal $\langle K_\alpha^\ell \otimes 1 - 1 \otimes 1 : \alpha \in \Pi\rangle$, that is,

$$U_\zeta = U_\zeta(\mathfrak{g}) := U_k/\langle K_\alpha^\ell \otimes 1 - 1 \otimes 1 : \alpha \in \Pi\rangle.$$

In Definition 1.6, we could have equivalently defined the algebra U_k via $U_k = U_{\mathsf{B}} \otimes_{\mathsf{B}} k$. The algebras U_k and U_ζ are Hopf algebras, with structure maps induced by those of $\mathbb{U}_k(\mathfrak{g})$. For each $\alpha \in \Pi$ we have in U_k the relations $E_\alpha^\ell = F_\alpha^\ell = K_\alpha^{2\ell} - 1 = 0$. If $p > 0$, then we additionally have $(E_\alpha^{(p^i\ell)})^p = (F_\alpha^{(p^i\ell)})^p = 0$ in U_k for all $i \geq 0$. The algebras U_k and U_ζ are then generated as algebras by the set

$$\left\{E_\alpha, E_\alpha^{(p^i\ell)}, F_\alpha, F_\alpha^{(p^i\ell)}, K_\alpha : \alpha \in \Pi, i \geq 0\right\}, \tag{1.1.9}$$

where by convention we set $0^0 = 1$ if $p := \mathrm{char}(k) = 0$.

Definition 1.7. Define $u_k = u_k(\mathfrak{g})$ to be the subalgebra of U_k generated by

$$\{E_\alpha, F_\alpha, K_\alpha : \alpha \in \Pi\}, \tag{1.1.10}$$

and define $u_\zeta = u_\zeta(\mathfrak{g})$ to be the image of u_k in U_ζ under the natural projection map $U_k \twoheadrightarrow U_\zeta$. We call $u_\zeta(\mathfrak{g})$ the Frobenius–Lusztig kernel of $U_\zeta(\mathfrak{g})$.

The algebras u_k and u_ζ are finite-dimensional Hopf algebras, with Hopf algebra structure maps inherited from U_k. We have $\dim u_\zeta = \ell^{\dim(\mathfrak{g})}$.

Let $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ denote the universal enveloping algebra of the complex Lie algebra $\mathfrak{g}_\mathbb{C}$. Fix a Chevalley basis $\{X_\alpha, \alpha \in \Phi; h_i, 1 \leq i \leq n\}$ for $\mathfrak{g}_\mathbb{C}$. The Kostant $\mathbb{Z}$-form $\mathfrak{U}(\mathfrak{g}_\mathbb{C})_\mathbb{Z}$ of $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ is the $\mathbb{Z}$-subalgebra of $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ generated by the set of divided powers

$$\left\{X_\alpha^{(n)} := X_\alpha^n/(n!) : \alpha \in \Phi, n \in \mathbb{N}\right\}.$$

We have $\Delta(X_\alpha^{(n)}) = \sum_{i=0}^n X_\alpha^{(i)} \otimes X_\alpha^{(n)}$ and $S(X_\alpha^{(n)}) = (-1)^n X_\alpha^{(n)}$ (recall that $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ is a Hopf algebra), so $\mathfrak{U}(\mathfrak{g}_\mathbb{C})_\mathbb{Z}$ is also a Hopf algebra. Now set $\mathrm{hy}(G) = \mathfrak{U}(\mathfrak{g}_\mathbb{C})_\mathbb{Z} \otimes_\mathbb{Z} k$. The algebra $\mathrm{hy}(G)$ is called the hyperalgebra of G. (The algebra $\mathrm{hy}(G)$ is also known as the *algebra of distributions on* G, and denoted by $\mathrm{Dist}(G)$; see [37, I.7, II.1.12] for details.) It inherits from $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ the structure of a cocommutative Hopf algebra. The category of rational G-modules is naturally isomorphic to the category of locally finite $\mathrm{hy}(G)$-modules [60]. If $k = \mathbb{C}$, then $\mathrm{hy}(G) \cong \mathfrak{U}(\mathfrak{g}_\mathbb{C})$, though in general the structure of $\mathrm{hy}(G)$ is more complicated.

The finite-dimensional Hopf algebra $u_\zeta(\mathfrak{g})$ defined above is a normal subalgebra of U_ζ (the normality condition can be checked, for example, using [49, Lemma 8.5], and then a base-change argument if $\mathrm{char}(k) \neq 0$). We have $U_\zeta//u_\zeta \cong \mathrm{hy}(G)$ as Hopf algebras. The quotient map $\mathrm{Fr} : U_\zeta \twoheadrightarrow \mathrm{hy}(G)$ is induced by the Frobenius morphism discovered by Lusztig in [49, §§8.10–8.16]. (This explains the terminology "Frobenius–Lusztig kernel.") Given a $\mathrm{hy}(G)$-module M, denote by $M^{[1]}$ the U_ζ-module obtained by pulling back the $\mathrm{hy}(G)$-module structure on M to U_ζ via Fr. Conversely, if N is U_ζ-module on which u_ζ acts trivially, then $N = M^{[1]}$ for some $\mathrm{hy}(G)$-module M.

If $p := \mathrm{char}(k) > 0$, then $\mathrm{hy}(G)$ is a directed union of finite-dimensional Hopf-subalgebras $\mathrm{hy}(G_r)$ $(r \geq 1)$. The algebra $\mathrm{hy}(G_r)$ is the hyperalgebra of G_r, the r-th

Frobenius kernel of G. As a subalgebra of $\mathrm{hy}(G) = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$, $\mathrm{hy}(G_r)$ is generated by the set

$$\left\{ X_\alpha^{(n)} \otimes 1 : \alpha \in \Phi, 1 \leq n < p^r \right\}.$$

If $p > 0$, then there exist corresponding finite-dimensional Hopf-subalgebras in $U_\zeta(\mathfrak{g})$.

Definition 1.8. Assume $p := \mathrm{char}(k) > 0$, and fix $r \in \mathbb{N}$. Define $U_\zeta(G_r)$ to be the (finite-dimensional) Hopf-subalgebra of $U_\zeta(\mathfrak{g})$ generated by

$$\left\{ E_\alpha, E_\alpha^{(p^i\ell)}, F_\alpha, F_\alpha^{(p^i\ell)}, K_\alpha : \alpha \in \Pi, 0 \leq i \leq r-1 \right\}. \tag{1.1.11}$$

We call $U_\zeta(G_r)$ the r-th Frobenius–Lusztig kernel of $U_\zeta(\mathfrak{g})$.

Note that $U_\zeta(G_r) = u_\zeta(\mathfrak{g})$ if $r = 0$. For convenience of notation, set

$$U_\zeta(G_r) = U_\zeta(\mathfrak{g}) \quad \text{if } r = \infty.$$

Collectively, we refer to the algebras $U_\zeta(G_r)$ $(r \geq 1)$ as the higher Frobenius–Luszig kernels of $U_\zeta(\mathfrak{g})$. We have $\mathrm{Fr}(U_\zeta(G_r)) = \mathrm{hy}(G_r)$ for all $r \geq 1$.

1.2 Root vectors and distinguished subalgebras

For each $\alpha \in \Pi$, Lusztig has defined an automorphism T_α of $\mathbb{U}_k$ [49]. We follow the notation of [36, Chapter 8]. Given $\alpha \in \Pi$, the automorphism T_α satisfies the following relations (for $\beta \in \Pi$, $\beta \neq \alpha$, and $r = -(\beta, \alpha^\vee)$):

$$T_\alpha(K_\mu) = K_{s_\alpha(\mu)} \qquad \text{for all } \mu \in \mathbb{Z}\Phi, \tag{1.2.1}$$

$$T_\alpha(E_\alpha) = -F_\alpha K_\alpha, \tag{1.2.2}$$

$$T_\alpha(F_\alpha) = -K_\alpha^{-1} E_\alpha, \tag{1.2.3}$$

$$T_\alpha(E_\beta) = \sum_{i=0}^{r} (-1)^i q_\alpha^{-i} E_\alpha^{(r-i)} E_\beta E_\alpha^{(i)}, \tag{1.2.4}$$

$$T_\alpha(F_\beta) = \sum_{i=0}^{r} (-1)^i q_\alpha^{i} F_\alpha^{(i)} F_\beta F_\alpha^{(r-i)}. \tag{1.2.5}$$

The automorphisms T_α satisfy the braid relations for W, hence if $w = s_{\beta_1} \cdots s_{\beta_n}$ is a reduced expression for $w \in W$ in terms of simple reflections $s_{\beta_i} \in W$ $(\beta_i \in \Pi)$, then $T_w := T_{\beta_1} \cdots T_{\beta_n}$ is a well-defined automorphism of $\mathbb{U}_k$, independent of the particular reduced expression for $w \in W$.

Let $w_0 \in W$ denote the longest word in W, and fix a reduced expression $w_0 = s_{\beta_1} \cdots s_{\beta_N}$. (So $N = |\Phi^+|$.) Set $\gamma_1 = \beta_1$, and, for $1 < i \leq N$, set $\gamma_i = s_{\beta_1} \cdots s_{\beta_{i-1}}(\beta_i)$. Then $\Phi^+ = \{\gamma_1, \ldots, \gamma_N\}$. This is the convex ordering of the roots in Φ^+ corresponding to the chosen reduced expression for w_0 (i.e., given $1 \leq i < j \leq N$, if $\gamma_i + \gamma_j = \gamma_k$, then $i < k < j$; cf. [20, Proposition 8.2]). We can define a linear order $\prec$ on Φ^+ by $\alpha \prec \beta$ if $\alpha = \gamma_i$, $\beta = \gamma_j$, and $i < j$.

For $\gamma = \gamma_i \in \Phi^+$, define the root vector $E_\gamma \in \mathbb{U}_k$ by

$$E_\gamma = E_{\gamma_i} := T_{\beta_1} \cdots T_{\beta_{i-1}} (E_{\beta_i})$$

Then $E_\gamma \in \mathbb{U}_k^+$, and if $\gamma \in \Pi$, then this expression equals the original generator $E_\gamma \in \mathbb{U}_k(\mathfrak{g})$ [36, Proposition 8.20].

The antiautomorphism κ of $\mathbb{U}_k(\mathfrak{g})$ defined in Lemma 1.3 commutes with the automorphisms T_α (i.e., $\kappa \circ T_\alpha = T_\alpha \circ \kappa$ for each $\alpha \in \Pi$). Given $\gamma \in \Phi^+$, define the root vector $F_\gamma \in \mathbb{U}_k^-$ by

$$F_\gamma = F_{\gamma_i} := \kappa(E_{\gamma_i}) = T_{\beta_1} \cdots T_{\beta_{i-1}} (F_{\beta_i}).$$

Since $\kappa(\mathbb{U}_k^+) = \mathbb{U}_k^-$, we have $F_\gamma \in \mathbb{U}_k^-$, and if $\gamma \in \Pi$, then this expression equals the original generator $F_\gamma \in \mathbb{U}_k(\mathfrak{g})$.

Let $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{N}^N$. Define $E^{\mathbf{m}} \in \mathbb{U}_k^+$ by $E^{\mathbf{m}} = E_{\gamma_1}^{m_1} \cdots E_{\gamma_N}^{m_N}$. Then the monomials $\left\{E^{\mathbf{m}} : \mathbf{m} \in \mathbb{N}^N\right\}$ constitute a PBW-type basis for the subalgebra $\mathbb{U}_k^+$ of $\mathbb{U}_k$ [36, Theorem 8.24]. Defining the divided power $E_\gamma^{(m)} \in \mathbb{U}_k^+$ by

$$E_\gamma^{(m)} = T_{\beta_1} \cdots T_{\beta_{i-1}} (E_{\beta_i}^{(m)}),$$

the corresponding monomials of divided powers $\left\{E^{(\mathbf{m})} : \mathbf{m} \in \mathbb{N}^N\right\}$ form an A-basis for the algebra U_{A}^+, hence also a B-basis for U_{B}^+ and k-bases for U_k^+ and U_ζ^+ [49, Theorem 6.7(d)]. (The k-algebras U_k^+ and U_ζ^+ are isomorphic.) Application of the antiautomorphism κ gives analogous results for the algebras $\mathbb{U}_k^-, U_{\mathsf{A}}^-, U_{\mathsf{B}}^-, U_k^-, U_\zeta^-$.

For $\alpha \in \Pi$, $n \in \mathbb{N}$ and $a \in \mathbb{Z}$, define

$$\begin{bmatrix} K_\alpha; a \\ n \end{bmatrix} = \prod_{i=1}^{n} \frac{K_\alpha q_\alpha^{a-i+1} - K_\alpha^{-1} q_\alpha^{-(a-i+1)}}{q_\alpha^i - q_\alpha^{-i}}.$$

(If $n = 0$, set the right side of the above expression equal to 1.) Then U_{A}^0 admits an A-basis consisting of all monomials

$$\prod_{\alpha \in \Pi} \left(K_\alpha^{\delta_\alpha} \begin{bmatrix} K_\alpha; 0 \\ n_\alpha \end{bmatrix} \right)$$

with $n_\alpha \in \mathbb{N}$ and $\delta_\alpha \in \{0, 1\}$ [49, Theorem 6.7(c)]. Combining this with the previously mentioned A-bases for U_{A}^+ and U_{A}^-, we obtain the following A-basis for U_{A}:

$$\left\{ F^{(\mathbf{r})} \prod_{\alpha \in \Pi} \left(K_\alpha^{\delta_\alpha} \left[\begin{smallmatrix} K_\alpha; 0 \\ n_\alpha \end{smallmatrix}\right] \right) E^{(\mathbf{s})} : \mathbf{r}, \mathbf{s} \in \mathbb{N}^N, n_\alpha \in \mathbb{N}, \delta_\alpha \in \{0, 1\} \right\}. \tag{1.2.6}$$

Given a subset $J \subseteq \Pi$, let $\Phi_J = \Phi \cap \mathbb{Z}J$ denote the corresponding subroot system of Φ, and let $W_J = \langle s_\alpha : \alpha \in J \rangle$ denote the corresponding subgroup of W. Let

$w_{0,J} \in W_J$ denote the longest word in W_J. We can choose a reduced expression for w_0 beginning with one for $w_{0,J}$, say, $w_0 = w_{0,J} w_J$ for some reduced word $w_J \in W$ satisfying $\ell(w_0) = \ell(w_{0,J}) + \ell(w_J)$. Set $M = \left|\Phi_J^+\right|$. Then $\{\gamma_1, \ldots, \gamma_M\}$ lists the positive roots in Φ_J^+, and $\{\gamma_{M+1}, \ldots, \gamma_N\}$ lists the remaining positive roots in Φ^+.

Fix $J \subseteq \Pi$, and consider the corresponding standard Levi and parabolic subalgebras $\mathfrak{l}_J$ and $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{u}_J$ of $\mathfrak{g}$. There exist corresponding subalgebras $\mathbb{U}_k(\mathfrak{l}_J)$ and $\mathbb{U}_k(\mathfrak{p}_J)$ of $\mathbb{U}_k(\mathfrak{g})$. Specifically, $\mathbb{U}_q(\mathfrak{l}_J)$ is the subalgebra of $\mathbb{U}_k(\mathfrak{g})$ generated by $\{E_\alpha, F_\alpha : \alpha \in J\} \cup \{K_\alpha, K_\alpha^{-1} : \alpha \in \Pi\}$, and $\mathbb{U}_q(\mathfrak{p}_J)$ is the subalgebra of $\mathbb{U}_k(\mathfrak{g})$ generated by $\{E_\alpha : \alpha \in J\} \cup \{F_\alpha, K_\alpha, K_\alpha^{-1} : \alpha \in \Pi\}$. Taking divided powers and specializing $q \mapsto \zeta$, we obtain the subalgebras $U_\zeta(\mathfrak{l}_J)$, $U_\zeta(\mathfrak{p}_J)$, $u_\zeta(\mathfrak{l}_J)$ and $u_\zeta(\mathfrak{p}_J)$ of U_ζ. Note that $U_\zeta^0 \subset U_\zeta(\mathfrak{l}_J)$. Applying the Frobenius morphism, we get $\mathrm{Fr}(U_\zeta(\mathfrak{l}_J)) = \mathrm{hy}(L_J)$ and $\mathrm{Fr}(U_\zeta(\mathfrak{p}_J)) = \mathrm{hy}(P_J)$, the hyperalgebras of L_J and P_J, the standard Levi and parabolic subgroups of G corresponding to the subset $J \subseteq \Pi$.

Given $J \subseteq \Pi$, define $\mathbb{U}_k(\mathfrak{u}_J^+)$ to be the $k(q)$-subspace of $\mathbb{U}_k(\mathfrak{g})$ spanned by all mononomials $\left\{E_{\gamma_{M+1}}^{a_{M+1}} \cdots E_{\gamma_N}^{a_N} : a_i \in \mathbb{N}\right\}$, and define $\mathbb{U}_k(\mathfrak{u}_J)$ to be the $k(q)$-subspace of $\mathbb{U}_k(\mathfrak{g})$ spanned by all mononomials $\left\{F_{\gamma_N}^{a_N} \cdots F_{\gamma_{M+1}}^{a_{M+1}} : a_i \in \mathbb{N}\right\}$. According to [36, Proposition 8.22], the space $\mathbb{U}_k(\mathfrak{u}_J)$ depends only on $w_J = w_{0,J} w_0$, and not on the chosen reduced expression for w_J. It will follow from Lemma 3.1 that $\mathbb{U}_k(\mathfrak{u}_J)$ is in fact a subalgebra of $\mathbb{U}_k(\mathfrak{g})$. Taking divided powers and specializing $q \mapsto \zeta$, we obtain corresponding subalgebras $u_\zeta(\mathfrak{u}_J) \subset U_\zeta(\mathfrak{u}_J) \subset U_\zeta(\mathfrak{g})$ that depend only on J and not on the chosen reduced expression for w_J.

Remark 1.9. Given $\mu \in \mathbb{Z}\Phi$, let $\mathbb{U}_{k,\mu}$ denote the μ-graded component of $\mathbb{U}_k(\mathfrak{g})$. According to [36, 8.14(9)], for each $\alpha \in \Pi$, we have $\omega \circ T_\alpha = (-q_\alpha)^{(\mu,\alpha^\vee)} T_\alpha \circ \omega$ as functions on $\mathbb{U}_{k,\mu}$. This implies that, up to plus or minus a power of q, $\omega(E_\gamma)$ is equivalent to F_γ for each $\gamma \in \Phi^+$. In particular, $\omega(\mathbb{U}_k(\mathfrak{p}_J^+)) = \mathbb{U}_k(\mathfrak{p}_J)$ and $\omega(\mathbb{U}_k(\mathfrak{u}_J^+)) = \mathbb{U}_k(\mathfrak{u}_J)$. Conversely, the equalities $\kappa(\mathbb{U}_k(\mathfrak{p}_J^+)) = \mathbb{U}_k(\mathfrak{p}_J)$ and $\kappa(\mathbb{U}_k(\mathfrak{u}_J^+)) = \mathbb{U}_k(\mathfrak{u}_J)$ are immediate from the fact that κ commutes with the automorphisms T_α.

1.3 Representation theory

Let $\mathscr{A}$ be an algebra admitting the structure of a left A-module. Recall that a $U_{\mathscr{A}}$-module M is said to be an integrable $U_{\mathscr{A}}$-module if M admits a weight space decomposition for $U_{\mathscr{A}}^0$ in the sense of [5, §1.2], and if for all $\alpha \in \Pi$, the operators $E_\alpha^{(n)}, F_\alpha^{(n)}$ are locally nilpotent on M (i.e., if for each fixed $m \in M$, $E_\alpha^{(n)} m = F_\alpha^{(n)} m = 0$ for all $n \gg 0$). The category of integrable modules for $U_{\mathscr{A}}$ decomposes as a direct sum of isomorphic subcategories $\mathcal{C}_\sigma$, indexed by the set of functions $\Sigma = \{\sigma : \Pi \to \{\pm 1\}\}$. We call $\mathcal{C}_\sigma$ the category of type σ integrable modules for $U_{\mathscr{A}}$.

Let $\mathcal{C}$ denote the category of type 1 integrable modules for U_k (i.e., $\mathcal{C}$ is the subcategory indexed by the constant function $\sigma(\alpha) \equiv 1$). Then $\mathcal{C}$ identifies with the category of integrable modules for U_ζ. Let $\mathcal{C}^{\leq}$ denote the category of integrable representations for the Borel subalgebra $U_\zeta(\mathfrak{b})$. The restriction functor $\mathrm{res} : \mathcal{C} \to \mathcal{C}^{\leq}$ possesses a right adjoint $H^0(-) = H^0(U_\zeta/U_\zeta(\mathfrak{b}), -) : \mathcal{C}^{\leq} \to \mathcal{C}$, called the induction functor. The induction functor is also commonly denoted by $\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})}(-)$. The counit $\varepsilon : \mathrm{res} \circ H^0(M) \to M$ of the adjunction $\eta : \mathrm{res} \dashv H^0(-)$ is called the evaluation map. It is a homomorphism of $U_\zeta(\mathfrak{b})$-modules.

Let $\lambda \in X^+$. Then the induced module $\nabla_\zeta(\lambda) := H^0(U_\zeta/U_\zeta(\mathfrak{b}), \lambda)$ has an irreducible socle, denoted $L_\zeta(\lambda)$, and every irreducible module in $\mathcal{C}$ arises in this fashion. The formal character of $\nabla_\zeta(\lambda)$ is given by Weyl's character formula. In general, $L_\zeta(\lambda) \neq \nabla_\zeta(\lambda)$, though we do have equality if $\lambda = (\ell - 1)\rho$, where $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. We denote the module $\nabla_\zeta((\ell - 1)\rho)$ by St_ℓ, and call it the Steinberg module. The Steinberg module is injective and projective for $u_\zeta(\mathfrak{g})$.

Given $J \subseteq \Pi$, let $X_J^+ \subset X$ be the set of J-dominant weights (i.e., $\lambda \in X_J^+$ if $(\lambda, \alpha^\vee) \in \mathbb{N}$ for all $\alpha \in J$). The set X_J^+ indexes the (isomorphism classes of) irreducible integrable $U_\zeta(\mathfrak{l}_J)$-modules. Given $\lambda \in X_J^+$, $\nabla_{\zeta,J}(\lambda) := H^0(U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{b}), \lambda)$ has irreducible socle isomorphic to $L_{\zeta,J}(\lambda)$, the irreducible integrable $U_\zeta(\mathfrak{l}_J)$-module of highest weight λ. The algebra $U_\zeta(\mathfrak{u}_J)$ (hence also $u_\zeta(\mathfrak{u}_J)$) acts trivially on $\nabla_{\zeta,J}(\lambda)$. If $(\lambda, \alpha^\vee) = \ell - 1$ for each $\alpha \in J$, then we call λ a J-Steinberg weight. If λ is a J-Steinberg weight, then $\nabla_{\zeta,J}(\lambda)$ is irreducible for $U_\zeta(\mathfrak{l}_J)$, and is injective and projective for $u_\zeta(\mathfrak{l}_J)$.

The category $\mathcal{C}$ is a proper subcategory of the category of all $U_\zeta(\mathfrak{g})$-modules (the left regular module for $U_\zeta(\mathfrak{g})$ is not integrable, for instance, because $E_\alpha^{(n)}, F_\alpha^{(n)}$ act by left multiplication on $U_\zeta(\mathfrak{g})$ as non-zero operators for all $n \in \mathbb{N}$), though all finite-dimensional $U_\zeta(\mathfrak{g})$-modules are objects in $\mathcal{C}$ [24, Appendix 3]. In contrast to the usual convention for treating the representation theory of algebraic groups and quantized enveloping algebras, in this paper we do not assume all of our modules to be integrable. In those cases where we do assume a module to be integrable, the assumption will be stated explicitly. By convention, every module for the finite-dimensional subalgebra $U_\zeta(G_r)$ $(0 \leq r < \infty)$ is called integrable.

Chapter 2

Cohomology of augmented algebras

In this chapter we assemble various results concerning the cohomology of augmented algebras. The content of this chapter will be utilized heavily in the cohomology calculations of Chapters 4 and 5.

2.1 Actions on cohomology groups

In this section, let A be a k-algebra with augmentation $\varepsilon : A \to k$. Let B denote a fixed normal subalgebra of A, and let V be an A-module. Since B is normal in A, the subspace $V^B = \{v \in V : bv = \varepsilon(b)v \;\forall\, b \in B\} \cong \mathrm{Hom}_B(k, V)$ is an A-submodule of V. Then the map $-^B : V \mapsto V^B$ defines an endofunctor on the category of A-modules. Of course, the A-module structure on V^B factors through the quotient $A//B$, so we also view $-^B$ as a functor from the category of A-modules to the category of $A//B$-modules, a full subcategory of the category of A-modules.

The cohomology $H^n(B, W)$ of B with coefficients in the B-module W is defined by $H^n(B, W) = \mathrm{Ext}^n_B(k, W) = R^n(\mathrm{Hom}_B(k, -))(W)$. In this context, $\mathrm{Hom}_B(k, -)$ is considered as a functor from the category of B-modules to the category of k-vector spaces. The right derived functors of $\mathrm{Hom}_B(k, -)$ are defined in terms of B-injective resolutions, whereas the right derived functors of $-^B$ are defined in terms of A-injective resolutions. The following lemma tells us that, given an A-module V, we can identify $R^n(-^B)(V)$ with $H^n(B, V)$ as k-vector spaces, provided A is flat as a right B-module.

Lemma 2.1. [7, Lemma I.4.3] Every injective A-module is injective for B if and only if A is flat as a right B-module.

Recall that, given a left exact functor $\mathcal{F}$ between abelian categories $\mathfrak{A}$ and $\mathfrak{B}$, the right derived functors $R^\bullet\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ form a universal cohomological δ-functor in the sense of [64, Chapter 2].

Lemma 2.2. Suppose that A is right B-flat. Then for each A-module V, there exists a unique natural extension of the action of A on V^B to an action of A on $H^\bullet(B,V)$. This action of A on $H^\bullet(B,V)$ factors through the quotient $A//B$.

Proof. The first statement follows from the observation that $-^B$, viewed as an endofunctor on the category of A-modules, is a universal cohomological δ-functor. To obtain the second statement, note that the endofunctor $-^B$ can be factored as a composition of functors $-^B = \mathcal{F}' \circ \mathcal{F}$, where $\mathcal{F}(V) = V^B$ is viewed as a functor from the category of A-modules to the category of $A//B$-modules, and $\mathcal{F}'$ is the full embedding of the category of $A//B$-modules into the category of A-modules. The functor $\mathcal{F}'$ is exact, so $\mathcal{F}' \circ R^m\mathcal{F} \cong R^m(\mathcal{F}' \circ \mathcal{F})$ for all $m \in \mathbb{N}$ [37, I.4.1(2)]. Since $R^m\mathcal{F}$ has image in the category of $A//B$-modules, this implies the lemma. □

We can provide one description for the action of A on $H^\bullet(B,V)$ as follows. Suppose A is right B-flat. For each A-module V and B-module W, there exists a natural isomorphism $\operatorname{Hom}_B(W,V) \cong \operatorname{Hom}_A(A \otimes_B W, V)$. In particular, if $W = k$ with the trivial B-action, then $\operatorname{Hom}_B(k,V) \cong \operatorname{Hom}_A(A//B, V)$. This implies the existence of a natural isomorphism $\operatorname{Ext}_B^\bullet(k,V) \cong \operatorname{Ext}_A^\bullet(A//B,V)$. Now, the cohomology groups $\operatorname{Ext}_A^\bullet(A//B,V)$ admit a natural A-module structure induced by the right multiplication of $A//B$ on itself. Identifying $H^\bullet(B,V) \cong \operatorname{Ext}_A^\bullet(A//B,V)$, we obtain a natural action of A on $H^\bullet(B,V)$ extending the action of A on V^B.

The following theorem provides a useful criterion for determining whether A is right B-free (in particular, whether A is right B-flat).

Theorem 2.3. [41, Corollary 1.3, Corollary 1.4] Assume that A is a Hopf algebra and that B is a normal Hopf subalgebra of A. Suppose the left regular representation of B can be extended to A. Let $\pi : A \to A//B$ be the natural projection map. Regard $\pi(A)$ as an A-module via π, and give $\pi(A) \otimes B$ and $B \otimes \pi(A)$ the diagonal A-module structures. Then $A \cong \pi(A) \otimes B$ as left A-modules. If the antipodes of A and B are bijective, then $A \cong B \otimes \pi(A)$ as left A-modules, and we can form a smash product $B\#\pi(A)$ in such a way that $A \cong B\#\pi(A)$ as algebras.

Our next goal is to investigate the actions of Hopf algebras on the cohomology groups $H^\bullet(B,V)$. When A is itself a Hopf algebra, this will give a new description for the action of A on $H^\bullet(B,V)$. First recall the notion of an H-module algebra.

Definition 2.4. Let H be a Hopf algebra over k with comultiplication Δ, counit ε, and antipode S. Given $h \in H$, write $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ (Sweedler notation).

(1) An algebra A is a left H-module algebra if, for all $h \in H$,

  (a) A is a left H-module via $h \otimes a \mapsto h \cdot a$,
  (b) $h \cdot (aa') = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot a')$ for all $a, a' \in A$, and
  (c) $h \cdot 1_A = \varepsilon(h) 1_A$.

(2) An algebra A is a right H-module algebra if, for all $h \in H$,

   (a) A is a right H-module via $a \otimes h \mapsto a \cdot h$,
   (b) $(aa') \cdot h = \sum (a \cdot h_{(1)})(a' \cdot h_{(2)})$ for all $a, a' \in A$, and
   (c) $1_A \cdot h = \varepsilon(h) 1_A$.

(We use the $\cdot$ notation to emphasize the action of H on A. This notation is cumbersome, so we will often omit the $\cdot$ and write $h \cdot a = ha$ or $a \cdot h = ah$.)

Definition 2.5. Let A be an augmented algebra over k, and let H be a Hopf algebra over k. Then we say that A is a left (resp. right) augmented H-module algebra if A is a left (resp. right) H-module algebra in the sense of Definition 2.4, and if additionally $\varepsilon(h \cdot a) = \varepsilon(h)\varepsilon(a)$ (resp. $\varepsilon(a \cdot h) = \varepsilon(h)\varepsilon(a)$) for all $a \in A$ and $h \in H$.

In this paper we are interested exclusively in augmented H-module algebras, so from now on we will drop the adjective "augmented" and refer to augmented H-module algebras simply as H-module algebras. If A is an (augmented) H-module algebra, then A_ε is (more or less by definition) an H-submodule of A.

Example 2.6. Let H be a Hopf algebra. Fix $h, k \in H$. The left adjoint action of h on k is given by

$$h \cdot k = \mathrm{Ad}_l(h)(k) = \sum h_{(1)} k S(h_{(2)}).$$

The right adjoint action of h on k is given by

$$k \cdot h = \mathrm{Ad}_r(h)(k) = \sum S(h_{(1)}) k h_{(1)}.$$

The linear maps $\mathrm{Ad}_l : H \to \mathrm{End}_k(H)$ and $\mathrm{Ad}_r : H \to \mathrm{End}_k(H)^{op}$ are algebra homomorphisms. We typically drop the subscript l and write $\mathrm{Ad} = \mathrm{Ad}_l$. The Hopf algebra H is both a left and a right H-module algebra via the left and right adjoint actions of H on itself [53, Example 4.1.9].

Our next definition is modeled on the special case $A = H$ of Example 2.6.

Definition 2.7. Let A be an augmented algebra over k, and let V be a left A-module. Let H be a Hopf algebra. Assume that A is an H-module algebra, and that V admits the structure of a left H-module.

(1) Suppose A is a left H-module algebra. We say that V admits compatible A and H-module structures if for all $v \in V$, $a \in A$ and $h \in H$, we have

$$h.(a.v) = \sum (h_{(1)} \cdot a).(h_{(2)}.v).$$

(2) Suppose A is a right H-module algebra. We say that V admits compatible A and H-module structures if for all $v \in V$, $a \in A$ and $h \in H$, we have

$$a.(h.v) = \sum h_{(1)}.\left((a \cdot h_{(2)}).v\right).$$

Example 2.8. Take $A = H$, and let V be a left H-module. We can consider A as either a left or right H-module algebra via the two adjoint actions of H on itself introduced in Example 2.6. In either case, the defining axioms for a Hopf algebra imply that V admits compatible A and H-module structures. More generally, suppose A is a subalgebra of H stable under the adjoint action of H on itself, and let V be an H-module, viewed as an A-module by restriction. Then V admits compatible A and H-module structures.

Example 2.9. The trivial module k admits compatible A and H-module structures.

Lemma 2.10. Let A be a right H-module algebra, and let V be a left A-module admitting a compatible left H-module structure. Then V^A is an H-submodule of V.

Proof. Let $v \in V^A$, and let $h \in H$. Then for all $a \in A$ we have

$$a.(h.v) = \sum h_{(1)}.\left((a \cdot h_{(2)}).v\right) = \varepsilon(a) \sum h_{(1)}\varepsilon(h_{(2)}).v = \varepsilon(a)h.v.$$

This proves the claim. □

We now recall certain chain complexes useful for the computation of cohomology. The left bar resolution $\mathsf{B}_\bullet(B)$ of B is the chain complex with $\mathsf{B}_n(B) = B \otimes B_\varepsilon^{\otimes n}$ and differential $d_n : \mathsf{B}_n(B) \to \mathsf{B}_{n-1}(B)$ defined for $n \geq 1$ by

$$d_n(b \otimes [b_1|\cdots|b_n]) = bb_1 \otimes [b_2|\cdots|b_n] + \sum_{i=1}^{n-1}(-1)^i b \otimes [b_1|\cdots|b_ib_{i+1}|\cdots|b_n]. \quad (2.1.1)$$

If we define $\varepsilon : \mathsf{B}_0(B) = B \to k$ to be the counit, then $\mathsf{B}_\bullet(B) \to k$ is a resolution of k by free left B-modules. Given a B-module W, set $C^\bullet(B, W) = \operatorname{Hom}_B(\mathsf{B}_\bullet(B), W)$, and let $\delta_n = \operatorname{Hom}_B(d_{n+1}, W) : C^n(B, W) \to C^{n+1}(B, W)$ denote the induced cochain map. Then $H^n(B, W) = H^n(C^\bullet(B, W), \delta)$.

Suppose that the augmented algebra A is a right H-module algebra, and that the normal subalgebra B of A is an H-submodule of A. Then the right action of H on B extends diagonally to an action of H on $\mathsf{B}_\bullet(B)$:

$$(b \otimes [b_1|\cdots|b_n]) \cdot h = \sum (bh_{(1)}) \otimes [b_1h_{(2)}|\cdots|b_nh_{(n+1)}]. \quad (2.1.2)$$

Since A is an H-module algebra, it follows that the H-action on $\mathsf{B}_\bullet(B)$ commutes with the differential of $\mathsf{B}_\bullet(B)$,

$$d_n(b \otimes [b_1|\cdots|b_n]) \cdot h = d_n(b \otimes [b_1|\cdots|b_n] \cdot h), \quad (2.1.3)$$

hence that $\mathsf{B}_\bullet(B)$ is a complex of right H-modules and H-module homomorphisms.

Recall that if M is a right H-module and if N is a left H-module, then we obtain a left H-module structure on $\mathrm{Hom}_k(M,N)$ by setting

$$(h.f)(m) = \sum h_{(1)}.f(m \cdot h_{(2)}). \tag{2.1.4}$$

If M is finite dimensional, then there exists an H-module isomorphism $\mathrm{Hom}_k(M,N) \cong N \otimes M^*$. The H-action on $N \otimes M^*$ is given by $h.(n \otimes f) = \sum h_{(1)}.n \otimes h_{(2)}.f$, and the H-action on M^* is given by $(h.f)(m) = f(m \cdot h)$.

Now let V be a left A-module admitting a compatible H-module structure. Consider the left action of H on $\mathrm{Hom}_B(\mathsf{B}_n(B),V) \subset \mathrm{Hom}_k(\mathsf{B}_n(B),V)$. We have

$$\begin{aligned}
(h.f)\left(b \otimes [b_1|\cdots|b_n]\right) &= \sum h_{(1)}.f\left(b \otimes [b_1|\cdots|b_n] \cdot h_{(2)}\right)\\
&= \sum h_{(1)}.f\left((bh_{(2)}) \otimes [b_1h_{(3)}|\cdots|b_nh_{(n+2)}]\right)\\
&= \sum h_{(1)}.\left((bh_{(2)}).f(1 \otimes [b_1h_{(3)}|\cdots|b_nh_{(n+2)}])\right)\\
&= \sum b.\left(h_{(1)}.f(1 \otimes [b_1h_{(2)}|\cdots|b_nh_{(n+1)}])\right) && (*)\\
&= b.(h.f)(1 \otimes [b_1|\cdots|b_n]).
\end{aligned}$$

The equality at line $(*)$ is obtained by using the compatible A and H-module structures on V. This calculation shows that the left action of H on $\mathrm{Hom}_k(\mathsf{B}_n(B),V)$ stabilizes the subspace $\mathrm{Hom}_B(\mathsf{B}_n(B),V)$. We can also show that the left action of H on $C^n(B,V) = \mathrm{Hom}_B(\mathsf{B}_n(B),V)$ commutes with the differential δ of the cochain complex $C^\bullet(B,V) = \mathrm{Hom}_B(\mathsf{B}_\bullet(B),V)$. Indeed, for $f \in C^{n-1}(B,V)$, we have

$$\begin{aligned}
(h.(\delta f))(b \otimes [b_1|\cdots|b_n]) &= \sum h_{(1)}.(\delta f)(b \otimes [b_1|\cdots|b_n] \cdot h_{(2)})\\
&= \sum h_{(1)}.f(d_n(b \otimes [b_1|\cdots|b_n] \cdot h_{(2)}))\\
&= \sum h_{(1)}.f(d_n(b \otimes [b_1|\cdots|b_n]) \cdot h_{(2)}) && \text{by (2.1.3)}\\
&= (h.f)(d_n(b \otimes [b_1|\cdots|b_n]))\\
&= (\delta(h.f))(b \otimes [b_1|\cdots|b_n]).
\end{aligned}$$

The above calculations establish the following theorem.

Theorem 2.11. Let A be an augmented algebra over k, and B a normal subalgebra of A. Assume that A is a right H-module algebra, and that B is an H-submodule of A. Let V be a left A-module with a compatible left H-module structure. Then (2.1.2) and (2.1.4) define a left H-module structure on $C^\bullet(B,V) = \mathrm{Hom}_B(\mathsf{B}_\bullet(B),V)$ such that $C^\bullet(B,V)$ is a complex of H-modules and H-module homomorphisms. The left action of H on $C^\bullet(B,V)$ induces a left action of H on $H^\bullet(B,V)$.

Definition 2.12. We call the left action of H on $H^\bullet(B,V)$ defined in Theorem 2.11 the adjoint action of H on $H^\bullet(B,V)$.

Remark 2.13. If M and N are left H-modules, then the standard diagonal action of H on $\mathrm{Hom}_k(M,N)$ is defined by $(h.f)(m) = \sum h_{(1)}.f(S(h_{(2)}).m)$. This definition is problematic for the purposes of defining actions of Hopf algebras on cohomology:

(1) If the algebra A in Lemma 2.10 were only a left H-module algebra, then it is not clear in general why V^A should be an H-submodule of V.

(2) Suppose the normal subalgebra B were only a left H-module algebra. The left action of H on B extends diagonally to a left action of H on $\mathsf{B}_\bullet(B)$. But then it is not clear in general that the standard diagonal action of H on $\mathrm{Hom}_k(\mathsf{B}_n(B),V)$ should stabilize the subspace $\mathrm{Hom}_B(\mathsf{B}_n(B),V)$, nor is it clear in general that the standard diagonal action of H on $\mathrm{Hom}_k(\mathsf{B}_n(B),V)$ should commute with the differential δ of the cochain complex $C^\bullet(B,V)$.

For these reasons, we do not consider the standard diagonal action of H on a Hom-set when defining Hopf algebra actions on cohomology.

Recall that the cup product defines a ring structure on $H^\bullet(B,k)$.

Lemma 2.14. Let A, B, H be as in Theorem 2.11. The adjoint action of H on the cohomology ring $H^\bullet(B,k)$ makes $H^\bullet(B,k)$ a left H-module algebra.

Proof. Set $C^\bullet(B) = C^\bullet(B,k) \cong \mathrm{Hom}_k(B_\varepsilon^{\otimes\bullet},k)$. Choose cocycles $f \in C^n(B)$, $g \in C^m(B)$, and let $\mathrm{cls}(f) \in H^n(B,k)$, $\mathrm{cls}(g) \in H^m(B,k)$ denote the images of f and g in $H^\bullet(B,k)$. The cup product of $\mathrm{cls}(f)$ and $\mathrm{cls}(g)$ in $H^\bullet(B,k)$ is equal to $\mathrm{cls}(f \smile g)$, where $f \smile g \in C^{n+m}(k)$ is the cocycle defined by

$$(f \smile g)([b_1|\cdots|b_{n+m}]) = f([b_1|\cdots|b_n])g([b_{n+1}|\cdots|b_{n+m}]) \tag{2.1.5}$$

Now for $h \in H$, we have

$$\begin{aligned}(h.(f \smile g))([b_1|\cdots|b_{n+m}]) &= (f \smile g)([b_1|\cdots|b_{n+m}]\cdot h)\\ &= \sum (f \smile g)([b_1h_{(1)}|\cdots|b_{n+m}h_{(n+m)}])\\ &= \sum f([b_1h_{(1)}|\cdots|b_nh_{(n)}])g([b_{n+1}h_{(n+1)}|\cdots|b_{n+m}h_{(n+m)}])\\ &= \sum (h_{(1)}.f)([b_1|\cdots|b_n])(h_{(2)}.g)([b_{n+1}|\cdots|b_{n+m}])\end{aligned}$$

So $h.\mathrm{cls}(f \smile g) = \sum (h_{(1)}.\mathrm{cls}(f))(h_{(2)}.\mathrm{cls}(g))$. □

Example 2.15. Let Π be a group and let $\Gamma \trianglelefteq \Pi$ be a normal subgroup. Set $A = H = k\Pi$, and set $B = k\Gamma$. Let V be a Π-module. It follows from [50, Exercise XI.9.3] that the adjoint action of $k\Pi$ on $H^\bullet(\Gamma,A) = H^\bullet(k\Gamma,A)$ is induced by the usual action of Π on $H^\bullet(\Gamma,A)$ (which is itself induced by the conjugation action of Π on Γ). Since the action of Π on $H^\bullet(\Gamma,A)$ factors through the quotient Π/Γ, we conclude that the adjoint action of $k\Pi$ on $H^\bullet(\Gamma,A)$ factors through the quotient $k\Pi//k\Gamma = k(\Pi/\Gamma)$.

More generally, we have:

Example 2.16. Suppose $A = H$ is a Hopf algebra, and that the normal subalgebra B of A is stable under the right adjoint action of A on itself. Then A is a right H-module algebra via the right adjoint action of A on itself, B is an H-submodule of A, and every left A-module V admits compatible A and H-module structures. We see that the adjoint action of A on $H^\bullet(B, V)$ is natural in V and extends the given action of A on V^B. If A is right B-flat, it follows then that the adjoint action of A on $H^\bullet(B, V)$ is equivalent to the action of A on $H^\bullet(B, V)$ described in Lemma 2.2. In particular, the adjoint action of A on $H^\bullet(B, V)$ factors through the quotient $A//B$.

The previous two examples are special cases of a more general situation, which we describe below. First, let A be a right H-module algebra, and let V be a left A-module admitting a compatible H-module structure. We describe an A-injective resolution $Q_\bullet = Q_\bullet(V)$ of V such that each Q_n admits a compatible H-module structure, and such that the differential δ of $Q_\bullet$ is an H-module homomorphism.

Recall the bimodule bar resolution $\mathsf{B}_\bullet(A, A) = A \otimes A_\varepsilon^{\otimes \bullet} \otimes A$, with differential $d_n : \mathsf{B}_n(A, A) \to \mathsf{B}_{n-1}(A, A)$, defined for $n > 0$ by

$$\begin{aligned} d_n(a \otimes [a_1|\cdots|a_n] \otimes a') = {} & aa_1 \otimes [a_2|\cdots|a_n] \otimes a' \\ & + \sum_{i=1}^{n-1} (-1)^i a \otimes [a_2|\cdots|a_i a_{i+1}|\cdots|a_n] \otimes a' \\ & + (-1)^n a \otimes [a_1|\cdots|a_{n-1}] \otimes a_n a', \end{aligned}$$

and $d_0 : A \otimes A \to A$ defined by $d_0(a \otimes a') = aa'$. Now form the complex $Q_\bullet(V) = \mathrm{Hom}_A(\mathsf{B}_\bullet(A, A), V)$, where $\mathrm{Hom}_A(-, V)$ is taken with respect to the left A-module structure of $\mathsf{B}_\bullet(A, A)$. The right A-module structure of $\mathsf{B}_n(A, A)$ induces the structure of a left A-module on $Q_n(V)$. Then $Q_\bullet(V)$ is an A-injective resolution of V. After Barnes [7, VI.2], we call $Q_\bullet(V)$ the coinduced resolution of V. For convenience, set $B_{-1}(A, A) = A$. Then $V = \mathrm{Hom}_A(B_{-1}(A, A), V) = Q_{-1}(V)$, and the inclusion $V \hookrightarrow Q_0(V)$ corresponds to the map $d_0 : B_0(A, A) \to A$. Clearly, the construction $V \mapsto Q_\bullet(V)$ is natural in V.

The right action of H on A extends diagonally to a right action of H on the complex $\mathsf{B}_\bullet(A, A) \to A$ such that the differentials are H-module homomorphisms. Using (2.1.4), we define a left H-module structure on $\mathrm{Hom}_k(\mathsf{B}_n(A, A), V)$, which restricts to a left H-module structure on $\mathrm{Hom}_A(\mathsf{B}_n(A, A), V) = Q_n(V)$. As in the calculation prior to Theorem 2.11, we easily check that the differential δ of $Q_\bullet(V)$ is an H-module homomorphism. It remains to show that the A and H-module structures on $Q_n(V)$ are compatible. Indeed, let $a \in A$, $h \in H$, and $f \in Q_n(V)$. For simplicity of notation, write $x = a' \otimes [a_1|\cdots|a_n] \otimes a'' = a'[a_1|\cdots|a_n]a''$. Then

$$\begin{aligned}(a.(h.f))(x) &= (h.f)(a'[a_1|\cdots|a_n]a''a) \\ &= \sum h_{(1)}.f(a'[a_1|\cdots|a_n]a''a \cdot h_{(2)}) \\ &= \sum h_{(1)}.f\left((a'h_{(2)})[a_1h_{(3)}|\cdots|a_nh_{(n+2)}](a''h_{(n+3)})(ah_{(n+4)})\right) \\ &= \sum h_{(1)}.((ah_{(n+4)}).f)\left((a'h_{(2)})[a_1h_{(3)}|\cdots|a_nh_{(n+2)}](a''h_{(n+3)})\right) \\ &= \sum (h_{(1)}.((ah_{(2)}).f))(x), \qquad (*)\end{aligned}$$

that is, $a.(h.f) = \sum h_{(1)}.\left((ah_{(2)}).f\right)$. (The equality at line $(*)$ is obtained using the identity $(\Delta^{n+3} \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta^{n+3}) \circ \Delta$, cf. [53, 1.4.2].) So the A and H-module structures on $Q_n(V)$ are compatible.

Now suppose that A, B, H, V satisfy the hypotheses of Theorem 2.11, and that A is right B-flat. Let $Q_\bullet = Q_\bullet(V)$ be the coinduced resolution of V defined above. Then the cohomology group $H^n(B, V)$ may be computed as either $H^n(\mathrm{Hom}_B(\mathsf{B}_\bullet(B), V)$ or as $H^n(\mathrm{Hom}_B(k, Q_\bullet))$. From Lemma 2.10 and Theorem 2.11, we conclude the existence of two (possibly inequivalent) H-module structures on $H^n(B, V)$, namely, the adjoint action of H on $H^n(B, V)$, and an H-module structure induced by the H-module structure of Q_n. That these two H-module structures are equivalent is the content of the following proposition.

Proposition 2.17. Maintain the notations and assumptions of the previous paragraph. The two left H-module structures on $H^\bullet(B, V)$ are equivalent via a natural isomorphism $H^n(\mathrm{Hom}_B(\mathsf{B}_\bullet(B), V)) \xrightarrow{\sim} H^n(\mathrm{Hom}_B(k, Q_\bullet))$.

Proof. The proof is a corollary of the proof of [54, Proposition 3.11]. Set $\mathsf{B}_{-1}(B) = k$, and set $Q_{-1} = V$. Form the first quadrant double complex $C = C^{\bullet,\bullet}$ by setting $C^{i,j} = \mathrm{Hom}_B(\mathsf{B}_{i-1}(B), Q_{j-1})$ if $(i, j) \neq (0, 0)$, and $C^{0,0} = 0$. Then C is a complex of H-modules and H-module homomorphisms. All rows and columns of C are exact, except possibly the bottom row and the leftmost column. The homology of the bottom row and that of the leftmost column both compute $H^\bullet(B, V) = \mathrm{Ext}^\bullet_B(k, V)$.

Denote the vertical and horizontal differentials of $C^{\bullet,\bullet}$ by $d_{i,j} : C^{i,j} \to C^{i,j+1}$ and $\partial_{i,j} : C^{i,j} \to C^{i+1,j}$. Define $Z_n \subset \bigoplus_{i=0}^n C^{i,n-i}$ by $(x_0, \ldots, x_n) \in Z_n$ if and only if $\partial(x_j) = d(x_{j+1})$, $d(x_0) = 0$ and $\partial(x_n) = 0$. Set $B_0 = 0$, and for $n > 0$, define $B_n \subset \bigoplus_{i=0}^n C^{i,n-i}$ to be the set of all $(y_0, \ldots, y_n) \in \bigoplus_{i=0}^n C^{i,n-i}$ such that there exists $(z_0, \ldots, z_{n-1}) \in \bigoplus_{i=0}^{n-1} C^{i,n-i-1}$ satisfying $y_j = d(z_j) + \partial(z_{j-1})$, $y_0 = d(z_0)$ and $y_n = \partial(z_{n-1})$. In the proof of [54, Proposition 3.11], Osborne shows $B_n \subset Z_n$. The projections onto the first and last coordinates define maps $Z_n \to \ker(d_{0,n})$ and $Z_n \to \ker(\partial_{n,0})$, respectively. Osborne shows that when composed with the projections $\ker(d_{0,n}) \twoheadrightarrow \ker(d_{0,n})/\operatorname{im}(d_{0,n-1})$ and $\ker(\partial_{n,0}) \twoheadrightarrow \ker(\partial_{n,0})/\operatorname{im}(\partial_{n-1,0})$, these maps induce isomorphisms

$$Z_n/B_n \cong H^n(C^{0,\bullet}, d_{0,\bullet}) \quad \text{and} \tag{2.1.6}$$
$$Z_n/B_n \cong H^n(C^{\bullet,0}, \partial_{\bullet,0}). \tag{2.1.7}$$

Since all of the differentials in C are H-module homomorphisms, it follows that (2.1.6) and (2.1.7) are isomorphisms of H-modules. We have $H^n(\mathrm{Hom}_B(\mathsf{B}_\bullet(B), V)) = H^{n+1}(C^{\bullet,0}, \partial_{\bullet,0})$, and $H^n(\mathrm{Hom}_B(k, Q_\bullet)) = H^{n+1}(C^{0,\bullet}, d_{0,\bullet})$. Finally, the naturality of the isomorphism is clear, because the construction of $Q_\bullet(V)$ and the construction of C were natural in V. □

Remark 2.18. Maintain the notations and assumptions of Proposition 2.17. The A and H-module structures on $H^\bullet(B, V)$ are compatible in the sense of Definition 2.7, because they are induced by the A and H-module structures of $Q_n = Q_n(V)$, and the A and H-module structures on $Q_n(V)$ are compatible.

We end this section by reproving the well-known result that the action of A on $H^\bullet(B, k)$ is trivial whenever B is central in A. First consider the following more general situation: Let $a \in A$, and suppose that the inner derivation $D_a = [a, -]$ of A descends to a derivation of B. Set H to be the universal enveloping algebra of the one dimensional Lie subalgebra of A generated by a. Then B is naturally a right H-module algebra, with $a \in H$ acting on $b \in B$ via $b \cdot a = [b, a] = -D_a(b)$.

Proposition 2.19. [30, Lemma 5.2.2] Suppose A is right B-flat. Let $a \in A$, and suppose $D_a(B) \subset B$. Let $\overline{a}$ denote the image of a in $A//B$. Then, for each A-module V, the action of $\overline{a}$ on $H^\bullet(B, V)$ in Lemma 2.2 is equivalent to the action of a on $H^\bullet(B, V)$ induced by D_a.

Proof. Apply Proposition 2.17. □

Corollary 2.20. Suppose A is right B-flat, and that B is central in A. Then $A//B$ acts trivially on $H^\bullet(B, k)$.

Proof. For each $a \in A$, D_a restricts to the zero map on B. Then for any $a \in A$ and any cocycle $f \in \mathrm{Hom}_B(\mathsf{B}_n(B), k)$, we have $(a.f)(x) = a.f(x) + f(x \cdot a) = \varepsilon(a).f(x) + f(0) = (\varepsilon(a)f)(x)$, and the result follows. □

2.2 Spectral sequences

We begin this section by recalling the construction of the spectral sequence associated to a filtered differential graded module. (Recall that a spectral sequence $\{E_r, d_r\}_{r\geq 1}$ in an abelian category $\mathfrak{A}$ is a collection of objects E_r in $\mathfrak{A}$ and morphisms $d_r : E_r \to E_r$ such that $E_{r+1} \cong H(E_r, d_r)$.) We then look at the Lyndon–Hochschild–Serre spectral sequence associated to an augmented algebra A and a fixed normal subalgebra B of

A. When H is a Hopf algebra acting on A and B as in §2.1, we show that the LHS spectral sequence associated to A and B is a spectral sequence of left H-modules. Our primary references for this section are [7, 52].

Let R be a ring. An R-module A is a filtered differential graded R-module if A is a direct sum of submodules, $A = \bigoplus_{n=0}^{\infty} A^n$, if there exists an R-linear mapping $d : A \to A$ of degree $+1$ ($d : A^n \to A^{n+1}$) satisfying $d \circ d = 0$, and if A admits a decreasing $\mathbb{Z}$-filtration F compatible with the differential d (i.e., $F^{p+1}A \subseteq F^pA$ for all $p \in \mathbb{Z}$, and $d(F^pA) \subseteq F^pA$). Since the differential d respects the filtration, the cohomology group $H(A, d) = \ker d / \operatorname{im} d$ inherits a $\mathbb{Z}$-filtration: Let $\iota_p : F^pA \hookrightarrow A$ denote the natural inclusion. Then $F^pH(A, d) = \operatorname{im} H(\iota_p) \subset H(A, d)$, that is, $F^pH(A, d)$ is the image of $H(F^pA, d)$ in $H(A, d)$ under the map induced by ι_p.

Define for $r \in \mathbb{N}$ and $p, q \in \mathbb{Z}$ the following R-submodules of A:

$$Z_r^{p,q} = F^pA^{p+q} \cap d^{-1}(F^{p+r}A^{p+q+1}) \tag{2.2.1}$$
$$B_r^{p,q} = F^pA^{p+q} \cap d(F^{p-r}A^{p+q-1}) \tag{2.2.2}$$
$$Z_\infty^{p,q} = \ker d \cap F^pA^{p+q} \tag{2.2.3}$$
$$B_\infty^{p,q} = \operatorname{im} d \cap F^pA^{p+q}, \tag{2.2.4}$$

and for $r \in \mathbb{N} \cup \{\infty\}$, define

$$E_r^{p,q} = Z_r^{p,q} / (Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}). \tag{2.2.5}$$

Then the differential d induces an R-linear map $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ ($r \in \mathbb{N}$).

Theorem 2.21. [52, Theorem 2.6] Each filtered differential graded R-module (A, d, F) determines a spectral sequence $\{E_r^{\bullet,\bullet}, d_r\}$ with $E_r^{\bullet,\bullet}$ as defined in (2.2.5), d_r induced by d, and

$$E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+1}A).$$

Suppose further that the filtration on A is bounded, that is, that for each $n \in \mathbb{N}$, there exist integers $s = s(n)$ and $t = t(n)$ such that $F^pA^n = 0$ for $p > s$, and $F^pA^n = A^n$ for $p \leq t$. Then the spectral sequence converges to $H(A, d)$, that is,

$$E_\infty^{p,q} \cong F^pH^{p+q}(A, d)/F^{p+1}H^{p+q}(A, d).$$

If A is a filtered differential graded algebra, then (2.2.5) defines a spectral sequence of algebras, and if the filtration on A is bounded, then the spectral sequence converges as an algebra to $H(A, d)$ [52, Theorem 2.14]. The condition that the filtration on A should be bounded can be weakened. It is sufficient to assume that the filtration is exhaustive (i.e., that $A = \bigcup_p F^pA$) and that $\bigcap_p F^pA = \{0\}$ [52, §3.1].

Now let $C = C^{\bullet,\bullet}$ be a double complex of R-modules, that is, a $\mathbb{Z} \times \mathbb{Z}$-indexed set of R-modules equipped with an endomorphism d' of bidegree $(1, 0)$ (the horizontal differential) and an endomorphism d'' of bidegree $(0, 1)$ (the vertical differential), such that $d' \circ d' = 0$, $d'' \circ d'' = 0$, and $d' \circ d'' + d'' \circ d' = 0$. Set $d = d' + d''$. Then $d \circ d = 0$, and $\{C, d\}$ becomes a differential graded R-module, with degree n graded part equal to $\bigoplus_{r+s=n} C^{r,s}$. We denote the differential graded R-module $\{C, d\}$ by $\mathrm{Tot}(C)$. (Note that the condition $d' \circ d'' + d'' \circ d' = 0$ is used to guarantee the equality $d \circ d = 0$. In the definition of a double complex one could instead require $d' \circ d'' - d'' \circ d' = 0$. In this case, one defines the total differential d by $d = d' + (-1)^i d''$, i.e., replace the vertical differential along the i-th column by $(-1)^i d''$.)

There exist two canonical filtrations on $\mathrm{Tot}(C)$, the column-wise filtration, defined by $F_I^p \mathrm{Tot}(C)_n = \bigoplus_{r \geq p} C^{r,n-r}$, and the row-wise filtration, defined by $F_{II}^p \mathrm{Tot}(C)_n = \bigoplus_{r \geq p} C^{n-r,r}$. Each filtration makes $\mathrm{Tot}(C)$ a filtered differential graded R-module, hence gives rise to a spectral sequence as in Theorem 2.21. We state the theorem describing the two spectral sequences below, but first some notation: Define bigraded R-modules $H_I^{\bullet,\bullet}(C) = H(C^{\bullet,\bullet}, d')$ and $H_{II}^{\bullet,\bullet}(C) = H(C^{\bullet,\bullet}, d'')$. Note that $H_I^{\bullet,\bullet}(C)$ and $H_{II}^{\bullet,\bullet}(C)$ are each differential bigraded modules, with differentials $\overline{d''}$ and $\overline{d'}$ induced by d'' and d', respectively. Set $H_I^{\bullet,\bullet} H_{II}(C) = H(H_{II}^{\bullet,\bullet}(C), \overline{d'})$, and set $H_{II}^{\bullet,\bullet} H_I(C) = H(H_I^{\bullet,\bullet}(C), \overline{d''})$.

Theorem 2.22. [52, Theorem 2.15] Let $\{C^{\bullet,\bullet}, d', d''\}$ be a double complex of R-modules. Then there exist spectral sequences $\{{}_IE_2^{\bullet,\bullet}, {}_Id_r\}$ and $\{{}_{II}E_2^{\bullet,\bullet}, {}_{II}d_r\}$, with

$$ {}_IE_2^{\bullet,\bullet} \cong H_I^{\bullet,\bullet} H_{II}(C) \quad \text{and} \quad {}_{II}E_2^{\bullet,\bullet} \cong H_{II}^{\bullet,\bullet} H_I(C). $$

If $C^{r,s} = \{0\}$ when either $r < 0$ or $s < 0$ (i.e., if C is a first quadrant double complex), then both spectral sequences converge to $H(\mathrm{Tot}(C), d)$.

Now let A be an augmented algebra over k, and let B be a normal subalgebra of A such that A is right B-flat. The preceding algebraic machinery can be applied to yield a spectral sequence relating the cohomology groups for the augmented algebras B and $A//B$ to those of A.

Theorem 2.23 (Lyndon–Hochschild–Serre Spectral Sequence)**.** Let A be an augmented algebra over k, and let B be a normal subalgebra of A such that A is right B-flat. Let V be a left A-module. Then there exists a spectral sequence satisfying

$$ E_2^{i,j} = H^i(A//B, H^j(B, V)) \Rightarrow H^{i+j}(A, V). \tag{2.2.6} $$

Summary of the construction of the LHS spectral sequence. We follow the approach of [7, Chapter VI]; for the equivalence of this approach to other constructions, consult [7, Chapter VIII]. Let $P^\bullet = \mathsf{B}_\bullet(A//B)$ be the left bar resolution of $A//B$, an $A//B$-free resolution of k, and let $Q_\bullet = Q_\bullet(V)$ be the coinduced resolution of V defined in §2.1, an A-injective resolution of V. Form the first quadrant double complex $C = C^{\bullet,\bullet}$ with $C^{i,j} = \mathrm{Hom}_A(P^i, Q_j)$. According to Theorem 2.22, the two canonical filtrations $F_I^\bullet$ and $F_{II}^\bullet$ on $\mathrm{Tot}(C)$ (the column-wise filtration and the row-wise filtration, respectively) each give rise to spectral sequences converging to $H(\mathrm{Tot}(C), d)$.

The spectral sequence determined by $F_{II}^{\bullet}$ collapses at the E_2-page and converges to $H^{\bullet}(A,V)$, while the $E_2^{i,j}$-term of the spectral sequence determined by $F_I^{\bullet}$ is as identified in (2.2.6); see [7, Chapter VI] for details. Thus, the desired spectral sequence is the one determined by the column-wise filtration $F_I^{\bullet}$ of $\mathrm{Tot}(C)$. □

Now suppose that A is a right H-module algebra, and that B is an H-submodule of A. Then $A//B$ inherits from A the structure of a right H-module algebra. The right H-module structure on $A//B$ extends diagonally to a right H-module structure on $P^{\bullet} = \mathsf{B}_{\bullet}(A//B)$ such that $P^{\bullet}$ is a complex of right H-modules and H-module homomorphisms. Suppose that the left A-module V admits a compatible H-module structure. Then, just as in §2.1, we see (for fixed $i,j \in \mathbb{N}$) that (2.1.4) defines a left H-module structure on $C^{i,j}$, and that $C^{\bullet,\bullet} = \mathrm{Hom}_A(P^{\bullet}, Q_{\bullet})$ is a double complex of H-modules and H-module homomorphisms. So $\mathrm{Tot}(C)$ is a left H-module, and, for each $p \in \mathbb{N}$, $F_I^p \,\mathrm{Tot}(C)$ (resp. $F_{II}^p \,\mathrm{Tot}(C)$) is an H-submodule of $\mathrm{Tot}(C)$. From the construction of (2.2.6) presented above, we deduce the following theorem.

Theorem 2.24. Maintain the notations and assumptions of Theorem 2.23 and of the previous paragraph. The spectral sequence (2.2.6) is a spectral sequence of left H-modules. The H-module actions on the E_2 page and on the abutment are the adjoint actions of H defined in Definition 2.12.

We conclude this section with the following well-known result.

Lemma 2.25. [30, Corollary 5.3] Suppose A is right B-flat and that B is central in A. Then $H^0(A//B, H^{\bullet}(B,k)) = H^{\bullet}(B,k)$, and if $V = k$ in the spectral sequence (2.2.6), then the image of $H^1(B,k)$ under the differential $d_2^{0,1}$ is central in $H^{\bullet}(A//B,k)$.

Chapter 3

Integral results for quantized enveloping algebras

In this chapter we lay the necessary groundwork in order to apply the results and techniques of Chapter 2 to the cohomology calculations of Chapters 4 and 5. Some of the results we obtain are well-known when the coefficient field k is of characteristic zero; with some extra work, we establish these results when the coefficient field k is of (almost) arbitrary characteristic.

3.1 An integral commutation formula

The following lemma generalizes an observation of Levendorskii and Soibelman [46, Proposition 5.5.2].

Lemma 3.1. Let $S \subset \mathsf{A} = \mathbb{Z}[q, q^{-1}]$ be the multiplicatively closed set generated by

$$\begin{array}{ll} \{1\} & \text{if } \Phi \text{ has type } ADE, \\ \left\{q^2 - q^{-2}\right\} & \text{if } \Phi \text{ has type } BCF, \\ \left\{q^2 - q^{-2}, q^3 - q^{-3}\right\} & \text{if } \Phi \text{ has type } G. \end{array}$$

Set $\mathscr{A} = S^{-1}\mathsf{A}$, the localization of A at S. Let $\Phi^+ = \{\gamma_1, \ldots, \gamma_N\}$ be an enumeration of Φ^+ as in §1.2, and let $1 \le i < j \le N$. Then in $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ we have

(a) $E_{\gamma_i} E_{\gamma_j} = q^{(\gamma_i, \gamma_j)} E_{\gamma_j} E_{\gamma_i} + (*)$, where $(*)$ is an $\mathscr{A}$-linear combination of monomials $E_{\gamma_1}^{m_1} \cdots E_{\gamma_N}^{m_N}$ with $m_s = 0$ unless $i < s < j$.

(b) $F_{\gamma_i} F_{\gamma_j} = q^{(\gamma_i, \gamma_j)} F_{\gamma_j} F_{\gamma_i} + (*)$, where $(*)$ is an $\mathscr{A}$-linear combination of monomials $F^{\mathbf{m}}$ with $m_s = 0$ unless $i < s < j$.

Proof. Part (b) is equivalent to part (a); apply the algebra anti-automorphism κ defined in §1.2 to infer one from the other. Part (a) is easily verified for $\mathfrak{g}$ of rank 2 using the `QuaGroup` package of the computer program `GAP`, with which all of the "commutators" $E_{\gamma_i}E_{\gamma_j} - q^{(\gamma_i,\gamma_j)}E_{\gamma_j}E_{\gamma_i}$ can be explicitly computed (cf. also the calculations in [49, §5.1], though the automorphisms constructed by Lusztig there differ slightly from those defined in [36, Chapter 8]; see the warning in [36, Remark 8.14]). From the rank 2 case we deduce the result for $\mathfrak{g}$ of arbitrary rank by the arguments in the proof of [20, Theorem 9.3(iv)]. □

Remark 3.2.

(1) The formulas in Lemma 3.1 are stated for the algebra $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$, but similar formulas for $\mathbb{U}_k(\mathfrak{g})$ are deduced via the identification $\mathbb{U}_k(\mathfrak{g}) = U_{k(q)} = U_{\mathsf{A}} \otimes_{\mathsf{A}} k(q)$.

(2) It follows from Lemma 3.1 that the algebra $\mathbb{U}_k(\mathfrak{u}_J)$ defined in §1.2 is a subalgebra of $\mathbb{U}_k(\mathfrak{g})$. More generally, given $w \in W$, Lemma 3.1 establishes that the subspaces $U^+[w], U^-[w]$ defined in [36, §8.24] are subalgebras of $\mathbb{U}_k(\mathfrak{g})$, since any reduced expression for w can be completed to a reduced expression for w_0.

Lemma 3.1 appears in the literature with our choice for the ring $\mathscr{A}$ replaced by $\mathbb{Q}[q, q^{-1}]$ (cf. [21, Lemma 1.7] or [20, Theorem 9.3(iv)]). This formulation is incorrect if the root system Φ has two root lengths, as the following two examples show.

Example 3.3. Let Φ be of type B_2, and write $\Pi = \{\alpha, \beta\}$ with α long. Choose the reduced expression $s_\alpha s_\beta s_\alpha s_\beta$ for w_0, so that the positive roots of Φ written in convex order are

$$\Phi^+ = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \{\alpha, \alpha + \beta, \alpha + 2\beta, \beta\}\,.$$

According to the `QuaGroup` package of `GAP`, the following relation holds in $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$:

$$F_{\gamma_3}F_{\gamma_1} = F_{\gamma_1}F_{\gamma_3} + (-1 + q^{-2})F_{\gamma_2}^{(2)}.$$

We have $\gamma_2 = s_\alpha(\beta)$, so by the definition of the divided power $F_{\gamma_2}^{(2)}$, $F_{\gamma_2}^{(2)} = F_{\gamma_2}^2/[2]^!_\beta$. Since β is short, $q_\beta = q^{(\beta,\beta)/2} = q$. Then $[2]^!_\beta = (q^2 - q^{-2})/(q - q^{-1}) = q + q^{-1}$, hence

$$\frac{(-1 + q^{-2})}{[2]^!_\beta} = \frac{(-1 + q^{-2})}{(q + q^{-1})} = \frac{-(q^2 - 1)}{q(q^2 + 1)},$$

which is not an element of $\mathbb{Q}[q, q^{-1}]$.

Example 3.4. Let Φ be of type G_2, and write $\Pi = \{\alpha, \beta\}$ with β long. Choose the reduced expression $s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta$ for w_0, so that the positive roots of Φ written in convex order are

$$\Phi^+ = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\} = \{\alpha, 3\alpha + \beta, 2\alpha + \beta, 3\alpha + 2\beta, \alpha + \beta, \beta\}\,.$$

According to the `QuaGroup` package of `GAP`, the following relation holds in $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$:

$$F_{\gamma_4}F_{\gamma_2} = q^{-3}F_{\gamma_2}F_{\gamma_4} + q^{-6}(q - 1)^2(q + 1)^2(q^2 + 1)F_{\gamma_3}^{(3)}.$$

We have $\gamma_3 = s_\alpha s_\beta(\alpha)$, so by the definition of the divided power $F_{\gamma_3}^{(3)}$, $F_{\gamma_3}^{(3)} = F_{\gamma_3}^3/[3]_\alpha^!$. Since α is short, $q_\alpha = q^{(\alpha,\alpha)/2} = q$. Then

$$[3]_\alpha^! = \prod_{n=1}^{3} \frac{q_\alpha^n - q_\alpha^{-n}}{q_\alpha - q_\alpha^{-1}} = \prod_{n=1}^{3} \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-3}(q^2+1)(q^2+q+1)(q^2-q+1),$$

hence

$$\frac{q^{-6}(q-1)^2(q+1)^2(q^2+1)}{[3]_\alpha^!} = \frac{(q-1)^2(q+1)^2}{q^3(q^2+q+1)(q^2-q+1)},$$

which is not an element of $\mathbb{Q}[q,q^{-1}]$.

Following De Concini and Kac [21, §1.7], we use Lemma 3.1 to define a multiplicative filtration on $\mathbb{U}_k(\mathfrak{g})$. For $\mathbf{r},\mathbf{s} \in \mathbb{N}^N$ and $u \in \mathbb{U}_k^0$, define $M_{\mathbf{r},\mathbf{s},u} = F^{\mathbf{r}} u E^{\mathbf{s}} \in \mathbb{U}_k(\mathfrak{g})$. (Recall the monomials $F^{\mathbf{r}}, E^{\mathbf{s}}$ were defined in §1.2.) Define the total height of the monomial $M_{\mathbf{r},\mathbf{s},u}$ by $\operatorname{ht}(M_{\mathbf{r},\mathbf{s},u}) = \sum_{i=1}^N (r_i + s_i) \operatorname{ht}(\gamma_i) \in \mathbb{N}$, where by $\operatorname{ht}(\gamma)$ we mean the usual height of $\gamma \in \Phi^+$ with respect to the chosen basis $\Pi \subset \Phi$. (The symbol $\sum$ in the preceding definition is printed incorrectly as $\prod$ in [21]; the definition is printed correctly in [20, §10.1].) Now define the degree of the monomial $M_{\mathbf{r},\mathbf{s},u}$ by

$$d(M_{\mathbf{r},\mathbf{s},u}) = (r_N, r_{N-1}, \dots, r_1, s_1, \dots, s_N, \operatorname{ht}(M_{\mathbf{r},\mathbf{s},u})). \tag{3.1.1}$$

View $\mathbb{N}^{2N+1}$ as a totally ordered semigroup via the reverse lexicographic ordering (i.e., the lexicographic order $<$ on $\mathbb{N}^{2N+1}$ such that $u_1 < u_2 < \cdots < u_{2N+1}$, where $u_i = (\delta_{i,1}, \dots, \delta_{i,2N+1})$). Given $\eta \in \mathbb{N}^{2N+1}$, define $\mathbb{U}_k(\mathfrak{g})_\eta$ to be the linear span in $\mathbb{U}_k(\mathfrak{g})$ of all monomials $M_{\mathbf{r},\mathbf{s},u}$ with $d(M_{\mathbf{r},\mathbf{s},u}) \le \eta$.

Proposition 3.5. [21, Proposition 1.7] The subspaces $\mathbb{U}_k(\mathfrak{g})_\eta$ for $\eta \in \mathbb{N}^{2N+1}$ define a multiplicative filtration of $\mathbb{U}_k(\mathfrak{g})$.

Proof. This follows from Lemma 3.1, the defining relations for $\mathbb{U}_k(\mathfrak{g})$ in Definition 1.1, and Remark 1.2. □

3.2 The De Concini–Kac integral form

In §1.1 we defined integral forms $U_{\mathsf{A}} \subset \mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ and $U_{\mathsf{B}} \subset \mathbb{U}_k(\mathfrak{g})$, due to Lusztig, which after base change to k resemble the hyperalgebra of an algebraic group. In this section we define integral forms $\mathcal{U}_{\mathcal{A}} \subset \mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ and $\mathcal{U}_{\mathcal{B}} \subset \mathbb{U}_k(\mathfrak{g})$, which after base change to k resemble the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. The construction of these integral forms is due to De Concini and Kac [21].

Define $S \subset \mathsf{A} = \mathbb{Z}[q,q^{-1}]$ to be the multiplicatively closed subset generated by

$$\begin{aligned} &\{q-q^{-1}\} && \text{if } \Phi \text{ has type } ADE,\\ &\{q-q^{-1}, q^2-q^{-2}\} && \text{if } \Phi \text{ has type } BCF,\\ &\{q-q^{-1}, q^2-q^{-2}, q^3-q^{-3}\} && \text{if } \Phi \text{ has type } G, \end{aligned}$$

and set $\mathcal{A} = S^{-1}\mathsf{A}$. Now define $\mathcal{U}_{\mathcal{A}} = \mathcal{U}_{\mathcal{A}}(\mathfrak{g})$ to be the $\mathcal{A}$-subalgebra of $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ generated by $\{E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1} : \alpha \in \Pi\}$. Then $\mathbb{U}_k(\mathfrak{g}) \cong \mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} k(q)$. In particular, $\mathcal{U}_{\mathcal{A}}$ is an $\mathcal{A}$-form of $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$.

It is clear that the Hopf algebra structure on $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ induces the structure of a Hopf algebra on $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$. It is also clear from the definition of $\mathcal{A}$ and from the formulas for the automorphisms T_α given in §1.2 that for each $\gamma \in \Phi^+$, we have $E_\gamma, F_\gamma \in \mathcal{U}_{\mathcal{A}}(\mathfrak{g})$. It follows then that the set of all monomials

$$\left\{F^{\mathbf{r}} K_\mu E^{\mathbf{s}} : \mathbf{r}, \mathbf{s} \in \mathbb{N}^N, \mu \in \mathbb{Z}\Phi\right\} \tag{3.2.1}$$

forms an $\mathcal{A}$-basis for $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$. (The monomials $F^{\mathbf{r}}, K_\mu, E^{\mathbf{s}}$ were defined in §§1.1–1.2. It is clear that the given monomials are linearly independent in $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$ because they are linearly independent in $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$. To show that they span $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$, use Lemma 3.1 and the defining relations for $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ to show that the subspace of $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$ spanned by the given monomials is stable under left multiplication by the generators for $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$.)

Viewing $S \subset \mathsf{A}$ instead as a subset of $\mathsf{B} = k[q, q^{-1}]$, set $\mathcal{B} = S^{-1}\mathsf{B}$. Define $\mathcal{U}_{\mathcal{B}}$ to be the $\mathcal{B}$-subalgebra of $\mathbb{U}_k(\mathfrak{g})$ generated by $\{E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1} : \alpha \in \Pi\}$. Of course, $\mathcal{B}$ admits a natural $\mathcal{A}$-module structure induced by the natural A-module structure of B. Then $\mathcal{U}_{\mathcal{B}} \cong \mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B}$, and $\mathcal{U}_{\mathcal{B}}$ admits a basis of the form (3.2.1). Under our assumption on the order of $\zeta \in k^\times$, the field k is naturally an $\mathcal{A}$-module (resp. $\mathcal{B}$-module) under the specialization $q \mapsto \zeta$.

Definition 3.6. Let $\zeta \in k^\times$ be a primitive ℓ-th root of unity with ℓ satisfying Assumption 1.5. Define $\mathcal{U}_k = \mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} k$ (equivalently, $\mathcal{U}_k = \mathcal{U}_{\mathcal{B}} \otimes_{\mathcal{B}} k$), where the $\mathcal{A}$-module structure on k is obtained via the specialization $q \mapsto \zeta$, and define $\mathcal{U}_\zeta = \mathcal{U}_\zeta(\mathfrak{g})$ to be the quotient of $\mathcal{U}_k$ by the two-sided ideal $\langle K_\alpha^\ell \otimes 1 - 1 \otimes 1 : \alpha \in \Pi\rangle$, that is,

$$\mathcal{U}_\zeta = \mathcal{U}_\zeta(\mathfrak{g}) := \mathcal{U}_\zeta / \langle K_\alpha^\ell \otimes 1 - 1 \otimes 1 : \alpha \in \Pi\rangle.$$

We call $\mathcal{U}_k$ and $\mathcal{U}_\zeta$ the De Concini–Kac quantum algebras with parameter q specialized to $\zeta \in k$.

For each subset $J \subseteq \Pi$, we also define distinguished subalgebras $\mathcal{U}_\zeta(\mathfrak{l}_J)$, $\mathcal{U}_\zeta(\mathfrak{p}_J)$, $\mathcal{U}_\zeta(\mathfrak{u}_J)$ of $\mathcal{U}_\zeta(\mathfrak{g})$ similarly to the way we did in §1.2.

Remark 3.7. Some authors define the integral form $\mathcal{U}_q \subset \mathbb{U}_k(\mathfrak{g})$ to be the $k[q, q^{-1}]$-subalgebra of $\mathbb{U}_k(\mathfrak{g})$ generated by $\{E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1} : \alpha \in \Pi\}$. In this case, for an arbitrary positive root $\gamma \in \Phi^+$, we may not have $E_\gamma, F_\gamma \in \mathcal{U}_q$ unless Φ has only one root length, though we will always have $sE_\gamma, s'F_\gamma \in \mathcal{U}_q$ for some $s, s' \in S$. For practical purposes, the choice of using either $\mathcal{U}_{\mathcal{A}}$ or $\mathcal{U}_q$ to define the algebra $\mathcal{U}_k$ is irrelevant, because under our standard assumption on $\zeta \in k^\times$, the denominators in S do not vanish under the specialization $q \mapsto \zeta$.

We now prove several structural results concerning the algebras $\mathcal{U}_k$ and $\mathcal{U}_\zeta$ that will be needed later. These results are well-known if $k = \mathbb{C}$ and if ζ is chosen to be a complex root of unity; cf. [21, Corollaries 3.1, 3.3]. We continue to make our standard assumptions on k and ζ, namely, that k is a field of characteristic $p \neq 2$ (and $p \neq 3$ if Φ has type G_2), and that $\zeta \in k^\times$ is a primitive ℓ-th root of unity with ℓ satisfying Assumption 1.5.

Lemma 3.8. For all $\gamma \in \Phi^+$, $\alpha \in \Pi$, the elements $E_\gamma^\ell, F_\gamma^\ell, K_\alpha^\ell, K_\alpha^{-\ell}$ are central in $\mathcal{U}_k$.

Proof. The ring homomorphism $\mathcal{A} \to k$ sending $q \mapsto \zeta$ factors through the quotient $\mathcal{A} \twoheadrightarrow \widetilde{\mathcal{A}} = \mathcal{A}/(\phi_\ell)\mathcal{A}$, where $\phi_\ell \in \mathbb{Z}[q]$ is the ℓ-th cyclotomic polynomial. The image of q in $\mathcal{A}/(\phi_\ell)\mathcal{A}$ is a primitive ℓ-th root of unity. By abuse of notation, we denote the image of q in $\mathcal{A}/(\phi_\ell)\mathcal{A}$ by ζ. The ring $\widetilde{\mathcal{A}}$ is naturally a subring of its field of fractions, the cyclotomic field $\mathbb{Q}(\zeta)$, hence $\mathcal{U}_{\widetilde{\mathcal{A}}} = \mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}}$ is naturally a subalgebra of $\mathcal{U}_{\mathbb{Q}(\zeta)}$. Now, for all $\gamma \in \Phi^+$ and $\alpha \in \Pi$, the elements $E_\gamma^\ell, F_\gamma^\ell, K_\alpha^\ell, K_\alpha^{-\ell}$ are central in $\mathcal{U}_{\mathbb{Q}(\zeta)}$ by [21, Corollary 3.1], hence they must also be central in $\mathcal{U}_{\widetilde{\mathcal{A}}}$. Since $\mathcal{U}_k = \mathcal{U}_{\widetilde{\mathcal{A}}} \otimes_{\widetilde{\mathcal{A}}} k$, we conclude that $E_\gamma^\ell, F_\gamma^\ell, K_\alpha^\ell, K_\alpha^{-\ell}$ must be central in $\mathcal{U}_k$. ☐

Let $\mathcal{Z}$ denote the central subalgebra of $\mathcal{U}_k$ generated by

$$\left\{E_\gamma^\ell, F_\gamma^\ell, K_\alpha^\ell, K_\alpha^{-\ell} : \gamma \in \Phi^+, \alpha \in \Pi\right\}.$$

Let Z^+, Z and Z^0 denote the usual positive, negative, and toral subalgebras of $\mathcal{Z}$.

Lemma 3.9.

(a) The algebra $\mathcal{U}_k$ (resp. $\mathcal{U}_k^+$, $\mathcal{U}_k^-$, $\mathcal{U}_k^0$) is a free $\mathcal{Z}$-module (resp. Z^+-module, Z-module, Z^0-module), with basis given by all monomials $F^{\mathbf{s}} K_{\alpha_1}^{t_1} \cdots K_{\alpha_n}^{t_n} E^{\mathbf{r}}$ (resp. $E^{\mathbf{r}}$, $F^{\mathbf{s}}$, $K_{\alpha_1}^{t_1} \cdots K_{\alpha_n}^{t_n}$) satisfying $0 \leq r_i, s_i, t_i < \ell$ for all i.

(b) $\mathcal{U}_\zeta(\mathfrak{g})//\mathcal{Z} \cong u_\zeta(\mathfrak{g})$, and a corresponding statement is true if we replace each algebra with its positive, negative, or toral variation.

Proof. If $k = \mathbb{Q}(\zeta)$, then part (a) is just [21, Corollary 3.3(b)]. In the general case, if we only require $r_i, s_i \in \mathbb{N}$ and $t_i \in \mathbb{Z}$, then the monomials described in part (a) form $\mathcal{A}$-bases for $\mathcal{U}_{\mathcal{A}}$ and its positive, negative, and toral variations. Now base-change to k and apply Lemma 3.8.

To prove part (b), note first that we have an inclusion of $\mathcal{B}$-forms $\mathcal{U}_{\mathcal{B}} \subset U_{\mathcal{B}} = U_{\mathsf{B}} \otimes_B \mathcal{B}$. Let $\iota : \mathcal{U}_{\mathcal{B}} \hookrightarrow U_{\mathcal{B}}$ be the inclusion map. Base-changing to k, we obtain an algebra homomorphism $\iota \otimes \mathrm{id} : \mathcal{U}_k = \mathcal{U}_{\mathcal{B}} \otimes_{\mathcal{B}} k \to U_k = U_{\mathcal{B}} \otimes_{\mathcal{B}} k$. Composing with the projection map $U_k \twoheadrightarrow U_\zeta$, we obtain an algebra homomorphism $\mathcal{U}_k \to U_\zeta$ with image equal to u_ζ and kernel equal to the two-sided ideal in $\mathcal{U}_k$ generated by $\mathcal{Z}$, that is, an isomorphism $\mathcal{U}_\zeta//\mathcal{Z} \xrightarrow{\sim} u_\zeta(\mathfrak{g})$. ☐

3.3 Adjoint actions on integral forms

Recall from Example 2.6 the left and right adjoint actions of a Hopf algebra on itself. Our goal in this section is to show that the left and right adjoint actions of $\mathbb{U}_k(\mathfrak{g})$ on itself induce H-module algebra structures on the De Concini–Kac quantum algebra $\mathcal{U}_\zeta(\mathfrak{g})$, on the Frobenius–Lusztig kernel $u_\zeta(\mathfrak{g})$, and on the distinguished subalgebras of $\mathcal{U}_\zeta(\mathfrak{g})$ and $u_\zeta(\mathfrak{g})$ corresponding to a fixed subset of simple roots $J \subseteq \Pi$.

Recall from §1.1 the two Hopf algebra structures on $\mathbb{U}_k(\mathfrak{g})$ defined by (1.1.7) and (1.1.8). From the two Hopf algebra structures we get two left adjoint actions of $\mathbb{U}_k(\mathfrak{g})$ on itself, and two right adjoint actions of $\mathbb{U}_k(\mathfrak{g})$ on itself. We write

$$\begin{aligned} \mathrm{Ad} = \mathrm{Ad}_l : \mathbb{U}_k(\mathfrak{g}) &\to \mathrm{End}_k(\mathbb{U}_k(\mathfrak{g})) \quad \text{and} \\ \mathrm{Ad}_r : \mathbb{U}_k(\mathfrak{g}) &\to \mathrm{End}_k(\mathbb{U}_k(\mathfrak{g}))^{op} \end{aligned}$$

to denote the left and right adjoint actions corresponding to (1.1.7), and we write

$$\begin{aligned} \overline{\mathrm{Ad}} = \overline{\mathrm{Ad}}_l : \mathbb{U}_k(\mathfrak{g}) &\to \mathrm{End}_k(\mathbb{U}_k(\mathfrak{g})) \quad \text{and} \\ \overline{\mathrm{Ad}}_r : \mathbb{U}_k(\mathfrak{g}) &\to \mathrm{End}_k(\mathbb{U}_k(\mathfrak{g}))^{op} \end{aligned}$$

to denote the left and right adjoint actions corresponding to (1.1.8). Collectively, we refer to these four adjoint actions as the adjoint actions of $\mathbb{U}_k(\mathfrak{g})$ on itself. In this paper we are principally interested in the right adjoint action defined by Ad_r, but for the sake of completeness (and since it requires little extra effort), we consider the other adjoint actions as well.

For fixed $x \in \mathbb{U}_k(\mathfrak{g})$, we have

$$\overline{\mathrm{Ad}}(x) = \omega \circ \mathrm{Ad}(\omega(x)) \circ \omega, \tag{3.3.1}$$

$$\overline{\mathrm{Ad}}_r(x) = \omega \circ \mathrm{Ad}_r(\omega(x)) \circ \omega, \tag{3.3.2}$$

$$\mathrm{Ad}_r(x) = \kappa \circ \mathrm{Ad}(\kappa(x)) \circ \kappa, \tag{3.3.3}$$

$$\overline{\mathrm{Ad}}_r(x) = \kappa \circ \overline{\mathrm{Ad}}(\kappa(x)) \circ \kappa. \tag{3.3.4}$$

Using these relations, it is sufficient to prove some results first for the adjoint action defined by Ad, and to then infer that a corresponding version of the result holds for the other adjoint actions. (Some care is typically required in making the transition, because κ is merely a k-algebra antiautomorphism of $\mathbb{U}_k(\mathfrak{g})$, not a $k(q)$-algebra antiautomorphism.) In this spirit, we record here formulas for the Ad-actions of the generators of $\mathbb{U}_k(\mathfrak{g})$. Let $u \in \mathbb{U}_k(\mathfrak{g})$. Then

$$\mathrm{Ad}(E_\alpha^{(r)})(u) = \sum_{i=0}^{r} (-1)^i q_\alpha^{i(r-1)} E_\alpha^{(r-i)} K_\alpha^i u K_\alpha^{-i} E_\alpha^{(i)}, \tag{3.3.5}$$

$$\mathrm{Ad}(F_\alpha^{(r)})(u) = \sum_{i=0}^{r} (-1)^{r-i} q_\alpha^{-(r-i)(r-1)} F_\alpha^{(i)} u F_\alpha^{(r-i)} K_\alpha^r, \tag{3.3.6}$$

$$\mathrm{Ad}(K_\alpha^{\pm 1})(u) = K_\alpha^{\pm 1} u K_\alpha^{\mp 1}. \tag{3.3.7}$$

In what follows, fix $J \subseteq \Pi$.

Proposition 3.10. The following stability results hold (and are equivalent):

(a) The subalgebra $\mathbb{U}_k(\mathfrak{u}_J^+)$ is stable under the Ad-action of $\mathbb{U}_k(\mathfrak{p}_J^+)$ on itself.

(b) The subalgebra $\mathbb{U}_k(\mathfrak{u}_J)$ is stable under the Ad_r-action of $\mathbb{U}_k(\mathfrak{p}_J)$ on itself.

(c) The subalgebra $\mathbb{U}_k(\mathfrak{u}_J)$ is stable under the $\overline{\mathrm{Ad}}$-action of $\mathbb{U}_k(\mathfrak{p}_J)$ on itself.

(d) The subalgebra $\mathbb{U}_k(\mathfrak{u}_J^+)$ is stable under the $\overline{\mathrm{Ad}}_r$-action of $\mathbb{U}_k(\mathfrak{p}_J^+)$ on itself.

Proof. Parts (a) and (c) are the content of [9, Proposition 2.7.1]. (Technically, the authors of [9] prove the result for $k = \mathbb{Q}(\zeta)$ a cyclotomic field, but the same proof works under our more general setup.) The equivalence of parts (a)–(d) is a consequence of equations (3.3.1–3.3.4) and Remark 1.9. □

Corollary 3.11. (cf. [9, Corollary 2.7.2]) The algebra $\mathbb{U}_k(\mathfrak{u}_J)$ is normal in $\mathbb{U}_k(\mathfrak{p}_J)$. Normality also holds for the specializations

$$U_\zeta(\mathfrak{u}_J) \subset U_\zeta(\mathfrak{p}_J),\ u_\zeta(\mathfrak{u}_J) \subset u_\zeta(\mathfrak{p}_J),\ \mathcal{U}_\zeta(\mathfrak{u}_J) \subset \mathcal{U}_\zeta(\mathfrak{p}_J).$$

The quotient maps associated to each normal subalgebra induce algebra isomorphisms

$$U_\zeta(\mathfrak{l}_J) \cong U_\zeta(\mathfrak{p}_J)//U_\zeta(\mathfrak{u}_J),\ u_\zeta(\mathfrak{l}_J) \cong u_\zeta(\mathfrak{p}_J)//u_\zeta(\mathfrak{u}_J),\ \mathcal{U}_\zeta(\mathfrak{l}_J) \cong \mathcal{U}_\zeta(\mathfrak{p}_J)//\mathcal{U}_\zeta(\mathfrak{u}_J).$$

Proof sketch. Specializing (3.3.5) and (3.3.6) to the case $r = 1$ and applying (3.3.1), we get (for $u \in \mathbb{U}_k(\mathfrak{g})$):

$$\begin{aligned}\overline{\mathrm{Ad}}(E_\alpha)(u) &= (E_\alpha u - uE_\alpha)K_\alpha^{-1}, \quad \text{and}\\ \overline{\mathrm{Ad}}(F_\alpha)(u) &= F_\alpha u - K_\alpha^{-1} u K_\alpha F_\alpha.\end{aligned}$$

Using the above formulas, the normality of $\mathbb{U}_k(\mathfrak{u}_J)$ in $\mathbb{U}_k(\mathfrak{p}_J)$ follows from Proposition 3.10(b). By restriction, normality holds for the integral forms $U_{\mathsf{A}}(\mathfrak{u}_J) \subset U_{\mathsf{A}}(\mathfrak{p}_J)$ and $\mathcal{U}_{\mathcal{A}}(\mathfrak{u}_J) \subset \mathcal{U}_{\mathcal{A}}(\mathfrak{p}_J)$, hence also for the specializations $U_\zeta(\mathfrak{u}_J) \subset U_\zeta(\mathfrak{p}_J)$, $u_\zeta(\mathfrak{u}_J) \subset u_\zeta(\mathfrak{p}_J)$, and $\mathcal{U}_\zeta(\mathfrak{u}_J) \subset \mathcal{U}_\zeta(\mathfrak{p}_J)$. The stated isomorphisms are then easily deduced from the PBW-type bases for the relevant algebras. □

The next proposition is of fundamental importance for what follows. First recall that there exists an inclusion of $\mathcal{B}$-forms $\mathcal{U}_{\mathcal{B}} \subset U_{\mathcal{B}}$. Set $\mathfrak{B} = \mathcal{B}_{(\langle q-\zeta \rangle)}$, the localization of $\mathcal{B}$ at the maximal ideal generated by $q - \zeta$. Then there exists inclusion of $\mathfrak{B}$-forms $\mathcal{U}_{\mathfrak{B}} \subset U_{\mathfrak{B}}$, and $U_k = U_{\mathfrak{B}}/(q-\zeta)U_{\mathfrak{B}}$ and $\mathcal{U}_k = \mathcal{U}_{\mathfrak{B}}/(q-\zeta)\mathcal{U}_{\mathfrak{B}}$.

Proposition 3.12. The adjoint actions $(\mathrm{Ad}, \mathrm{Ad}_r, \overline{\mathrm{Ad}}, \overline{\mathrm{Ad}}_r)$ of $U_{\mathfrak{B}}$ on itself stabilize the subspace $\mathcal{U}_{\mathfrak{B}}$, hence induce actions of U_k on $\mathcal{U}_k$ (which we also call adjoint actions, and which we denote by the same symbols). The adjoint actions of U_k on $\mathcal{U}_k$ factor through U_ζ.

Proof. We begin by proving the proposition for the left adjoint action of $U_{\mathfrak{B}}$ on itself defined by Ad. Let $C \subset U_{\mathfrak{B}}$ be the set of generators identified in (1.1.9). It suffices to show, for each $c \in C$, that $\mathrm{Ad}(c)$ stabilizes $\mathcal{U}_{\mathfrak{B}}$. The algebra $\mathcal{U}_{\mathfrak{B}}$ is a Hopf algebra in its own right, and $\{E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1} : \alpha \in \Pi\} \subset \mathcal{U}_{\mathfrak{B}}$, so the adjoint action of these elements will stabilize $\mathcal{U}_{\mathfrak{B}}$. It remains to check, for each $\alpha \in \Pi$ and $i \geq 0$, that the adjoint actions of $E_\alpha^{(p^i\ell)}$ and $F_\alpha^{(p^i\ell)}$ stabilize $\mathcal{U}_{\mathfrak{B}}$.

Set $p = \mathrm{char}(k)$. Consider the multiplicity s with which the factor $(q-\zeta)$ appears in the denominator of

$$E_\alpha^{(p^i\ell)} = E_\alpha^{p^i\ell}/[p^i\ell]_\alpha^! = E_\alpha^{p^i\ell} \cdot \prod_{j=1}^{p^i\ell} \frac{q_\alpha - q_\alpha^{-1}}{q_\alpha^j - q_\alpha^{-j}} = E_\alpha^{p^i\ell} \cdot \prod_{j=1}^{p^i\ell} \frac{q_\alpha^{j-1}(q_\alpha^2 - 1)}{q_\alpha^{2j} - 1}.$$

If $j = p^e m\ell$ with $(p, m) = 1$, then $(q-\zeta)$ appears as a factor in $q_\alpha^{2j} - 1 = q^{j\cdot(\alpha,\alpha)} - 1$ with multiplicity p^e. If $\ell \nmid j$, then ζ is not a root of $q_\alpha^{2j} - 1$. It follows then that the multiplicity s is equal to

$$s = (p^i - p^{i-1}) + p(p^{i-1} - p^{i-2}) + \cdots + p^{i-1}(p-1) + p^i = p^i + \sum_{j=1}^{i} p^{i-1}(p-1).$$

This formula is valid even if $p = 0$, provided we accept the convention $0^0 = 1$. To prove the formula for s, observe that in $\{1, \ldots, p^i\ell\}$ there are $p^i - p^{i-1}$ multiples of ℓ that are not multiples of p, there are $p^{i-1} - p^{i-2}$ multiples of $p\ell$ that are not multiples of p^2, etc., and there is precisely one multiple of $p^i\ell$.

The divided power $E_\alpha^{(p^i\ell)}$ differs from $E_\alpha^{p^i\ell}/(q-\zeta)^s$ by a unit in $\mathfrak{B}$. We want to show, for each $u \in \mathcal{U}_{\mathfrak{B}}$, that $\mathrm{Ad}(E_\alpha^{p^i\ell})(u) \in (q-\zeta)^s \cdot \mathcal{U}_{\mathfrak{B}}$, for then $\mathrm{Ad}(E_\alpha^{(p^i\ell)})$ will define a $\mathfrak{B}$-linear endormorphism of $\mathcal{U}_{\mathfrak{B}}$. We first prove the result for the generator $E_\alpha^{(\ell)} \in U_{\mathfrak{B}}$. Given $u \in \mathcal{U}_{\mathfrak{B}}$, we have by (3.3.5)

$$\mathrm{Ad}(E_\alpha^\ell)(u) = \sum_{i=0}^{\ell} (-1)^i q_\alpha^{i(\ell-1)} \begin{bmatrix} \ell \\ i \end{bmatrix}_\alpha E_\alpha^{\ell-i} K_\alpha^i u K_\alpha^{-i} E_\alpha^i.$$

The right hand side of this equation is zero in $\mathcal{U}_k = \mathcal{U}_{\mathfrak{B}}/(q-\zeta)\mathcal{U}_{\mathfrak{B}}$ by standard properties of Gaussian binomial coefficients evaluated at roots of unity (cf. [47, Proposition 3.2]) and by Lemma 3.8, so we conclude that $\mathrm{Ad}(E_\alpha^\ell)(u) \in (q-\zeta)\cdot\mathcal{U}_{\mathfrak{B}}$, as desired. Similarly, $\mathrm{Ad}(F_\alpha^\ell)(u) \in (q-\zeta)\cdot\mathcal{U}_{\mathfrak{B}}$. If $\mathrm{char}(k) = 0$, then this proves the claim for the left Ad-action, so assume now that $p := \mathrm{char}(k) \neq 0$.

Let $u \in \mathcal{U}_{\mathcal{A}}$ and let $E_\alpha^\ell \in \mathcal{U}_{\mathcal{A}} \subset U_{\mathcal{A}}$. Consider the special case $k = \mathbb{C}$ of the above paragraph. In this case we have $\mathfrak{B} = \mathbb{C}[q, q^{-1}]_{(\langle q-\zeta\rangle)}$, and $\mathcal{U}_{\mathcal{A}}$ and $U_{\mathcal{A}}$ are naturally subalgebras of $\mathcal{U}_{\mathfrak{B}}$ and $U_{\mathfrak{B}}$, respectively. Of course, $\mathrm{Ad}(E_\alpha^\ell)(u) \in \mathcal{U}_{\mathcal{A}}$ because $\mathcal{U}_{\mathcal{A}}$ is a Hopf algebra and $E_\alpha^\ell \in \mathcal{U}_{\mathcal{A}}$. But by the previous paragraph we also have, for each primitive ℓ-th root of unity $z \in \mathbb{C}$ (of which there are $\varphi(\ell)$, where φ is Euler's totient function), $\mathrm{Ad}(E_\alpha^\ell)(u) \in (q-z)\cdot \mathcal{U}_{\mathfrak{B}}$. It follows then that $\mathrm{Ad}(E_\alpha^\ell)(u) \in \phi_\ell \cdot \mathcal{U}_{\mathcal{A}}$, where $\phi_\ell \in \mathbb{Z}[q]$ is the ℓ-th cyclotomic polynomial. (Recall that ϕ_ℓ factors over $\mathbb{C}$ as $\prod(q-z)$, where z ranges over all primitive ℓ-th roots of unity in $\mathbb{C}$.)

Return now to the situation of an arbitrary field k. Let $i \geq 1$, and consider $\mathrm{Ad}(E_\alpha^{p^i\ell})(u) \in \mathcal{U}_{\mathcal{A}}$. By the assumptions on ℓ and p made in §1.1, the product $p^i\ell$ also satisfies Assumption 1.5. Recall also that the adjoint map $\mathrm{Ad} : U_{\mathcal{A}} \to \mathrm{End}_{\mathcal{A}}(U_{\mathcal{A}})$ is an algebra homomorphism. Then, by the observation of the previous paragraph, we have for each $0 \leq j \leq i$,

$$\mathrm{Ad}(E_\alpha^{p^i\ell})(u) = \mathrm{Ad}((E_\alpha^{p^j\ell})^{p^{i-j}})(u) = \mathrm{Ad}(E_\alpha^{p^j\ell})^{p^{i-j}}(u) \in (\phi_{p^j\ell})^{p^{i-j}} \cdot \mathcal{U}_{\mathcal{A}}.$$

It follows that $\mathrm{Ad}(E_\alpha^{p^i\ell})(u) \in \phi \cdot \mathcal{U}_{\mathcal{A}}$, where $\phi = \prod_{j=0}^{i}(\phi_{p^j\ell})^{p^{i-j}} \in \mathbb{Z}[q]$.

By abuse of notation, we denote the image of ϕ in $\mathbb{F}_p[q] \subset k[q]$ by the same symbol. Since $\mathcal{U}_{\mathfrak{B}} = \mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{B}$ and $U_{\mathfrak{B}} = U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{B}$, we conclude that the equation $\mathrm{Ad}(E_\alpha^{p^i\ell})(u) \in \phi \cdot \mathcal{U}_{\mathfrak{B}}$ also holds for all $u \in \mathcal{U}_{\mathfrak{B}}$, viewing $E_\alpha^{p^i\ell}$ now as an element of $\mathcal{U}_{\mathfrak{B}} \subset U_{\mathfrak{B}}$. We wish to determine the multiplicity with which the factor $(q-\zeta)$ occurs in $\phi \in k[q]$. According to [32], the polynomials $\phi_{p^j\ell}$ factors over $\mathbb{F}_p$ as $(\phi_\ell)^{\varphi(p^j)}$. The polynomial ϕ_ℓ is separable over $\mathbb{F}_p$, hence $(q-\zeta) \in k[q]$ occurs as a factor of the image of $\phi_{p^j\ell}$ in $k[q]$ exactly $\varphi(p^j) = p^{j-1}(p-1)$ times. It follows then that the multiplicity of $(q-\zeta)$ in ϕ is equal to $\sum_{j=0}^{i} p^{i-j}\varphi(p^j) = p^i + \sum_{j=1}^{i} p^{i-1}(p-1) = s$, and $\mathrm{Ad}(E_\alpha^{p^i\ell})(u) \in (q-\zeta)^r \cdot \mathcal{U}_{\mathfrak{B}} = (q-\zeta)^s \cdot \mathcal{U}_{\mathfrak{B}}$.

We have shown that $\mathrm{Ad}(E_\alpha^{p^i\ell})(u) \in (q-\zeta)^s \cdot \mathcal{U}_{\mathfrak{B}}$. By a similar argument (replacing E_α by F_α in the previous three paragraphs), we can also show that $\mathrm{Ad}(F_\alpha^{p^i\ell})(u) \in (q-\zeta)^s \cdot \mathcal{U}_{\mathfrak{B}}$. This proves that the Ad-action of $U_{\mathfrak{B}}$ on itself induces an action of U_k on $\mathcal{U}_k$. Since $\zeta^\ell = 1$ in k, it is clear that the adjoint action of $K_\alpha^\ell \otimes 1 \in U_k$ on $\mathcal{U}_k$ is trivial for all $\alpha \in \Pi$, hence that the adjoint action of U_k on $\mathcal{U}_k$ factors through the quotient $U_\zeta = U_k / \langle K_\alpha^\ell \otimes 1 - 1 \otimes 1 : \alpha \in \Pi \rangle$. This proves the proposition for the left adjoint action defined by Ad.

To prove the proposition for the remaining three adjoint actions, apply equations (3.3.1–3.3.4). For example, the algebras $U_{\mathfrak{B}}$ and $\mathcal{U}_{\mathfrak{B}}$ are each stable under the automorphism ω, so (3.3.1) implies that the result for the Ad-action is equivalent to that for the $\overline{\mathrm{Ad}}$-action. Similarly, (3.3.2) implies that the result for the Ad_r-action is equivalent to that for the $\overline{\mathrm{Ad}}_r$-action. To establish the claim for the Ad_r-action, apply (3.3.3) and use the fact that the claim for the Ad-action remains true if we replace ζ by ζ^{-1}, and replace $\mathfrak{B}$ by $\mathcal{B}_{(q-\zeta^{-1})}$; the details are left to the reader. □

Remark 3.13. If $\mathrm{char}(k) = 0$, then Proposition 3.12 is essentially just [6, Proposition 2.9.2(i)], though the proof appearing in [6] appears to be incorrect. (In the notation of their proof, the elements $\frac{c}{q-\zeta}$ do not generate the $\mathcal{A}$-algebra $\mathsf{U}_{\mathcal{A}}$.)

Corollary 3.14. Fix $J \subseteq \Pi$.

(a) The subalgebra $u_\zeta(\mathfrak{g})$ is stable under the adjoint actions of $U_\zeta(\mathfrak{g})$ on itself.

(b) The subalgebras $u_\zeta(\mathfrak{u}_J^+)$ and $u_\zeta(\mathfrak{p}_J^+)$ are stable under the Ad-action of $U_\zeta(\mathfrak{p}_J^+)$.

(c) The subalgebras $u_\zeta(\mathfrak{u}_J)$ and $u_\zeta(\mathfrak{p}_J)$ are stable under the Ad_r-action of $U_\zeta(\mathfrak{p}_J)$.

(d) The subalgebras $u_\zeta(\mathfrak{u}_J)$ and $u_\zeta(\mathfrak{p}_J)$ are stable under the $\overline{\mathrm{Ad}}$-action of $U_\zeta(\mathfrak{p}_J)$.

(e) The subalgebras $u_\zeta(\mathfrak{u}_J^+)$ and $u_\zeta(\mathfrak{p}_J^+)$ are stable under the $\overline{\mathrm{Ad}}_r$-action of $U_\zeta(\mathfrak{p}_J^+)$.

Proof. Parts (b)–(e) follow from part (a) and Proposition 3.10. We prove part (a) for the left adjoint action defined by Ad, and leave to the reader the details of modifying our argument to accommodate the other three adjoint actions.

Recall that U_ζ is defined as a quotient of $U_k = U_{\mathfrak{B}} \otimes_{\mathfrak{B}} k$. We first prove that the left Ad-action of U_k on itself stabilizes the finite-dimensional subspace $u_k(\mathfrak{g})$. Let C be the set of generators for U_k identified in (1.1.9), and let D be the set of generators for u_k identified in (1.1.10). Recall from Example 2.6 that Ad makes U_k an H-module algebra over itself. Then to show that the adjoint action of U_k preserves the subspace u_k, it suffices to show $\mathrm{Ad}(c)(d) \in u_k$ whenever $c \in C$ and $d \in D$.

Let $c \in C$ and $d \in D$, considered now as elements of $U_{\mathfrak{B}}$. By Proposition 3.12, $\mathrm{Ad}(c)(d) \in \mathcal{U}_{\mathfrak{B}}$. Rewriting $\mathrm{Ad}(c)(d)$ in terms of the divided power basis of $U_{\mathfrak{B}}$, it follows that all monomials appearing in $\mathrm{Ad}(c)(d)$ with an exponent $\geq \ell$ will be preceded by an element of $\mathfrak{B}$ that vanishes upon the specialization $q \mapsto \zeta$. Since $U_k = U_{\mathfrak{B}} \otimes_{\mathfrak{B}} k$, it follows that after base change to k, $\mathrm{Ad}(c)(d) \in u_k$. This proves that the Ad-action of U_k on itself stabilizes the subspace u_k.

Now since $\zeta^\ell = 1$ in k, the Ad-action of $K_\alpha^\ell \otimes 1$ on U_k is trivial for each $\alpha \in \Pi$, hence the algebra homomorphism $\mathrm{Ad} : U_k \to \mathrm{End}_k(u_k)$ factors through the quotient $U_\zeta = U_k/\langle K_\alpha^\ell \otimes 1 - 1 \otimes 1\rangle$. We leave it to the reader to check that the ideal $\langle K_\alpha^\ell \otimes 1 - 1 \otimes 1\rangle$ is an $\mathrm{Ad}(U_\zeta)$-stable subspace of U_k. (First show that U_k acts trivially on the vector $K_\alpha^\ell \otimes 1 - 1 \otimes 1$, and then use the fact that U_k is a Hopf module algebra over itself.) Then Ad induces an action of U_ζ on itself which stabilizes the subspace $u_\zeta(\mathfrak{g})$. This proves part (a) for the left adjoint action defined by Ad. □

Proposition 3.15. Let V be a left $U_\zeta(\mathfrak{p}_J)$-module. Then

$$V^{u_\zeta(\mathfrak{u}_J)} = \{v \in V : uv = \varepsilon(u)v \,\forall\, u \in u_\zeta(\mathfrak{u}_J)\}$$

is a $U_\zeta(\mathfrak{p}_J)$-submodule of V.

Proof. Apply Corollary 3.14(c), Example 2.8, and Lemma 2.10. □

Recall the central subalgebra $\mathcal{Z} \subset \mathcal{U}_k$ defined in §3.2. Given $J \subseteq \Pi$, define $Z_J^+ \subset \mathcal{Z}$ to be the subalgebra generated by $\left\{E_\gamma^\ell : \gamma \in \Phi^+\backslash\Phi_J^+\right\}$, and define $Z_J \subset \mathcal{Z}$ to be the subalgebra generated by $\left\{F_\gamma^\ell : \gamma \in \Phi^+\backslash\Phi_J^+\right\}$. Part (a) of the next result is a generalization of [6, Proposition 2.9.2(ii)], which is stated there without proof under the assumption $\mathrm{char}(k) = 0$ and $J = \emptyset$.

Proposition 3.16. Fix $J \subseteq \Pi$.

(a) The subalgebra $\mathcal{Z} \subset \mathcal{U}_k$ is stable under the adjoint actions of U_ζ.

(b) The adjoint actions of $u_\zeta(\mathfrak{g})$ on $\mathcal{Z}$ are trivial.

(c) The subalgebra $Z_J^+ \subset \mathcal{U}_k$ is stable under the Ad-action of $U_\zeta(\mathfrak{p}_J^+)$.

(d) The subalgebra $Z_J \subset \mathcal{U}_k$ is stable under the Ad_r-action of $U_\zeta(\mathfrak{p}_J)$.

(e) The subalgebra $Z_J \subset \mathcal{U}_k$ is stable under the $\overline{\mathrm{Ad}}$-action of $U_\zeta(\mathfrak{p}_J)$.

(f) The subalgebra $Z_J^+ \subset \mathcal{U}_k$ is stable under the $\overline{\mathrm{Ad}}_r$-action of $U_\zeta(\mathfrak{p}_J^+)$.

Proof. Parts (c)–(f) follow from part (a) and Proposition 3.10, while part (b) is an easy consequence of the fact that $\mathcal{Z}$ is central in $\mathcal{U}_k$ and that the adjoint actions of the elements $\{E_\alpha, F_\alpha, K_\alpha : \alpha \in \Pi\}$ on $\mathcal{U}_k$ are the same whether we consider these generators as elements of u_ζ or as elements of $\mathcal{U}_k$. Now we prove part (a). Once again, we prove the result for the left adjoint action defined by Ad, and leave to the reader the straight-forward task of modifying our argument to accommodate the remaining adjoint actions.

Clearly, the Ad-action of U_ζ^0 on $\mathcal{U}_k$ stabilizes $\mathcal{Z}$. (The generators of $\mathcal{Z}$ are weight vectors for the adjoint actions of U_ζ^0.) Since the Ad-action of u_ζ on $\mathcal{Z}$ is trivial by part (b), to prove the claim it suffices to show, for each $\alpha \in \Pi$ and $n \in \mathbb{N}$, that the Ad-actions of the generators $E_\alpha^{(n)}, F_\alpha^{(n)} \in U_\zeta$ on $\mathcal{U}_k$ stabilize $\mathcal{Z}$.

Fix $\alpha \in \Pi$. Define linear maps $e_\alpha, e'_\alpha : U_{\mathfrak{B}} \to U_{\mathfrak{B}}$ by $e_\alpha(u) = [E_\alpha^{(\ell)}, u]$ (the usual commutator) and $e'_\alpha(u) = E_\alpha^{(\ell)} u - q_\alpha^{\ell(\ell-1)} K_\alpha^\ell u K_\alpha^{-\ell} E_\alpha^{(\ell)}$. De Concini and Kac have shown [21, §3.4] that the derivation e_α descends to a derivation of $\mathcal{U}_k$, and that the induced map $\mathcal{U}_k \to \mathcal{U}_k$ stabilizes the central subalgebra $\mathcal{Z}$. (Technically, they do this for $k = \mathbb{C}$, but all of the same calculations for $\mathbb{U}_k(\mathfrak{g})$ go through under our more general setup; cf. the calculations in [49] and [36, Chatper 8].) The linear map $(e_\alpha - e'_\alpha)$ also descends to a linear endomorphism of $\mathcal{U}_k$: Given $u \in \mathcal{U}_{\mathfrak{B}}$ of weight μ for U_ζ^0, we have $(e_\alpha - e'_\alpha)(u) = (q_\alpha^{\ell(\ell-1+(\mu,\alpha^\vee))} - 1) u E_\alpha^{(\ell)}$. Since ζ is a root for the leading polynomial, we can cancel out the factor of $(q - \zeta)$ appearing in the denominator of $E_\alpha^{(\ell)}$. Then $(e_\alpha - e'_\alpha)$ induces a linear endomorphism of $\mathcal{U}_k$, and from the above formula it is clear that this map map stabilizes the subspace $\mathcal{Z}$. Now $e'_\alpha = e_\alpha - (e_\alpha - e'_\alpha)$ induces a linear map on $\mathcal{U}_k$ stabilizing the subalgebra $\mathcal{Z}$. Equation (3.3.5) and Lemma 3.8 imply that the linear map $(\mathrm{Ad}(E_\alpha^{(\ell)}) - e'_\alpha)$ does the same, hence $\mathrm{Ad}(E_\alpha^{(\ell)}) = e'_\alpha + (\mathrm{Ad}(E_\alpha^{(\ell)}) - e'_\alpha)$ does as well.

By an argument completely analogous to that performed in the preceding paragraph, we can show that $\mathrm{Ad}(F_\alpha^{(\ell)})$ stabilizes the central subalgebra $\mathcal{Z}$. If $\mathrm{char}(k) = 0$, then this proves the claim, because in that case U_ζ is generated as an algebra by u_ζ and the elements $\left\{E_\alpha^{(\ell)}, F_\alpha^{(\ell)} : \alpha \in \Pi\right\}$. So now assume $\mathrm{char}(k) \neq 0$.

Let $c = F^{\mathbf{r}} K_\mu E^{\mathbf{s}} \in \mathcal{U}_{\mathcal{A}}$ be a basis monomial as in (3.2.1). Assume that all parts of $\mathbf{r}, \mathbf{s} \in \mathbb{N}^N$ are divisible by ℓ, and assume that $\mu \in \ell\mathbb{Z}\Phi$. Then the image of c in $\mathcal{U}_k$ is an element of $\mathcal{Z}$, and $\mathcal{Z}$ is spanned by all such monomials. Fix $n \in \mathbb{N}$, and consider $E_\alpha^n \in \mathcal{U}_{\mathcal{A}} \subset U_{\mathcal{A}}$. Since $\mathcal{U}_{\mathcal{A}}$ is a Hopf algebra, we have $\mathrm{Ad}(E_\alpha^n)(c) \in \mathcal{U}_{\mathcal{A}}$. Write $\mathrm{Ad}(E_\alpha^n)(c) = \sum a'_{\mathbf{b},\mathbf{c},\lambda} F^{\mathbf{b}} K_\lambda E^{\mathbf{c}}$, a sum of monomial basis elements in $\mathcal{U}_{\mathcal{A}}$. Then

$$\mathrm{Ad}(E_\alpha^{(n)})(c) = \sum a_{\mathbf{b},\mathbf{c},\lambda} F^{\mathbf{b}} K_\lambda E^{\mathbf{c}}, \tag{3.3.8}$$

where $a_{\mathbf{b},\mathbf{c},\lambda} = a'_{\mathbf{b},\mathbf{c},\lambda}/[n]^!_\alpha \in (1/[n]^!_\alpha)\mathcal{A}$. From Proposition 3.12 we conclude that in fact $a_{\mathbf{b},\mathbf{c},\lambda} \in \mathbb{Z}[q]_{(\langle \phi_\ell \rangle)}$, the localization of $\mathbb{Z}[q]$ at the prime ideal generated by the ℓ-th cyclotomic polynomial ϕ_ℓ.

Strictly speaking, (3.3.8) is an equation in $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$, but if we replace each $a_{\mathbf{b},\mathbf{c},\lambda}$ by its image in $k(q)$, then we obtain a corresponding valid equation in $\mathbb{U}_k(\mathfrak{g})$. (Note that it does indeed make sense to consider the image of each $a_{\mathbf{b},\mathbf{c},\lambda}$ in $k(q)$.) In this context, Proposition 3.12 asserts that the coefficients $a_{\mathbf{b},\mathbf{c},\lambda}$ are actually elements of $\mathfrak{B} = \mathcal{B}_{(\langle q-\zeta \rangle)} \subset k(q)$, and if we specialize $q \mapsto \zeta$ in (3.3.8), then we obtain a formula for the adjoint action of $E_\alpha^{(n)} \in U_\zeta$ on $c \in \mathcal{Z} \subset \mathcal{U}_k$.

Set $\mathfrak{p}$ to be the kernel of the unique ring homomorphism $\mathcal{A} \to k$ sending $q \mapsto \zeta$. Then $\mathfrak{p}$ is a prime ideal (because k is an integral domain). Let $\mathcal{A}_{\mathfrak{p}}$ denote the localization of $\mathcal{A}$ at $\mathfrak{p}$. Then $a_{\mathbf{b},\mathbf{c},\lambda} \in \mathcal{A}_{\mathfrak{p}}$, and the map $\mathcal{A}_{\mathfrak{p}} \to k$ factors through the quotient $\mathcal{A}_{\mathfrak{p}}/(\phi_\ell)\mathcal{A}_{\mathfrak{p}}$, which identifies with a subalgebra of the ring $\mathbb{Q}[q]/(\phi_\ell)\mathbb{Q}[q]$, a cyclotomic field. We have already proven the claim in (a) for fields of characteristic zero, so we know that the image of $a_{\mathbf{b},\mathbf{c},\lambda}$ in $\mathcal{A}_{\mathfrak{p}}/(\phi_\ell)\mathcal{A}_{\mathfrak{p}}$ will be zero unless all parts of $\mathbf{b}, \mathbf{c} \in \mathbb{N}^N$ are divisible by ℓ and unless $\lambda \in \ell\mathbb{Z}\Phi$. This implies that the adjoint action of $E_\alpha^{(n)} \in U_\zeta$ on $c \in \mathcal{Z}$ will have image in $\mathcal{Z}$, as desired. □

Remark 3.17. For each $J \subseteq \Pi$, the projection map $\mathcal{U}_k(\mathfrak{g}) \twoheadrightarrow \mathcal{U}_\zeta(\mathfrak{g})$ induces algebra isomorphisms $\mathcal{U}_k(\mathfrak{u}_J) \xrightarrow{\sim} \mathcal{U}_\zeta(\mathfrak{u}_J)$ and $\mathcal{U}_k(\mathfrak{u}_J^+) \xrightarrow{\sim} \mathcal{U}_\zeta(\mathfrak{u}_J^+)$, so Proposition 3.12 can be interpreted as saying that there is an adjoint action of $U_\zeta(\mathfrak{p}_J)$ on $\mathcal{U}_\zeta(\mathfrak{u}_J)$. But to conclude that the action of $U_\zeta(\mathfrak{g})$ on $\mathcal{U}_k(\mathfrak{g})$ induces an action of U_ζ on all of $\mathcal{U}_\zeta(\mathfrak{g})$, we would need to show that the ideal $\langle K_\alpha^\ell \otimes 1 - 1 \otimes 1 : \alpha \in \Pi \rangle \subset \mathcal{U}_k$ is a U_ζ-stable subspace. This is false, as the following example illustrates. (It is important here not to become confused by the ambiguity of our notation. As a subset of $\mathcal{U}_k$, the set $\{K_\alpha \otimes 1 - 1 \otimes 1 : \alpha \in \Pi\}$ does not generate an $\mathrm{Ad}(U_\zeta)$-stable subspace. But as a subset of U_k, it does generate an $\mathrm{Ad}(U_\zeta)$-stable subspace, as was remarked in the proof of Corollary 3.14.)

Example 3.18. Let $\mathfrak{g} = \mathfrak{sl}_2$. Write $\Phi^+ = \{\alpha\}$. Then $q_\alpha = q$. Set $E = E_\alpha$, $K = K_\alpha$. Let $\ell = 3$, and set $\zeta = e^{2\pi i/3} \in \mathbb{C}$, a primitive third root of unity. Set $k = \mathbb{Q}(\zeta)$. Then in $\mathbb{U}_k(\mathfrak{g})$ we have

$$\begin{aligned}\operatorname{Ad}(E^{(\ell)})(K^{\ell}) &= \sum_{i=0}^{\ell}(-1)^i q^{i(\ell-1)} E^{(\ell-i)} K^i K^{\ell} K^{-i} E^{(i)} \\ &= \sum_{i=0}^{3}(-1)^i q^{2i} E^{(3-i)} q^{6i} E^{(i)} K^3 \\ &= \sum_{i=0}^{3}(-1)^i q^{8i} \begin{bmatrix} 3 \\ i \end{bmatrix} E^{(3)} K^3 \\ &= -q^3(-1+q^2)^3(1+q^2+2q^4+2q^6+2q^8+q^{10}+q^{12})E^3K^3\end{aligned}$$

Evaluating at $q=\zeta$, we get $\operatorname{Ad}(E^{(3)})(K^3) = 3i\sqrt{3}\cdot E^3K^3 \in \mathcal{Z}$, in agreement with Proposition 3.16. But then $\operatorname{Ad}(E^{(3)})(K^3-1) = 3i\sqrt{3}\cdot E^3K^3 - 0 = 3i\sqrt{3}\cdot E^3K^3$, which is not an element of the ideal $\langle K^3\otimes 1 - 1\otimes 1\rangle \subset \mathcal{U}_k$, because $3i\sqrt{3}\cdot E^3K^3$ is equivalent to $3i\sqrt{3}\cdot E^3 \neq 0$ in $\mathcal{U}_\zeta$.

The next result is immediate from Propositions 3.12 and 3.16 and Remark 3.17 (cf. [6, Proposition 2.9.2] and [9, Corollary 2.7.4] for the case $\operatorname{char}(k)=0$).

Corollary 3.19. Under the induced Ad_r-action of U_ζ on $\mathcal{U}_k$,

(a) $U_\zeta(\mathfrak{p}_J)$ stabilizes $\mathcal{U}_\zeta(\mathfrak{u}_J)$, Z_J, and $u_\zeta(\mathfrak{u}_J)$.

(b) The action of $u_\zeta(\mathfrak{p}_J)$ on Z_J is trivial.

(c) Ad_r induces an action of $\operatorname{hy}(P_J) = U_\zeta(\mathfrak{p}_J)//u_\zeta(\mathfrak{p}_J)$ on Z_J.

Let $Z_{J,\varepsilon}$ denote the augmentation ideal of Z_J. It is a submodule for the Ad_r-action of $U_\zeta(\mathfrak{p}_J)$ on Z_J. Since $\mathcal{U}_\zeta(\mathfrak{u}_J)$ is a Hopf module algebra for $U_\zeta(\mathfrak{p}_J)$, we conclude that the subspace $(Z_{J,\varepsilon})^2 \subset Z_{J,\varepsilon}$ is also stable under the Ad_r-action of $U_\zeta(\mathfrak{p}_J)$. This makes $Z_{J,\varepsilon}/(Z_{J,\varepsilon})^2$ a right $\operatorname{hy}(P_J) = U_\zeta(\mathfrak{p}_J)//u_\zeta(\mathfrak{p}_J)$-module. Equivalently, if we precompose the structure map for $Z_{J,\varepsilon}/(Z_{J,\varepsilon})^2$ with the antipode S of $U_\zeta(\mathfrak{p}_J)$, then we can view $Z_{J,\varepsilon}/(Z_{J,\varepsilon})^2$ as a left $\operatorname{hy}(P_J)$-module (or as a left P_J-module).

If $\operatorname{char}(k)=0$ and $J=\emptyset$, then the next proposition is just [6, Corollary 2.9.6].

Proposition 3.20. Fix $J \subseteq \Pi$.

(a) As left modules for $\operatorname{hy}(P_J)$, $Z_{J,\varepsilon}/(Z_{J,\varepsilon})^2 \cong \mathfrak{u}_J$, where the left action of $\operatorname{hy}(P_J)$ on $\mathfrak{u}_J$ is induced by the usual adjoint action of the parabolic subgroup P_J on the Lie algebra $\mathfrak{u}_J$ (i.e., the adjoint action in the sense of algebraic groups).

(b) The Ad_r-action of $U_\zeta(\mathfrak{p}_J)$ on Z_J induces a $U_\zeta(\mathfrak{p}_J)$-equivariant algebra isomorphism $H^\bullet(Z_J,k) \cong \Lambda^\bullet(\mathfrak{u}_J^*)^{[1]}$.

Proof. For any $J \subseteq \Pi$, there exists a natural isomorphism of $\mathbb{Z}\Phi^+$-graded vector spaces $Z_{J,\varepsilon}/(Z_{J,\varepsilon})^2 \cong \mathfrak{u}_J$. Suppose $J = \emptyset$ and $\mathrm{char}(k) = 0$. Then, by [6, Corollary 2.9.6(i)], the left Ad-action of $U_\zeta(\mathfrak{b}^+)$ on $Z_\varepsilon^+/(Z_\varepsilon^+)^2$ induces a left $U_\zeta(\mathfrak{b}^+)$-module isomorphism $Z_\varepsilon^+/(Z_\varepsilon^+)^2 \cong (\mathfrak{u}^+)^{[1]}$, where the action of $\mathrm{hy}(B^+) = U_\zeta(\mathfrak{b}^+)//u_\zeta(\mathfrak{b}^+)$ on $\mathfrak{u}^+$ is induced by the usual adjoint action (i.e., the adjoint action in the sense of algebraic groups) of the positive Borel subgroup B^+ on the Lie algebra $\mathfrak{u}^+$. Part (a) then follows from (3.3.3) via a base-change argument like the one employed in the proof of Proposition 3.16; the details are left to the reader.

Now suppose $J \subseteq \Pi$ and $\mathrm{char}(k)$ are arbitrary. By part (a), there exists by restriction a B-equivariant vector space isomorphism $Z_{J,\varepsilon}/(Z_{J,\varepsilon})^2 \xrightarrow{\sim} \mathfrak{u}_J$, with B acting on $\mathfrak{u}_J$ via the adjoint action. By Frobenius reciprocity, this isomorphism corresponds to an injective P_J-module homomorphism $Z_{J,\varepsilon}/(Z_{J,\varepsilon})^2 \hookrightarrow \mathrm{ind}_B^{P_J} \mathfrak{u}_J$. From the tensor identity, we get $\mathrm{ind}_B^{P_J} \mathfrak{u}_J \cong \mathfrak{u}_J \otimes \mathrm{ind}_B^{P_J} k = \mathfrak{u}_J \otimes k = \mathfrak{u}_J$, where P_J acts on the right hand side via the adjoint action. By dimension comparison, we conclude that $Z_{J,\varepsilon}/(Z_{J,\varepsilon})^2 \cong \mathfrak{u}_J$ as P_J-modules, hence that $Z_{J,\varepsilon}/(Z_{J,\varepsilon})^2 \cong \mathfrak{u}_J^{[1]}$ as left $U_\zeta(\mathfrak{p}_J)$-modules. (The argument in this paragraph appears in the proof of [9, Lemma 5.4.1].)

According to Lemma 2.14, the adjoint action of $U_\zeta(\mathfrak{p}_J)$ on Z_J makes $H^\bullet(Z_J, k)$ a left H-module algebra. Since Z_J is a polynomial algebra on generators F_β^ℓ, $\beta \in \Phi^+ \backslash \Phi_J^+$, there exists a graded algebra isomorphism $H^\bullet(Z_J, k) \cong \Lambda^\bullet(\mathfrak{u}_J^*)$. (Here $\Lambda^\bullet(\mathfrak{u}_J^*)$ denotes the exterior algebra on the vector space $\mathfrak{u}_J^*$.) This algebra is generated in degree one, so the action of $U_\zeta(\mathfrak{p}_J)$ on $H^\bullet(Z_J, k)$ is completely determined by the action of $U_\zeta(\mathfrak{p}_J)$ on $H^1(Z_J, k)$. Part (b) now follows from part (a) and the fact that there exists a natural isomorphism $H^1(Z_J, k) \cong (Z_{J,\varepsilon}/(Z_{J,\varepsilon})^2)^*$ compatible with the adjoint action of $U_\zeta(\mathfrak{p}_J)$. (The natural isomorphism is evident from the low degree terms in the cobar resolution computing $H^\bullet(Z_J, k)$; cf. also [31, Lemma 2.2].) $\square$

Chapter 4

Cohomology of the Frobenius–Lusztig kernel

In this chapter we apply the results of Chapters 2 and 3 to compute the structure of the cohomology ring $H^\bullet(u_\zeta(\mathfrak{g}), k)$ for the Frobenius–Lusztig kernel $u_\zeta(\mathfrak{g})$ of $U_\zeta(\mathfrak{g})$. Our strategy is essentially the same as that in [9], though we are primarily interested in the case $\mathrm{char}(k) > 0$. One significant difference between the exposition here and that in [9] is our implicit use, consistent with the exposition of Chapter 2, of adjoint actions on cohomology induced by the right adjoint action $\mathrm{Ad}_r : \mathbb{U}_k(\mathfrak{g}) \to \mathrm{End}_k(\mathbb{U}_k(\mathfrak{g}))^{op}$. The authors of [9] mistakenly consider adjoint actions on cohomology induced by the left adjoint action $\overline{\mathrm{Ad}} : \mathbb{U}_k(\mathfrak{g}) \to \mathrm{End}_k(\mathbb{U}_k(\mathfrak{g}))$ (see Remark 2.13 for a justification of the inadmissibility of $\overline{\mathrm{Ad}}$).

4.1 Cohomology of the De Concini–Kac quantum algebra

Fix $J \subseteq \Pi$. The algebra $\mathcal{U}_\zeta(\mathfrak{u}_J)$ is a right $U_\zeta(\mathfrak{p}_J)$-module algebra by Corollary 3.19, hence $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), k)$ is a left $U_\zeta(\mathfrak{p}_J)$-module algebra by Theorem 2.11. By restriction, $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), k)$ is a left U_ζ^0-module algebra.

Lemma 4.1. [9, Lemma 2.8.2, Corollary 2.8.3] For each $n \in \mathbb{N}$, the cohomology group $H^n(\mathcal{U}_\zeta(\mathfrak{u}_J), k)$ is a finite-dimensional weight module for U_ζ^0. For each $\lambda \in X$, the weight space $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), k)_\lambda$ is finite-dimensional.

Proof. Recall that $u_\zeta(\mathfrak{u}_J) \cong \mathcal{U}_\zeta(\mathfrak{u}_J)//Z_J$. The algebra $\mathcal{U}_\zeta(\mathfrak{u}_J)$ is free over the central subalgebra Z_J by Lemma 3.9. In particular, $\mathcal{U}_\zeta(\mathfrak{u}_J)$ is right Z_J-flat. It follows then from Corollary 3.19 and Theorem 2.24 that there exists a spectral sequence of $U_\zeta(\mathfrak{p}_J)$-modules satisfying

$$E_2^{i,j} = H^i(u_\zeta(\mathfrak{u}_J), H^j(Z_J, k)) \Rightarrow H^{i+j}(\mathcal{U}_\zeta(\mathfrak{u}_J), k).$$

By Corollary 2.20, the action of $u_\zeta(\mathfrak{u}_J)$ on $H^j(Z_J,k)$ is trivial, so we can rewrite the spectral sequence as

$$E_2^{i,j} = H^j(Z_J,k) \otimes H^i(u_\zeta(\mathfrak{u}_J),k) \Rightarrow H^{i+j}(\mathcal{U}_\zeta(\mathfrak{u}_J),k). \tag{4.1.1}$$

(The $H^j(Z_J,k)$ term should be written on the left side of the tensor product in order to preserve the $U_\zeta(\mathfrak{p}_J)$-module structure; cf. (2.1.4) for our convention on the diagonal action of a Hopf algebra on the cobar resolution.) We have $H^\bullet(Z_J,k) \cong \Lambda^\bullet(\mathfrak{u}_J^*)^{[1]}$ as a $U_\zeta(\mathfrak{p}_J)$-module algebra by Proposition 3.20, and for each $i \in \mathbb{N}$ the cohomology group $H^i(u_\zeta(\mathfrak{u}_J),k)$ is a finite-dimensional weight module for the adjoint action of U_ζ^0. (The cobar complex computing $H^\bullet(u_\zeta(\mathfrak{u}_J),k)$ is a complex of weight modules for U_ζ^0, so $H^\bullet(u_\zeta(\mathfrak{u}_J),k)$ is a weight module for U_ζ^0. Given $i \in \mathbb{N}$, the cohomology group $H^i(u_\zeta(\mathfrak{u}_J),k)$ is finite-dimensional because $u_\zeta(\mathfrak{u}_J)$ is a finite-dimensional algebra.)

Fix $n \in \mathbb{N}$. The differentials of the spectral sequence (4.1.1) are U_ζ^0-module homomorphisms, and each term of the E_2 page is a weight module for U_ζ^0. Since a submodule of a weight module is again a weight module, it follows that $H^n(\mathcal{U}_\zeta(\mathfrak{u}_J),k)$ identifies with a weight submodule of the finite-dimensional vector space $\bigoplus_{i+j=n} E_2^{i,j}$. This proves the first claim of the lemma.

Now fix $\lambda \in X$. To show that $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J),k)_\lambda$ is finite-dimensional, it suffices to show $(E_2^{i,j})_\lambda = 0$ for $i \gg 0$. First, $E_2^{i,j} \cong H^j(Z_J,k) \otimes H^i(u_\zeta(\mathfrak{u}_J),k)$. If $(E_2^{i,j})_\lambda \neq 0$, then $\lambda = \nu + \mu$ for some $\nu,\mu \in X$ with $H^j(Z_J,k)_\nu \neq 0$ and $H^i(u_\zeta(\mathfrak{u}_J),k)_\mu \neq 0$. The cohomology group $H^\bullet(Z_J,k)$ is a finite-dimensional weight module for U_ζ^0, so there are only finitely many possibilities for $\nu \in X$, hence only finitely many possibilities for $\mu = \lambda - \nu$. Considering the possible weights occurring in each particular degree of the cobar complex computing $H^\bullet(u_\zeta(\mathfrak{u}_J),k)$, it follows that for any fixed $\mu \in X$, $H^i(u_\zeta(\mathfrak{u}_J),k)_\mu = 0$ for all $i \gg 0$. This now implies the desired result for $(E_2^{i,j})_\lambda$. □

Set $A = \mathcal{U}_\zeta(\mathfrak{u}_J)$. Let $\mathsf{B}^\bullet(A) = A_\varepsilon^{\otimes\bullet}$ be the complex with differential $d : \mathsf{B}^n \to \mathsf{B}^{n-1}$ defined by

$$d([a_1|\dots|a_n]) = \sum_{i=1}^{n-1}(-1)^i[a_1|\dots|a_ia_{i+1}|\dots|a_n]. \tag{4.1.2}$$

Let $C = C^\bullet(A) = \operatorname{Hom}_k(\mathsf{B}^\bullet(A),k)$ denote the cobar complex computing $H^\bullet(A,k)$. (For convenience of notation, we have replaced the left bar resolution $\mathsf{B}_\bullet(A)$ discussed in §2.1 by the complex $\mathsf{B}^\bullet(A)$ defined here. Of course, $C^\bullet(A) \cong \operatorname{Hom}_A(\mathsf{B}_\bullet(A),k)$ as graded complexes. If we wanted to compute $H^\bullet(A,V)$ for some arbitrary (non-trivial) A-module V, then we would need to compute the cohomology of the cochain complex $C^\bullet(A,V) = \operatorname{Hom}_A(\mathsf{B}_\bullet(A),V)$, and not merely the that of the cochain complex $\operatorname{Hom}_k(\mathsf{B}^\bullet(A),V)$.) The cup product $\smile: C^n \otimes C^m \to C^{n+m}$ defined in (2.1.5) makes C a differential graded algebra. By the proof of Lemma 2.14, the Ad_r-action of $U_\zeta(\mathfrak{p}_J)$ on $\mathcal{U}_\zeta(\mathfrak{u}_J)$ makes C a differential graded $U_\zeta(\mathfrak{p}_J)$-module algebra.

For $n \in \mathbb{N}$, the set $C^n(A) = \operatorname{Hom}_k(A_\varepsilon^{\otimes n},k)$ is not a weight module for U_ζ^0. Let $C_f^n(A) \subset C^n(A)$ denote the space of all n-cochains with support in finite-dimensional subspaces of $A_\varepsilon^{\otimes n}$. It is the unique largest weight module in $C^n(A)$. Observe that $C_f^\bullet = C_f^\bullet(A)$ is a subcomplex of $C^\bullet(A)$ and is closed under the cup product. Using Lemma 4.1, we deduce that $C_f^\bullet(A)$ captures the cohomology of $C^\bullet(A)$.

Lemma 4.2. $H^\bullet(C_f^\bullet(A)) \cong H^\bullet(A,k)$.

Proof. The inclusion of complexes $\varphi : C_f^\bullet \hookrightarrow C^\bullet(A)$ induces an algebra homomorphism $H(\varphi) : H^\bullet(C_f^\bullet) \to H^\bullet(A,k)$. From Lemma 4.1 we know that $H^\bullet(A,k)$ is a weight module for U_ζ^0. To prove the surjectivity of $H(\varphi)$ it then suffices to verify the following claim: Given $\lambda \in X$ and $v \in H^n(A,k)_\lambda$, we can choose an n-cocycle $f \in C_f^n(A)_\lambda$ such that $\mathrm{cls}(\varphi f) = v$. Indeed, let $v \in H^n(A,k)_\lambda$, and choose an arbitrary n-cocylce $f \in C^n(A) = \mathrm{Hom}_k(\mathsf{B}^n(A),k)$ such that $\mathrm{cls}(f) = v$. The complex $\mathsf{B}^\bullet(A)$ decomposes as a direct sum of subcomplexes $\mathsf{B}^\bullet(A) = \bigoplus_{\mu\in X}\mathsf{B}^\bullet(A)_\mu$ (the weight space decomposition of $\mathsf{B}^\bullet(A)$ for the Ad_r-action of U_ζ^0), hence the cochain complex $C^\bullet(A)$ decomposes as a direct product of chain complexes

$$C^\bullet(A) = \mathrm{Hom}_k(\textstyle\bigoplus_{\mu\in X}\mathsf{B}^\bullet(A)_\mu, k) = \prod_{\mu\in X}\mathrm{Hom}_k(\mathsf{B}^\bullet(A)_\mu, k).$$

Note that $C_f^\bullet(A) = \bigoplus_{\mu\in X}\mathrm{Hom}_k(\mathsf{B}^\bullet(A)_\mu,k)$, and $C^\bullet(A)_\mu = \mathrm{Hom}_k(\mathsf{B}^\bullet(A)_\mu,k)$.

Now write $f = \sum_{\mu\in X} f_\mu$, where f_μ denotes the restriction of f to $\mathsf{B}^n(A)_\mu$. From the above observations it follows that f is a cocycle in $C^n(A)$ if and only if each f_μ is a cocycle in $C^n(A)$. Set $g = f - f_\lambda$. To prove $\mathrm{cls}(f_\lambda) = v$, it suffices to show that g is a coboundary in $C^n(A)$. Writing $g = \sum_{\mu\in X} g_\mu$, we must show that each g_μ is a coboundary in $C^n(A)_\mu = \mathrm{Hom}_k(\mathsf{B}^n(A)_\mu,k)$.

We have $\mathrm{cls}(g) \in H^n(A,k)_\lambda$, hence for each fixed $u \in U_\zeta^0$, the cocycle $u.g - \lambda(u)g$ must be a coboundary in $C^n(A)$. In particular, for fixed $\mu \in X$, $u.g_\mu - \lambda(u)g_\mu = (\mu(u) - \lambda(u))g_\mu$ is a coboundary in $C^n(A)_\mu$. We can choose $u \in U_\zeta^0$ such that $0 \neq y := \mu(u) - \lambda(u) \in k$. Then if $y \cdot g_\mu = \delta(h)$ for $h \in C^{n-1}(A)_\mu$, then we also have $g_\mu = \delta(y^{-1}\cdot h)$, that is, g_μ is a coboundary in $C^n(A)_\mu$. This proves that g is a coboundary in $C^n(A)$, hence that $\mathrm{cls}(\varphi f_\lambda) = \mathrm{cls}(f) = v$.

It remains to show that $H(\varphi)$ is injective. It suffices to show that if $f \in C^n(A)_\lambda$ is a cocycle, and if $f = \delta(g)$ for some $g \in C^{n-1}(A)$, then we can choose g to be an element of $C^{n-1}(A)_\lambda$. Indeed, suppose $f \in C^n(A)_\lambda$ and $g \in C^{n-1}(A)$ are as above. Write $g = g' + g''$ for some $g' \in C^{n-1}(A)_\lambda$ and some $g'' \in \mathrm{Hom}_k(W^{n-1},k)$. Here $W^\bullet = \bigoplus_{\mu\neq\lambda} B^\bullet(A)_\mu$. Note that $\mathrm{Hom}_k(W^\bullet,k)$ is a subcomplex of $C^\bullet(A)$. Now $\delta(g)$ and $\delta(g')$ are both elements of $C^n(A)_\lambda$, hence

$$\delta(g) - \delta(g') = \delta(g'') \in C^n(A)_\lambda \cap \mathrm{Hom}_k(W^n,k) = \{0\}\,.$$

So $f = \delta(g) = \delta(g')$. This proves the claim. □

Recall the multiplicative filtration of $\mathbb{U}_\mathbb{Q}(\mathfrak{g})$ indexed by $\mathbb{N}^{2N+1}$ defined in §3.1. This filtration restricts to a filtration on the integral form $\mathcal{U}_\mathcal{A} \subset \mathbb{U}_\mathbb{Q}(\mathfrak{g})$, hence induces a multiplicative filtration on the De Concini–Kac quantum algebra $\mathcal{U}_\zeta(\mathfrak{g})$. By restriction, we obtain a multiplicative filtration on the distinguished subalgebra $\mathcal{U}_\zeta(\mathfrak{u}_J)$.

Again, set $A = \mathcal{U}_\zeta(\mathfrak{u}_J)$. Applying Lemma 3.1, we obtain the following description for the associated graded algebra $\operatorname{gr} A$ (cf. [21, Proposition 1.7]): As an algebra, $\operatorname{gr} A$ is generated by indeterminates

$$\left\{X_\alpha : \alpha \in \Phi^+ \backslash \Phi_J^+\right\} \tag{4.1.3}$$

subject to the relations

$$X_\alpha X_\beta = \zeta^{(\alpha,\beta)} X_\beta X_\alpha \quad \text{if} \quad \alpha \prec \beta. \tag{4.1.4}$$

(We write $\alpha \prec \beta$ if, in the notation of §1.2, we have $\alpha = \gamma_i$, $\beta = \gamma_j$, and $i < j$.)

The filtration on $\mathcal{U}_\zeta(\mathfrak{u}_J)$ induces a multiplicative filtration on $u_\zeta(\mathfrak{u}_J) = \mathcal{U}_\zeta(\mathfrak{u}_J)//Z_J$. (Equivalently, the filtration on $\mathbb{U}_\mathbb{Q}(\mathfrak{g})$ restricts to a filtration on the integral form U_A, hence induces a filtration on $U_\zeta(\mathfrak{g})$ and its subalgebra $u_\zeta(\mathfrak{u}_J)$.) The associated graded algebra $\operatorname{gr} u_\zeta(\mathfrak{u}_J)$ is generated by indeterminates (4.1.3) subject to (4.1.4), as well as the following additional relations:

$$X_\alpha^\ell = 0 \quad \text{for each} \quad \alpha \in \Phi^+ \backslash \Phi_J^+. \tag{4.1.5}$$

So $\operatorname{gr} A$ is a twisted polynomial ring, while $\operatorname{gr} u_\zeta(\mathfrak{u}_J)$ is a truncated twisted polynomial ring. The Ad_r-action of U_ζ^0 on $\mathcal{U}_\zeta(\mathfrak{u}_J)$ induces right actions of U_ζ^0 on the graded algebras $\operatorname{gr}\mathcal{U}_\zeta(\mathfrak{u}_J)$ and $\operatorname{gr} u_\zeta(\mathfrak{u}_J)$, such that for each $\alpha \in \Phi^+ \backslash \Phi_J^+$, the generator X_α is of weight α for U_ζ^0 (i.e., in the notation of [3, §1.4], $X_\alpha \cdot u = \chi_\alpha(u) X_\alpha$ for all $u \in U_\zeta^0$).

By a similar sort of analysis as that used to prove Lemmas 4.1 and 4.2, one can prove the following result. The details are left to the reader.

Lemma 4.3. For each $n \in \mathbb{N}$, the cohomology group $H^n(\operatorname{gr}\mathcal{U}_\zeta(\mathfrak{u}_J), k)$ is a finite-dimensional weight module for U_ζ^0. For each $\lambda \in X$, the weight space $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), k)$ is finite-dimensional. The cohomology ring $H^\bullet(\operatorname{gr}\mathcal{U}_\zeta(\mathfrak{u}_J), k)$ can be computed as the cohomology of the cochain complex $C_f^\bullet(\operatorname{gr}\mathcal{U}_\zeta(\mathfrak{u}_J)) = \bigoplus_{\mu \in X} C^\bullet(\operatorname{gr}\mathcal{U}_\zeta(\mathfrak{u}_J))_\mu$.

Let $\Lambda_{\zeta,J}^\bullet$ denote the graded algebra with generators $\left\{x_\alpha : \alpha \in \Phi^+ \backslash \Phi_J^+\right\}$, each of graded degree one, subject to the relations

$$\begin{gathered} x_\alpha x_\beta + \zeta^{-(\alpha,\beta)} x_\beta x_\alpha = 0 \quad \text{if} \quad \alpha \prec \beta, \\ \text{and} \quad x_\alpha^2 = 0 \quad \text{for each} \quad \alpha \in \Phi^+ \backslash \Phi_J^+. \end{gathered}$$

Assigning the generator x_α to have weight α for U_ζ^0, the algebra $\Lambda_{\zeta,J}^\bullet$ becomes a left U_ζ^0-module algebra. The following lemma generalizes an observation of Ginzburg and Kumar. Certain details of the proof will be needed later in Chapter 5.

Lemma 4.4. [30, Proposition 2.1] There exists an isomorphism of graded U_ζ^0-module algebras $H^\bullet(\operatorname{gr}\mathcal{U}_\zeta(\mathfrak{u}_J), k) \cong \Lambda_{\zeta,J}^\bullet$.

Proof. Set $A = \mathcal{U}_\zeta(\mathfrak{u}_J)$. The relations (4.1.4) for $\operatorname{gr} A$ imply by [56, Theorem 5.3] that the algebra $\operatorname{gr} A$, viewed as an $\mathbb{N}$-graded algebra with generators (4.1.3) each of graded degree one, is a homogeneous Koszul algebra in the sense of [56, §2]. Then $H^\bullet(\operatorname{gr} A, k) \cong \Lambda^\bullet_{\zeta,J}$ as graded algebras by [56, Theorem 2.5]. Under this isomorphism, the element $x_\alpha \in \Lambda^1_{\zeta,J}$ corresponds to the class of the cocyle $f \in \operatorname{Hom}_k((\operatorname{gr} A)_\varepsilon, k)$ satisfying $f(X_\alpha) = 1$ and $f(v) = 0$ for all $v \in V \subset (\operatorname{gr} A)_\varepsilon$. Here $V \subset (\operatorname{gr} A)_\varepsilon$ is the vector subspace spanned by all monomials $\prod_{\beta \in \Phi^+ \setminus \Phi^+_J} X_\beta^{a_\beta}$ such that either $a_\alpha \neq 1$, or else $a_\alpha = 1$ and $a_\beta \neq 0$ for some $\beta \neq \alpha$. The statement on the U^0_ζ-module algebra structure of $H^\bullet(\operatorname{gr} A, k)$ follows from the fact that f has weight α for U^0_ζ. □

Remark 4.5. Priddy [56] computes the cohomology of $\mathbb{Z}$-graded algebras with coefficients in $\mathbb{Z}$-graded modules via a $\mathbb{Z}$-graded version of the Ext functor. That his calculation for the Ext-algebra $\operatorname{Ext}^\bullet_{\operatorname{gr}\mathcal{U}_\zeta(\mathfrak{u}_J)}(k,k) = H^\bullet(\operatorname{gr}\mathcal{U}_\zeta(\mathfrak{u}_J), k)$ agrees with the (usual) non-graded version of Ext that we consider in this paper is a consequence of the last statement in Lemma 4.3.

The following result can now be proved along the lines of [9, Proposition 2.9.1(b)].

Proposition 4.6. [9, Proposition 2.9.1(b)] For each $\lambda \in X$ and $n \in \mathbb{N}$, we have

$$\dim H^n(\mathcal{U}_\zeta(\mathfrak{u}_J), k)_\lambda \leq \dim(\Lambda^n_{\zeta,J})_\lambda.$$

Let ℓ be an odd positive integer satisfying Assumption 1.5. Define the subset $\Phi_0 \subseteq \Phi$ by

$$\Phi_0 = \Phi_{0,\ell} = \{\alpha \in \Phi : (\rho, \alpha^\vee) \equiv 0 \bmod \ell\}.$$

Theorem 4.7. [9, Theorem 3.5.1] Let ℓ be as in Assumption 1.5. Then there exists an element $w \in W$ and a subset $J \subseteq \Pi$ such that $w(\Phi_0) = \Phi_J$.

If $\ell \geq h$, h the Coxeter number of Φ, then $\Phi_0 = \emptyset$, $w = 1$ and $J = \emptyset$. For arbitrary ℓ satisfying Assumption 1.5, explicit descriptions for elements $w \in W$ and subsets $J \subseteq \Pi$ satisfying Theorem 4.7 are given in [13, §§3.3–3.7] and [9, §§3.3,9.1].

Set $M = H^0(U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{b}), w \cdot 0)^* = \nabla_{\zeta,J}(w \cdot 0)^* \cong \nabla_{\zeta,J}(-w_{0,J}(w \cdot 0))$. The highest weight of M is $-w_{0,J}(w \cdot 0)$. According to [9, Lemma 4.1.1], the weight $w \cdot 0$ is a J-Steinberg weight. Then M is irreducible for $U_\zeta(\mathfrak{l}_J)$, and is irreducible, injective, and projective for $u_\zeta(\mathfrak{l}_J)$. (If $\operatorname{char}(k) = 0$, then M is even injective in the category of finite-dimensional modules for $U_\zeta(\mathfrak{l}_J)$; see [5, §9.10]. This property fails if $\operatorname{char}(k) > 0$.) We have $M = L_{\zeta,J}(w \cdot 0)^* \cong L_{\zeta,J}(-w_{0,J}(w \cdot 0))$.

The Ad_r-action of $U_\zeta(\mathfrak{p}_J)$ on $\mathcal{U}_\zeta(\mathfrak{u}_J)$ makes $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), k)$ a left $U_\zeta(\mathfrak{p}_J)$-module algebra. Together with the defining action of $U_\zeta(\mathfrak{p}_J)$ on M, we obtain an action of $U_\zeta(\mathfrak{p}_J)$ on $\operatorname{Hom}_k(M, H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), k))$ (i.e., the standard diagonal action of $U_\zeta(\mathfrak{p}_J)$ on $\operatorname{Hom}_k(M, H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), k))$; see Remark 2.13).

Given a Hopf algebra A, a subalgebra B, and A-modules V and W, there is, a priori, no reason to expect that the standard diagonal action of A on $\operatorname{Hom}_k(V, W)$

should stabilize the subspace $\mathrm{Hom}_B(V,W)$. If, however, the antipode S of A is bijective, and if B is a normal Hopf subalgebra of A, then the result does hold. Indeed, if the antipode of A is bijective, then by [5, Proposition 2.9] we have $\mathrm{Hom}_B(V,W) = \mathrm{Hom}_k(V,W)^B$, and $\mathrm{Hom}_k(V,W)^B$ is an A-submodule of $\mathrm{Hom}_k(V,W)$ by the normality of B in A. In the present situation, we conclude that, for each $n \in \mathbb{N}$, $\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^n(\mathcal{U}_\zeta(\mathfrak{u}_J),k))$ is a $U_\zeta(\mathfrak{l}_J)$-module on which $u_\zeta(\mathfrak{l}_J)$ acts trivially, hence a (finite-dimensional, integrable) $U_\zeta(\mathfrak{l}_J)//u_\zeta(\mathfrak{l}_J) \cong \mathrm{hy}(L_J)$-module. In what follows we will explicitly determine the $\mathrm{hy}(L_J)$-module structure of this set.

Proposition 4.8. [9, Proposition 4.2.1(a)] Let ℓ be as in Assumption 1.5. Assume moreover that $\ell \nmid n+1$ when Φ is of type A_n, and $\ell \neq 9$ when Φ is of type E_6. Choose $w \in W$ such that $w(\Phi_0) = \Phi_J$, and let $\nu \in X$. Suppose $\gamma := -w_{0,J}(w \cdot 0) + \ell\nu$ is a J-dominant weight in $\Lambda^i_{\zeta,J}$. Then $\nu = 0$ and $i = \ell(w)$.

Theorem 4.9. (cf. [9, Theorem 4.3.1]) Let ℓ be as in Assumption 1.5. Assume moreover that $\ell \nmid n+1$ when Φ is of type A_n, and $\ell \neq 9$ when Φ is of type E_6. Choose $w \in W$ such that $w(\Phi_0) = \Phi_J$. Then there exists an isomorphism of $U_\zeta(\mathfrak{l}_J)$-modules

$$\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^i(\mathcal{U}_\zeta(\mathfrak{u}_J),k)) \cong \begin{cases} 0 & i \neq \ell(w) \\ k & i = \ell(w) \end{cases}.$$

Proof. Fix $i \in \mathbb{N}$, and set $V = \mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^i(\mathcal{U}_\zeta(\mathfrak{u}_J),k))$. Suppose $V \neq 0$. As remarked above, the $U_\zeta(\mathfrak{l}_J)$-action on V factors through $U_\zeta(\mathfrak{l}_J)//u_\zeta(\mathfrak{l}_J) \cong \mathrm{hy}(L_J)$. Then any $U_\zeta(\mathfrak{l}_J)$-composition factor of V will have the form $L_J(\nu)^{[1]}$ for some J-dominant weight ν. Here $L_J(\nu)$ is the simple L_J-module of highest weight $\nu \in X_J^+$.

Suppose $L_J(\nu)^{[1]}$ occurs as a $U_\zeta(\mathfrak{l}_J)$-composition factor of V. Then, in particular, $\ell\nu$ occurs as a weight of U_ζ^0 in V. Since the $\ell\nu$-weight space of V is naturally isomorphic to the space of $u_\zeta(\mathfrak{l}_J)U_\zeta^0$-equivariant linear maps $L_{\zeta,J}(-w_{0,J}(w \cdot 0)) \otimes \ell\nu \to H^i(\mathcal{U}_\zeta(\mathfrak{u}_J),k)$, we conclude that $\gamma := -w_{0,J}(w \cdot 0) + \ell\nu$ must occur as a weight of U_ζ^0 in $H^i(\mathcal{U}_\zeta(\mathfrak{u}_J),k)$. By Propositions 4.6 and 4.8, the only possibilities for ν and i are $\nu = 0$ and $i = \ell(w)$. So $V = 0$ if $i \neq \ell(w)$.

Assume $i = \ell(w)$. The weight $-w_{0,J}(w \cdot 0)$ occurs with multiplicity one in $\Lambda^{\ell(w)}_{\zeta,J}$, hence it occurs in $H^{\ell(w)}(\mathcal{U}_\zeta(\mathfrak{u}_J),k)$ with multiplicity at most one by Proposition 4.6. Then either $V = 0$, or else $V \cong k$ as a $U_\zeta(\mathfrak{l}_J)$-module. The fact that $V \neq 0$ is a corollary of the proof of Theorem 4.17; we defer the details until then. □

Remark 4.10. In the case $\mathrm{char}(k) = 0$, Bendel et al. [9] prove $V = k$ by a calculation involving formal characters. Their argument relies on the fact that, if γ is a J-Steinberg weight, then the induced module $\nabla_{\zeta,J}(\gamma)$ is injective for $U_\zeta(\mathfrak{l}_J)$. This property fails if $\mathrm{char}(k) > 0$, so we rely on a more indirect argument to show $V \neq 0$.

4.2 Cohomology of the Frobenius–Lusztig kernel

Our first goal in this section is to verify the equivalent conditions of Lemma 2.1 for certain pairs of algebras, hence to show that the theory in Chapter 2 can be applied to the computation of cohomology for the Frobenius–Lusztig kernel $u_\zeta(\mathfrak{g})$.

Lemma 4.11. Fix $J \subseteq \Pi$. The algebra $u_\zeta(\mathfrak{p}_J)$ is free for both the left and right regular actions of the subalgebras $u_\zeta(\mathfrak{l}_J)$ and $u_\zeta(\mathfrak{u}_J)$ on $u_\zeta(\mathfrak{p}_J)$.

Proof. Multiplication in $u_\zeta(\mathfrak{p}_J)$ induces isomorphisms of vector spaces

$$u_\zeta(\mathfrak{l}_J) \otimes u_\zeta(\mathfrak{u}_J) \xrightarrow{\sim} u_\zeta(\mathfrak{p}_J) \xleftarrow{\sim} u_\zeta(\mathfrak{u}_J) \otimes u_\zeta(\mathfrak{l}_J).$$

Then $u_\zeta(\mathfrak{p}_J)$ is a free (left or right) module for $u_\zeta(\mathfrak{l}_J)$ (resp. $u_\zeta(\mathfrak{u}_J)$), with basis given by any basis for $u_\zeta(\mathfrak{u}_J)$ (resp. $u_\zeta(\mathfrak{l}_J)$). □

The next result generalizes [16, Theorem 1] to quantized enveloping algebras of arbitrary rank.

Proposition 4.12. Fix $J \subseteq \Pi$. The algebra $U_\zeta(\mathfrak{p}_J)$ is a smash product of the hyperalgebra $\mathrm{hy}(P_J)$ and the Frobenius–Lusztig kernel $u_\zeta(\mathfrak{p}_J)$. The algebra $U_\zeta(\mathfrak{p}_J)$ is a free (in particular, flat) module for both the left and right regular actions of $u_\zeta(\mathfrak{p}_J)$ on $U_\zeta(\mathfrak{p}_J)$.

Proof. We prove the proposition by showing that the pair $(A, B) = (U_\zeta(\mathfrak{p}_J), u_\zeta(\mathfrak{p}_J))$ satisfies the hypotheses of Theorem 2.3. The statement on the left freeness of $U_\zeta(\mathfrak{p}_J)$ for the action of $u_\zeta(\mathfrak{p}_J)$ will then follow from Theorem 2.3; the statement on the right freeness is obtained by applying the antipode S of $U_\zeta(\mathfrak{p}_J)$.

We follow the strategy in the proof of [41, Theorem 5.1(i)]. Set $\mathrm{St}_\ell = L_\zeta((\ell-1)\rho)$, the Steinberg module for $U_\zeta(\mathfrak{g})$. It is well-known that $\mathrm{St}_\ell \cong u_\zeta^-$ as u_ζ^--modules (the Steinberg module St_ℓ is simultaneously the projective cover and the injective hull of the trivial module for u_ζ^-, cf. [3, §2.1]). Set $\mathfrak{p} = \mathfrak{p}_J$. Define the functor coind from the category of u_ζ^--modules to the category of $u_\zeta(\mathfrak{p})$-modules by

$$\mathrm{coind}(V) = u_\zeta(\mathfrak{p}) \otimes_{u_\zeta^-} V$$

(i.e., coind is the tensor induction functor from u_ζ^- to $u_\zeta(\mathfrak{p})$.) Then $u_\zeta(\mathfrak{p}) \cong \mathrm{coind}(u_\zeta^-)$.

If u_ζ^- were a Hopf-subalgebra of $u_\zeta(\mathfrak{p})$, then the tensor identity for the tensor induction functor would yield the series of $u_\zeta(\mathfrak{p})$-module isomorphisms

$$u_\zeta(\mathfrak{p}) \cong \mathrm{coind}(u_\zeta^-) \cong \mathrm{coind}(k \otimes \mathrm{St}_\ell|_{u_\zeta^-}) \cong \mathrm{coind}(k) \otimes \mathrm{St}_\ell\,. \tag{4.2.1}$$

But u_ζ^- is not a Hopf-subalgebra of $u_\zeta(\mathfrak{p})$. Still, in this case, the usual maps giving the inverse isomorphisms of the tensor identity yield the isomorphisms of (4.2.1).

Specifically, the following linear maps are well-defined $u_\zeta(\mathfrak{p})$-module isomorphisms:

$$\varphi : \text{coind}(k \otimes \text{St}_\ell) \to \text{coind}(k) \otimes \text{St}_\ell \qquad h \otimes (v \otimes w) \mapsto \sum (h_{(1)} \otimes v) \otimes h_{(2)} w$$
$$\psi : \text{coind}(k) \otimes \text{St}_\ell \to \text{coind}(k \otimes \text{St}_\ell) \qquad (h \otimes v) \otimes w \mapsto \sum h_{(1)} \otimes (v \otimes S(h_{(2)})w)$$

Here $h \in u_\zeta(\mathfrak{p})$, $v \in k$, and $w \in \text{St}_\ell$. Since u_ζ^- is not a Hopf-subalgebra of $u_\zeta(\mathfrak{p})$, the well-definedness of φ and ψ is dependent on the first factor in $\text{coind}(k \otimes \text{St}_\ell)$ being the trivial module.

So now $u_\zeta(\mathfrak{p}) \cong \text{coind}(k) \otimes \text{St}_\ell$ as a left $u_\zeta(\mathfrak{p})$-module. Let $L_1, \ldots, L_r$ be the $u_\zeta(\mathfrak{p})$-composition factors for $\text{coind}(k)$. Then the left regular representation of $u_\zeta(\mathfrak{p})$ admits a filtration with quotients $L_1 \otimes \text{St}_\ell, \ldots, L_r \otimes \text{St}_\ell$. Since St_ℓ is projective for $u_\zeta(\mathfrak{p})$, so is each $L_i \otimes \text{St}_\ell$, hence there exists an isomorphism of left $u_\zeta(\mathfrak{p})$-modules

$$u_\zeta(\mathfrak{p}) \cong (L_1 \oplus \cdots \oplus L_r) \otimes \text{St}_\ell\,.$$

Since each L_i can be lifted to a simple $U_\zeta(\mathfrak{p})$-module, we have $L_1 \oplus \cdots \oplus L_r \cong V|_{u_\zeta(\mathfrak{p})}$ for some completely reducible $U_\zeta(\mathfrak{p})$-module V. Then as a left $u_\zeta(\mathfrak{p})$-module, $u_\zeta(\mathfrak{p}) \cong (V \otimes \text{St}_\ell)|_{u_\zeta(\mathfrak{p})}$, the restriction to $u_\zeta(\mathfrak{p})$ of a $U_\zeta(\mathfrak{p})$-module. □

Remark 4.13. Given the results of Lemma 4.11 and Proposition 4.12, the results of Chapter 2 can now be applied to the pairs $(A, B) = (U_\zeta(\mathfrak{p}_J), u_\zeta(\mathfrak{p}_J))$, $(u_\zeta(\mathfrak{p}_J), u_\zeta(\mathfrak{u}_J))$, etc. In particular, given such a pair, and given an A-module V, there exists a natural isomorphism $R^n(-^B)(V) \cong H^n(B, V)$. (If $\text{char}(k) = 0$, and if we restrict our attention to finite-dimensional modules, then we can bypass Proposition 4.12 and note that the conclusion of Lemma 2.1 holds for (say) the pair $(U_\zeta(\mathfrak{g}), u_\zeta(\mathfrak{g}))$ by [5, Theorem 9.12]. This approach fails in general, for if $\text{char}(k) > 0$, then St_ℓ is not injective in the category of all finite-dimensional $U_\zeta(\mathfrak{g})$-modules.)

Theorem 4.14. (cf. [9, Theorem 5.1.1]) Let ℓ be as in Assumption 1.5, and let $w \in W$ be such that $w(\Phi_0) = \Phi_J$. Then there exists a first quadrant spectral sequence of G-modules satisfying

$$E_2^{i,j} = R^i \operatorname{ind}_{P_J}^G H^j(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \Rightarrow H^{i+j-\ell(w)}(u_\zeta(\mathfrak{g}), k). \tag{4.2.2}$$

Proof. The proof is essentially the same as that of [9, Theorem 5.1.1]. Set $\mathfrak{p} = \mathfrak{p}_J$, and set $P = P_J$. Define functors $\mathcal{F}_1, \mathcal{F}_2$ from the category of integrable $U_\zeta(\mathfrak{p})$-modules to the category of rational G-modules by setting

$$\mathcal{F}_1(-) = (-)^{u_\zeta(\mathfrak{g})} \circ H^0(U_\zeta/U_\zeta(\mathfrak{p}), -),$$
$$\mathcal{F}_2(-) = \operatorname{ind}_P^G(-) \circ (-)^{u_\zeta(\mathfrak{p})}.$$

The functors $\mathcal{F}_1, \mathcal{F}_2$ are naturally equivalent because they are both right adjoint to the functor $\mathcal{G}(-) = (-)^{[1]}|_{U_\zeta(\mathfrak{p})}$ from the category of rational G-modules to the category of integrable $U_\zeta(\mathfrak{p})$-modules.

We make some preliminary observations. First, the functor $(-)^{u_\zeta(\mathfrak{p})}$ is right adjoint to the (exact) forgetful functor $(-)^{[1]}$ from the category of rational P-modules to the category of integrable $U_\zeta(\mathfrak{p})$-modules, hence maps injective integrable modules for $U_\zeta(\mathfrak{p})$ to injective rational P-modules. Second, the induction functor $H^0(U_\zeta/U_\zeta(\mathfrak{p}), -)$ maps injective integrable modules for $U_\zeta(\mathfrak{p})$ to injective integrable modules for $U_\zeta(\mathfrak{g})$. Third, injective integrable modules for $U_\zeta(\mathfrak{g})$ (resp. $U_\zeta(\mathfrak{p})$) remain injective when considered as objects in the category of all $u_\zeta(\mathfrak{g})$-modules (resp. $u_\zeta(\mathfrak{p})$-modules). This last observation follows by a standard argument (see, e.g., [18, Proposition 2.1] or [55, Theorem 2.9.1]) and from the fact that the induction functors $H^0(U_\zeta/u_\zeta(\mathfrak{g}), -)$ and $H^0(U_\zeta(\mathfrak{p})/u_\zeta(\mathfrak{p}), -)$ are exact [3, Corollary 2.3].

In the definition of the functor $\mathcal{F}_1$, we are considering $(-)^{u_\zeta(\mathfrak{g})}$ as a functor from the category of integrable $U_\zeta(\mathfrak{g})$-modules to the category of rational G-modules (equivalently, the category of integrable $U_\zeta(\mathfrak{g})//u_\zeta(\mathfrak{g})$-modules). The right derived functors $R^i(-^{u_\zeta(\mathfrak{g})})$ are defined in terms of injective resolutions by integrable $U_\zeta(\mathfrak{g})$-modules. Because (and only because) we have observed that injective integrable modules for $U_\zeta(\mathfrak{g})$ restrict to injective modules for $u_\zeta(\mathfrak{g})$, we can identify the right derived functors $R^i(-^{u_\zeta(\mathfrak{g})})$ with the cohomology functors $H^i(u_\zeta(\mathfrak{g}), -)$. Similarly, we can identify the right derived functors of $(-)^{u_\zeta(\mathfrak{p})}$ with the cohomology functors $H^i(u_\zeta(\mathfrak{p}), -)$.

The observations of the above paragraphs imply for any integrable $U_\zeta(\mathfrak{p})$-module M the existence of spectral sequences

$$\begin{aligned} {}'E_2^{i,j} &= H^i(u_\zeta(\mathfrak{g}), H^j(U_\zeta/U_\zeta(\mathfrak{p}), M)) \Rightarrow (R^{i+j}\mathcal{F}_1)(M), \quad \text{and} \\ E_2^{i,j} &= R^i \operatorname{ind}_P^G H^j(u_\zeta(\mathfrak{p}), M) \Rightarrow (R^{i+j}\mathcal{F}_2)(M), \end{aligned}$$

necessarily converging to the same abutment. There also exists a spectral sequence as follows (cf. [37, I.4.5]):

$$R^i \operatorname{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})} R^j \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 \Rightarrow R^{i+j} \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})} w \cdot 0. \tag{4.2.3}$$

Since $w \cdot 0$ is J-dominant by [9, Lemma 4.1.1], we have $R^j \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 = 0$ for $j > 0$ by Kempf's vanishing theorem [57, Corollary 5.5]. (The version of Kempf's vanishing theorem proved in [57] admits the possibility $\operatorname{char}(k) > 0$.) Then the spectral sequence (4.2.3) collapses, and yields by [1, Corollary 3.8] that

$$R^i \operatorname{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})} \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 = \begin{cases} k & i = \ell(w) \\ 0 & i \neq \ell(w) \end{cases}$$

Then, taking $M = \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0$, the first spectral sequence ${}'E_2^{i,j}$ collapses to yield $(R^{\bullet+\ell(w)}\mathcal{F}_1)(\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \cong H^\bullet(u_\zeta(\mathfrak{g}), k)$. Combining this with the second spectral sequence $E_2^{i,j}$ yields the spectral sequence (4.2.2) of the theorem. □

Consider the spectral sequence of Theorem 2.24 with $A = u_\zeta(\mathfrak{p}_J)$, $B = u_\zeta(\mathfrak{u}_J)$, $H = U_\zeta(\mathfrak{p}_J)$, and $V = \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0$. We have

$$E_2^{i,j} = H^i(u_\zeta(\mathfrak{l}_J), H^j(u_\zeta(\mathfrak{u}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)) \Rightarrow H^{i+j}(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0). \quad (4.2.4)$$

The action of $U_\zeta(\mathfrak{u}_J)$ (in particular, the action of $u_\zeta(\mathfrak{u}_J)$) on V is trivial, so there exists an isomorphism of $U_\zeta(\mathfrak{p}_J)$-modules $H^\bullet(u_\zeta(\mathfrak{u}_J), V) \cong V \otimes H^\bullet(u_\zeta(\mathfrak{u}_J), k)$. Contrary to the notation of [9], to preserve the $U_\zeta(\mathfrak{p}_J)$-module structure, V must appear on the left hand side of the above tensor product, cf. the remark in the proof of Lemma 4.1. We may thus reidentify the E_2 term of (4.2.4) as

$$E_2^{i,j} = H^i(u_\zeta(\mathfrak{l}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 \otimes H^j(u_\zeta(\mathfrak{u}_J), k)) \Rightarrow H^{i+j}(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0). \quad (4.2.5)$$

We will use this form of (4.2.4) to describe the structure of $H^\bullet(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)$. First we make some preliminary observations.

The standard diagonal action of $U_\zeta(\mathfrak{g})$ on $V^* = \operatorname{Hom}_k(V, k)$ makes V^* a $U_\zeta(\mathfrak{g})$-module: If $f \in V^*$, $v \in V$, and $h \in U_\zeta(\mathfrak{g})$, then $(h.f)(v) = f(S(h).v)$. With this action of $U_\zeta(\mathfrak{g})$ on V^*, the contraction map $V^* \otimes V \xrightarrow{c} k$, $f \otimes v \mapsto f(v)$, is a $U_\zeta(\mathfrak{g})$-module homomorphism. The map $\tau : k \to V \otimes V^*$, defined to be the composition of the inclusion $k \hookrightarrow \operatorname{End}_k(V)$, $a \mapsto a \cdot \mathrm{id}_V$, with the canonical isomorphism $\operatorname{End}_k(V) \cong V \otimes V^*$, is also a homomorphism of $U_\zeta(\mathfrak{g})$-modules. In particular, c and τ are $u_\zeta(\mathfrak{l}_J)$-module homomorphisms. If W is an arbitrary $u_\zeta(\mathfrak{l}_J)$-module, it follows then that there exists a natural vector space isomorphism (cf. [5, Proposition 1.18]):

$$\operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(V^*, W) \cong \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(k, V \otimes W).$$

Now take $W = H^\bullet(u_\zeta(\mathfrak{u}_J), k)$. It is trivial as module for $u_\zeta(\mathfrak{u}_J)$ by Example 2.16. The action of $u_\zeta(\mathfrak{u}_J)$ on $V^* \cong (\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*$ is also trivial. Then

$$\operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(V^*, W) = \operatorname{Hom}_{u_\zeta(\mathfrak{p}_J)}(V^*, W).$$

By the comments made prior to Proposition 4.8, we conclude that $\operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(V^*, W)$ is a submodule for the standard diagonal action of $U_\zeta(\mathfrak{p}_J)$ on $\operatorname{Hom}_k(V^*, W)$. The action of $U_\zeta(\mathfrak{p}_J)$ factors through the quotient $\mathrm{hy}(P_J) = U_\zeta(\mathfrak{p}_J)//u_\zeta(\mathfrak{p}_J)$.

The "nilpotent" algebra $\mathbb{U}_k(\mathfrak{u}_J^+)$ is a left coideal subalgebra of $\mathbb{U}_k(\mathfrak{g})$, that is, it is a subalgebra of $\mathbb{U}_k(\mathfrak{g})$ and $\Delta(\mathbb{U}_k(\mathfrak{u}_J^+)) \subseteq \mathbb{U}_k(\mathfrak{g}) \otimes \mathbb{U}_k(\mathfrak{u}_J^+)$. Indeed, let $I \subset \mathbb{U}_k(\mathfrak{g})$ be the subalgebra generated by $\{E_\alpha : \alpha \in \Pi \backslash J\}$. It is a left coideal subalgebra. (This is clear from (1.1.7).) Let I' denote the subalgebra of $\mathbb{U}_k(\mathfrak{g})$ generated by $\operatorname{Ad}(\mathbb{U}_k(\mathfrak{l}_J^+))(I)$. It is a left coideal by [45, Lemma 1.2], and $I' \subset \mathbb{U}_k(\mathfrak{u}_J)$ by Proposition 3.10. Let $\mathfrak{n}_J^+ = \mathfrak{l}_J^+ \cap \mathfrak{u}^+$. Then $\mathbb{U}_k(\mathfrak{n}_J^+)$ is the subalgebra of $\mathbb{U}_k(\mathfrak{u}^+)$ generated by $\{E_\alpha : \alpha \in J\}$, and multiplication induces an isomorphism of vector spaces $\mathbb{U}_k(\mathfrak{l}_J^+) \cong \mathbb{U}_k(\mathfrak{n}_J^+) \otimes \mathbb{U}_k^0$. Since $\operatorname{Ad}(\mathbb{U}_k^0)(I) = I$, we have $I' = \operatorname{Ad}(\mathbb{U}_k(\mathfrak{n}_J^+))(I)$. To prove the equality $I' = \mathbb{U}_k(\mathfrak{u}_J^+)$, we first consider the smash product $I'\#\mathbb{U}_k(\mathfrak{n}_J^+)$. As a vector space, $I'\#\mathbb{U}_k(\mathfrak{n}_J^+) = I' \otimes \mathbb{U}_k(\mathfrak{n}_J^+)$, and the multiplication is given by $(x \otimes y)(x' \otimes y') = \sum x \operatorname{Ad}(y_{(1)})(x') \otimes y_{(2)} y'$ for $x, x' \in I'$ and $y, y' \in \mathbb{U}_k(\mathfrak{n}_J^+)$. Now there exists a natural algebra homomorphism $\mu : I'\#\mathbb{U}_k(\mathfrak{n}_J^+) \to \mathbb{U}_k(\mathfrak{u}^+)$, $x \otimes y \mapsto xy$, which is clearly surjective because its image contains a set of generators for $\mathbb{U}_k(\mathfrak{u}^+)$. The map μ is injective because of the triangular decomposition $\mathbb{U}_k(\mathfrak{u}^+) \cong \mathbb{U}_k(\mathfrak{u}_J^+) \otimes \mathbb{U}_k(\mathfrak{n}_J^+)$. So μ is an isomorphism. By comparing the dimensions of weight spaces in $\mathbb{U}_k(\mathfrak{u}^+)$ and $I'\#\mathbb{U}_k(\mathfrak{n}_J^+)$, we then conclude that $I' = \mathbb{U}_k(\mathfrak{u}_J^+)$.

Applying the identities $\Delta \circ \kappa = (\kappa \otimes \kappa) \circ T \circ \Delta$ (where $T : a \otimes b \mapsto b \otimes a$ is the transposition) and $\kappa(\mathbb{U}_k(\mathfrak{u}_J^+)) = \mathbb{U}_k(\mathfrak{u}_J)$, we conclude that $\mathbb{U}_k(\mathfrak{u}_J)$ is a right coideal subalgebra of $\mathbb{U}_k(\mathfrak{g})$ (i.e., it is a subalgebra and $\Delta(\mathbb{U}_k(\mathfrak{u}_J)) \subseteq \mathbb{U}_k(\mathfrak{u}_J) \otimes \mathbb{U}_k(\mathfrak{g})$). It follows that $u_\zeta(\mathfrak{u}_J)$ is a right coideal subalgebra of $u_\zeta(\mathfrak{g})$. Moreover, given $\gamma \in \Phi^+ \backslash \Phi_J^+$, we have $\Delta(F_\gamma) \in 1 \otimes F_\gamma + u_\zeta(\mathfrak{u}_J)_\varepsilon \otimes u_\zeta(\mathfrak{g})$. As usual, $u_\zeta(\mathfrak{u}_J)_\varepsilon$ denotes the augmentation ideal of $u_\zeta(\mathfrak{u}_J)$. This implies that the $U_\zeta(\mathfrak{p}_J)$-module $V \otimes W$ is trivial as a module for $u_\zeta(\mathfrak{u}_J)$, hence that $\operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(k, V \otimes W) = \operatorname{Hom}_{u_\zeta(\mathfrak{p}_J)}(k, V \otimes W)$. Then $\operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(k, V \otimes W)$ is also module for the diagonal action of $U_\zeta(\mathfrak{p}_J)$, and the action factors through the quotient $\operatorname{hy}(P_J) = U_\zeta(\mathfrak{p}_J)//u_\zeta(\mathfrak{p}_J)$.

Lemma 4.15. The vector space isomorphism

$$\operatorname{Hom}_{u_\zeta(\mathfrak{p}_J)}(V^*, W) \cong \operatorname{Hom}_{u_\zeta(\mathfrak{p}_J)}(k, V \otimes W) \tag{4.2.6}$$

is an isomorphism of $\operatorname{hy}(P_J)$-modules.

Proof. The explicit isomorphism is given by

$$\varphi : \operatorname{Hom}_{u_\zeta(\mathfrak{p}_J)}(k, V \otimes W) \to \operatorname{Hom}_{u_\zeta(\mathfrak{p}_J)}(V^*, W),$$

where if $g \in V^*$ and $f \in \operatorname{Hom}_{u_\zeta(\mathfrak{p}_J)}(k, V \otimes W)$, then $\varphi(f)(g) = (g \otimes \operatorname{id}_W) \circ f(1)$. The lemma now follows by a straight-forward calculation, using the fact the coproduct on $\operatorname{hy}(P_J)$ (which is induced by the coproduct on $U_\zeta(\mathfrak{p}_J)$) is cocommutative, and the fact that antipode S of $\operatorname{hy}(P_J)$ (which is induced by the antipode of $U_\zeta(\mathfrak{p}_J)$) squares to the identity. ◻

Proposition 4.16. (cf. [9, Proposition 5.2.1]) Let $w \in W$ (once again the Weyl group of Φ) and $J \subseteq \Pi$ be such that $w(\Phi_0) = \Phi_J$. Then for each $j \geq 0$, there exists an isomorphism of $\operatorname{hy}(P_J)$-modules

$$H^j(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \cong \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}((\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^j(u_\zeta(\mathfrak{u}_J), k). \tag{4.2.7}$$

Proof. Since $V = \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0$ is injective for $u_\zeta(\mathfrak{l}_J)$, in the spectral sequence (4.2.5) we have $E_2^{i,j} = 0$ for all $i > 0$. Then (4.2.5) collapses at the E_2-page, yielding the $U_\zeta(\mathfrak{p}_J)$-module isomorphism

$$H^j(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \cong \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(k, V \otimes H^j(u_\zeta(\mathfrak{u}_J), k).$$

Application of Lemma 4.15 then gives (4.2.7). ◻

Theorem 4.17. (cf. [9, Theorem 5.3.1]) Let ℓ be as in Assumption 1.5. Assume moreover that $\ell \nmid n+1$ when Φ is of type A_n, and $\ell \neq 9$ when Φ is of type E_6. Choose $w \in W$ such that $w(\Phi_0) = \Phi_J$. Then there exists a U_ζ^0-module isomorphism

$$H^s(u_\zeta(\mathfrak{p}_J), \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \cong S^{\frac{s-\ell(w)}{2}}(\mathfrak{u}_J^*)^{[1]}. \tag{4.2.8}$$

Proof. The proof follows the same strategy as the proof of [9, Theorem 5.3.1], namely, investigating the U_ζ^0-module structure of the Hom-set on the right side of the isomorphism (4.2.7). We will also show how the possibility $V = 0$ in the proof of Theorem 4.9 would lead a contradiction.

First recall the notation from [9, §5.3]: $M = (\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*$, and $\mathcal{G}(-)$ is the functor on the category of $U_\zeta(\mathfrak{p}_J)$-modules defined by $\mathcal{G}(-) = \mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, -)$. The module M is projective for $u_\zeta(\mathfrak{l}_J)$, so $\mathcal{G}$ is exact. We have $N = \left|\Phi^+ \backslash \Phi_J^+\right|$, and $f_1, \ldots, f_N$ is an arbitrary fixed ordering of the root vectors in $\mathcal{U}_\zeta(\mathfrak{u}_J)$ corresponding to the elements $\{\gamma_1, \ldots, \gamma_N\}$ of $\Phi^+ \backslash \Phi_J^+$. (The notation here is inconsistent with that of §1.2, but the reader should have no trouble making the readjustment for the remainder of the section.) For each $1 \leq i \leq N$, f_i^ℓ is central in $\mathcal{U}_\zeta(\mathfrak{u}_J)$ by Lemma 3.8, and for each $0 \leq i \leq N$, the algebra A_i is defined by $A_i = \mathcal{U}_\zeta(\mathfrak{u}_J)//\langle f_1^\ell, \ldots, f_i^\ell \rangle$, where $\langle \cdots \rangle$ denotes "the subalgebra generated by $\cdots$." Then $A_0 = \mathcal{U}_\zeta(\mathfrak{u}_J)$, and $A_N = u_\zeta(\mathfrak{u}_J)$. For $1 \leq i \leq N$, we have $B_i = \langle f_i^\ell \rangle \subset A_{i-1}$, a normal central subalgebra of A_{i-1} isomorphic to a polynomial algebra in one variable and over which A_{i-1} is free. Also, $A_{i-1}//B_i \cong A_i$. Finally, for $0 \leq i \leq N$, V_i denotes the i-dimensional vector space with basis $\{x_1, \ldots, x_i\}$. The vector space V_i is given the structure of a U_ζ^0-module by assigning x_i to have weight γ_i. Then $V_0 = \{0\}$, and $V_N \cong \mathfrak{u}_J^*$ as U_ζ^0-modules.

It follows from Corollary 3.19 that, for each $1 \leq i \leq N$, the algebra A_i is a right $u_\zeta(\mathfrak{l}_J)U_\zeta^0$-module algebra, and $B_i \subset A_i$ is a right $u_\zeta(\mathfrak{l}_J)U_\zeta^0$-submodule of A_i. Then for each $1 \leq i \leq N$, there exists by Theorem 2.24 a spectral sequence of $u_\zeta(\mathfrak{l}_J)U_\zeta^0$-modules satisfying

$$'E_2^{a,b} = H^a(A_i, H^b(B_i, k)) \Rightarrow H^{a+b}(A_{i-1}, k)$$

Since B_i is central in A_{i-1}, the action of $A_i = A_{i-1}//B_i$ on $H^\bullet(B_i, k)$ is trivial by Corollary 2.20, so we may rewrite the spectral sequence as

$$'E_2^{a,b} = H^b(B_i, k) \otimes H^a(A_i, k) \Rightarrow H^{a+b}(A_{i-1}, k) \tag{4.2.9}$$

(Once again, the factor $H^b(B_i, k)$ should go on the left side of the tensor product.) The functor $\mathcal{G}$ is exact on the category of $u_\zeta(\mathfrak{l}_J)$-modules; applying it to the spectral sequence (4.2.9), and using the fact that the adjoint action of $u_\zeta(\mathfrak{l}_J)$ on $H^\bullet(B_i, k)$ is trivial (cf. Corollary 3.19(b)), we obtain the new spectral sequence of U_ζ^0-modules:

$$E_2^{a,b} = H^b(B_i, k) \otimes \mathcal{G}\left(H^a(A_i, k)\right) \Rightarrow \mathcal{G}\left(H^{a+b}(A_{i-1}, k)\right). \tag{4.2.10}$$

Assuming the validity of Theorem 4.9, Bendel et al. use (4.2.10) in order to prove, by induction on i, that for each $0 \le i \le N$, there exists a U_ζ^0-module isomorphism

$$\mathcal{G}\left(H^s(A_i,k)\right) \cong \begin{cases} S^r(V_i)^{[1]} & \text{if } s = 2r + \ell(w) \\ 0 & \text{else.} \end{cases} \tag{4.2.11}$$

We sketch their proof of the claim, and also indicate how $V = 0$ in the proof of Theorem 4.9 would lead to the conclusion $\mathcal{G}(H^\bullet(A_i,k)) = 0$ for all $0 \le i \le N$.

First observe the following U_ζ^0-module isomorphism (cf. Lemma 4.4):

$$H^b(B_i,k) = \begin{cases} k & \text{if } b = 0 \\ k_i^{[1]} & \text{if } b = 1 \\ 0 & \text{else.} \end{cases}$$

Here k_i denotes the one-dimensional U_ζ^0-module of weight γ_i. Then $E_2^{a,b} = 0$ if $b \ge 2$, and the only possible non-zero differentials in (4.2.10) have the form $d_2 : E_2^{a,1} \to E_2^{a+2,0}$. Also, $E_2^{a,1} \cong E_2^{a,0} \otimes k_i^{[1]}$, so $E_2^{a,1} \neq 0$ if and only if $E_2^{a,0} \neq 0$.

First suppose $V = 0$ in the proof of Theorem 4.9, and suppose by way of induction that, for fixed $1 \le i \le N$, $\mathcal{G}(H^\bullet(A_j,k)) = 0$ for all $0 \le j \le i-1$. (The assumption $V = 0$ is equivalent to the base case $i = 1$.) Then $E_2^{0,0} = E_\infty^{0,0} \cong \mathcal{G}(H^0(A_{i-1},k)) = 0$, so $\mathcal{G}(H^0(A_i,k)) = 0$. Also, $E_2^{1,0} = 0$, so $\mathcal{G}(H^1(A_i,k)) = 0$. Now suppose, by induction, that $\mathcal{G}(H^j(A_i,k)) = 0$ for all $0 \le j \le a$, hence that $E_2^{j,b} = 0$ for all $0 \le j \le a$. Then $E_2^{a,1} = E_\infty^{a,1} = 0$ and $E_2^{a-1,1} = 0$, hence

$$E_2^{a+1,0} = E_2^{a+1,0}/d_2(E_2^{a-1,1}) = E_\infty^{a+1,0} \cong \mathcal{G}(H^{a+1}(A_{i-1},k)) = 0.$$

It follows that $\mathcal{G}(H^{a+1}(A_i,k)) = 0$. By induction, we conclude that $\mathcal{G}(H^\bullet(A_i,k)) = 0$, and then by induction on i we conclude that $\mathcal{G}(H^\bullet(A_i,k)) = 0$ for all $0 \le i \le N$. In particular,

$$0 = \mathcal{G}(H^\bullet(A_N,k)) = \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^\bullet(u_\zeta(\mathfrak{u}_J),k)) \cong H^\bullet(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0).$$

But then every term of the E_2-page of (4.2.2) is zero, which implies by Theorem 4.14 that $H^\bullet(u_\zeta(\mathfrak{g}),k) = 0$. This is an obvious contradiction, since $H^0(u_\zeta(\mathfrak{g}),k) = k \neq 0$, so we must conclude that $V \neq 0$ in the proof of Theorem 4.9.

Having established the validity of Theorem 4.9, we now summarize the proof of the claim (4.2.11). The argument is by induction on i: the base case $i = 0$ is Theorem 4.9, while the case $i = N$ is equivalent to the statement of Theorem 4.17.

Assume (4.2.11) is true for $i - 1$. Set

$$A = \min\left\{a \in \mathbb{N} : E_2^{a,0} \neq 0\right\} = \min\left\{a \in \mathbb{N} : E_2^{a,1} \neq 0\right\}.$$

Then $E_\infty^{A,0} \cong E_2^{A,0}/d_2(E_2^{A-1,1}) = E_2^{A,0} \neq 0$, so by induction, $A = \ell(w)$. In particular, $\mathcal{G}(H^a(A_i,k)) = 0$ for $a < \ell(w)$. Then

$$E_2^{\ell(w)+1,0} = E_2^{\ell(w)+1,0}/d_2(E_2^{\ell(w)-1,1}) \cong E_\infty^{\ell(w)+1,0} \subset \mathcal{G}(H^{\ell(w)+1}(A_{i-1},k)) = 0,$$

and by induction, $E_2^{\ell(w)+a,0} = 0 = E_2^{\ell(w)+a,1}$ for all odd $a > 0$.

Now, for even $a \geq 0$, we have

$$\ker(d_2^{\ell(w)+a,1}) \subseteq E_\infty^{\ell(w)+a,1} \subseteq \mathcal{G}(H^{\ell(w)+a+1}(A_{i-1},k)) = 0,$$

so $d_2 : E_2^{\ell(w)+a,1} \to E_2^{\ell(w)+a+2,0}$ is an injective U_ζ^0-module homomorphism. Then, for all even $a \geq 0$, we have the U_ζ^0-module isomorphisms

$$E_\infty^{\ell(w)+a,0} \cong \mathcal{G}(H^{\ell(w)+a}(A_{i-1},k)) = S^{a/2}(V_{i-1})^{[1]}.$$

It follows that there exists a short exact sequence of U_ζ^0-modules

$$0 \to E_2^{\ell(w)+a-2,1} \to E_2^{\ell(w)+a,0} \to S^{a/2}(V_{i-1})^{[1]} \to 0.$$

By induction on $a = 2r$, we can rewrite the short exact sequence as

$$0 \to k_i^{[1]} \otimes S^{r-1}(V_i)^{[1]} \to \mathcal{G}(H^{\ell(w)+2r}(A_i,k)) \to S^r(V_{i-1})^{[1]} \to 0.$$

Then, as a U_ζ^0-module,

$$\mathcal{G}(H^{\ell(w)+2r}(A_i,k)) \cong \left(k_i^{[1]} \otimes S^{r-1}(V_i)^{[1]}\right) \oplus S^r(V_{i-1})^{[1]}.$$

The left side of the direct sum identifies with the space of all degree-r homogeneous polynomials in the x_j ($1 \leq j \leq i$) containing at least one factor of x_i, and the right side of the direct sum identifies with all degree-r homogeneous polynomials in x_j with $1 \leq j \leq i-1$. So, as a U_ζ^0-module, $\mathcal{G}(H^{\ell(w)+2r}(A_i,k)) \cong S^r(V_i)^{[1]}$. This proves (4.2.11) for $s = 2r + \ell(w)$, and by induction we obtain it for all $s \geq 0$.

Now (4.2.11) holds for i, and by induction we obtain it for all $0 \leq i \leq N$. Application of Proposition 4.16 shows that the case $i = N$ is equivalent to the statement of the theorem. This completes the proof. □

Proposition 4.18. (cf. [9, Lemma 5.4.1]) The isomorphism (4.2.8) of Theorem 4.17 extends to an isomorphism of $U_\zeta(\mathfrak{p}_J)$-modules, where the action of $U_\zeta(\mathfrak{p}_J)$ on $S^\bullet(\mathfrak{u}_J^*)^{[1]}$ is that induced by the coadjoint action of $\mathrm{hy}(P_J)$ on $S^\bullet(\mathfrak{u}_J^*)$.

Proof. We follow the same strategy used by Bendel et al. to prove [9, Lemma 5.4.1], namely, we show by induction on s that the $U_\zeta(\mathfrak{p}_J)$-module structure on the right side of the isomorphism (4.2.8) is induced by the coadjoint action of $\mathrm{hy}(P_J)$ on $S^\bullet(\mathfrak{u}_J^*)$. The authors of [9] are not explicit about the details of their induction argument; we fill in the details here, and obtain as a byproduct new information on the cohomology algebra $H^\bullet(u_\zeta(\mathfrak{u}_J),k)$. Retain the notations $M = (\operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*$ and $\mathcal{G}(-) = \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(M,-)$ from the proof of Theorem 4.17.

Consider the spectral sequence of $U_\zeta(\mathfrak{p}_J)$-modules from Lemma 4.1:

$$'E_2^{a,b} = H^b(Z_J,k) \otimes H^a(u_\zeta(\mathfrak{u}_J),k) \Rightarrow H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}_J),k). \tag{4.2.12}$$

It is a spectral sequence of algebras. In particular, the differential d_2 is a derivation: if $x \in H^b(Z_J,k)$ and $y \in H^a(u_\zeta(\mathfrak{u}_J),k)$, then $d_2(xy) = d_2(x)y + (-1)^b x.d_2(y) = d_2(x)y$. (We have $'E_2^{a,0} \subseteq \ker(d_2)$ because (4.2.12) is a first quadrant spectral sequence.) Evidently, the differential d_2 is completely determined by its restriction to the column $'E_2^{0,\bullet} = H^\bullet(Z_J,k)$. Also, the E_2-page of (4.2.12) is a module for the algebra $H^\bullet(Z_J,k)$. The algebra structure of (4.2.12) makes $'E_2^{\bullet,\bullet}$ a $U_\zeta(\mathfrak{p}_J)$-module algebra. Of course, the $U_\zeta(\mathfrak{p}_J)$-module algebra structure of $H^\bullet(Z_J,k)$ is known by Proposition 3.20: $H^\bullet(Z_J,k) \cong \Lambda^\bullet(\mathfrak{u}_J^*)^{[1]}$, with $\operatorname{hy}(P_J)$ acting on $S^\bullet(\mathfrak{u}_J^*)$ via the coadjoint action.

The functor $\mathcal{G}$ is exact, so applying it to (4.2.12) yields the new spectral sequence

$$E_2^{a,b} = \mathcal{G}('E_2^{a,b}) \Rightarrow \mathcal{G}(H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}_J),k)). \tag{4.2.13}$$

The $U_\zeta(\mathfrak{p}_J)$-modules M and $'E_2^{a,b}$ are both trivial for $u_\zeta(\mathfrak{u}_J)$ (cf. Example 2.16), so $E_2^{a,b} = \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(M,'E_2^{a,b}) = \operatorname{Hom}_{u_\zeta(\mathfrak{p}_J)}(M,'E_2^{a,b})$. Then $E_2^{a,b}$ is naturally a $U_\zeta(\mathfrak{p}_J)$-module (cf. the comments made prior to Proposition 4.8), and the action of $u_\zeta(\mathfrak{p}_J)$ on $E_2^{a,b}$ is trivial. Since the differentials of (4.2.12) are $U_\zeta(\mathfrak{p}_J)$-module homomorphisms, it follows that the E_2 page of (4.2.13) is a complex of $U_\zeta(\mathfrak{p}_J)$-modules and $U_\zeta(\mathfrak{p}_J)$-module homomorphisms.

The adjoint action of $u_\zeta(\mathfrak{p}_J)$ on $H^\bullet(Z_J,k)$ is trivial, so we can rewrite $E_2^{a,b}$ as

$$E_2^{a,b} = H^b(Z_J,k) \otimes \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^a(u_\zeta(\mathfrak{u}_J),k)). \tag{4.2.14}$$

(The factor $H^b(Z_J,k)$ goes on the left to preserve the $U_\zeta(\mathfrak{p}_J)$-module structure.) The E_2-page of (4.2.13) also retains the structure of a left module for $H^\bullet(Z_J,k)$. More generally, let $\varphi \in \mathcal{G}('E_2^{a,b}) = \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(M,'E_2^{a,b})$, and let $z \in ('E_2^{c,d})^{u_\zeta(\mathfrak{l}_J)}$. (For example, take $z \in d_2('E_2^{0,\bullet})$.) Let $L_z : 'E_2^{a,b} \to 'E_2^{a,b}$ denote the corresponding left multiplication operator. Then $L_z \circ \varphi \in \mathcal{G}('E_2^{a+c,b+d}) = \operatorname{Hom}_{u_\zeta(\mathfrak{l}_J)}(M,'E_2^{a+c,b+d})$. If $z \in H^b(Z_J,k)$, then this is just a restatement of the fact that (4.2.14) is equivalent to (4.2.13). It is clear that the action of z on $E_2^{a,b}$ is compatible (in the sense of §2.1) with the left $U_\zeta(\mathfrak{p}_J)$-module structure of $E_2^{a,b}$. It also follows that if $b=0$ (so that $\varphi \in E_2^{a,0}$), then $d_2(z\varphi) = d_2(z)\varphi$.

By Theorem 4.9, $E_\infty^{a,b}$ is nonzero only if $a+b = \ell(w)$, in which case it is the trivial module k. Then $E_\infty^{\ell(w)+a,0} = 0$ for all $a \geq 1$. Also, by Theorem 4.17, $E_2^{a,\bullet} = 0$ for $a < \ell(w)$, and $E_2^{\ell(w)+a,\bullet} = 0$ for all odd a. Then the equalities $E_\infty^{\ell(w),1} = 0$ and $E_\infty^{\ell(w)+2,0} = 0$ imply that the differential $d_2 : E_2^{\ell(w),1} \to E_2^{\ell(w)+2,0}$ is an isomorphism.

Let $\{x_1, \dots, x_N\}$ be a basis for $V_N \cong \mathfrak{u}_J^*$ as in the proof of Theorem 4.17, and let $\{z_1, \dots, z_n\}$ denote the corresponding generators for the algebra $H^\bullet(Z_J, k) \cong \Lambda^\bullet(\mathfrak{u}_J^*)^{[1]} \cong \Lambda^\bullet(V_N)^{[1]}$. (So z_i has weight $\ell\gamma_i$ for U_ζ^0.) Let φ now denote an arbitrary non-zero basis vector for $E_2^{\ell(w),0} \cong k$. The proof of the lemma will be complete once we verify the following claim: The U_ζ^0-module isomorphism $S^r(\mathfrak{u}_J^*)^{[1]} \cong E_2^{\ell(w)+2r,0}$ can be realized via the the map $\psi : S^r(\mathfrak{u}_J^*)^{[1]} \to E_2^{\ell(w)+2r,0}$, where ψ maps the homogeneous polynomial $f(x_1, \dots, x_N) \in S^r(V_N) \cong S^r(\mathfrak{u}_J^*)$ to $f(d_2(z_1), \dots, d_2(z_N)).\varphi \in E_2^{\ell(w)+2r,0}$. (The map ψ is well-defined, because the $d_2(z_i)$, viewed as operators on $E_2^{\bullet,\bullet}$, commute; cf. Lemma 2.25.) Indeed, if the claim is true, then the statement on the $U_\zeta(\mathfrak{p}_J)$-module structure for $E_2^{\ell(w)+2r,0}$ follows from the known $U_\zeta(\mathfrak{p}_J)$-module structure of $H^\bullet(Z_J, k)$, the $U_\zeta(\mathfrak{p}_J)$-module isomorphism $d_2 : E_2^{\ell(w),1} \to E_2^{\ell(w)+2,0}$, and the remark made above on the compatibility of the $U_\zeta(\mathfrak{p}_J)$-module structure for $E_2^{\bullet,\bullet}$ with respect to the left action by elements of $({}'E_2^{a,b})^{u_\zeta(\mathfrak{l}_J)}$.

Since $S^r(\mathfrak{u}_J^*)$ and $E_2^{\ell(w)+2r,0}$ each have the same finite dimension, to prove the claim it suffices to show that ψ is surjective. This is done by induction on r. The case $r = 0$ is trivial, and the case $r = 1$ follows from the isomorphism $d_2 : E_2^{\ell(w),1} \to E_2^{\ell(w)+2,0}$. (When $r = 1$, ψ *is* the differential d_2.) So assume $r \geq 2$, and that the claim is true for all $0 \leq s < r$.

For $0 \leq i \leq N$, let $A_i = \mathcal{U}_\zeta(\mathfrak{u}_J)//\langle f_1^\ell, \dots, f_N^\ell \rangle$ and $B_i = \langle f_i^\ell \rangle \subset A_{i-1}$ be as defined in the proof of Theorem 4.17. We have $A_{N-1}//B_N \cong u_\zeta(\mathfrak{u}_J) \cong \mathcal{U}_\zeta(\mathfrak{u}_J)//Z_J$. The surjection $\mathcal{U}_\zeta(\mathfrak{u}_J) \twoheadrightarrow A_{N-1}$ maps Z_J onto B_N, and induces a morphism η' of spectral sequences:

$$\begin{array}{ccc}
{}_{I'}E_2^{a,b} := H^b(B_N, k) \otimes H^a(u_\zeta(\mathfrak{u}_J), k) & \Longrightarrow & H^{i+j}(A_{N-1}, k) \\
\downarrow \eta' & & \downarrow \eta' \\
{}_{II'}E_2^{a,b} := H^b(Z_J, k) \otimes H^a(u_\zeta(\mathfrak{u}_J), k) & \Longrightarrow & H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}_J), k)
\end{array}$$

(cf. the proof of Theorem 4.17 and the construction presented in §2.2 for the LHS spectral sequence). The morphism η' restricts to the identity on $H^\bullet(u_\zeta(\mathfrak{u}_J), k)$. Identifying $H^\bullet(B_N, k)$ with the exterior algebra on the one-dimensional subspace of V_N spanned by z_N, the effect of η' on $H^\bullet(B_N, k)$ is the natural inclusion $H^\bullet(B_N, k) \subset H^\bullet(Z_J, k) \cong \Lambda^\bullet(V_N)^{[1]}$.

Applying the functor $\mathcal{G}$ to each spectral sequence, we obtain the induced morphism η of spectral sequences:

$$\begin{array}{ccc}
{}_{I}E_2^{a,b} := H^b(B_N, k) \otimes \mathcal{G}(H^a(u_\zeta(\mathfrak{u}_J), k)) & \Longrightarrow & \mathcal{G}(H^{i+j}(A_{N-1}, k)) \\
\downarrow \eta & & \downarrow \eta \\
{}_{II}E_2^{a,b} := H^b(Z_J, k) \otimes \mathcal{G}(H^a(u_\zeta(\mathfrak{u}_J), k)) & \Longrightarrow & \mathcal{G}(H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}_J), k))
\end{array}$$

By the proof of Theorem 4.17, $\eta\circ d_2({}_IE_2^{\ell(w)+2(r-1),1}) = d_2(z_N).S^{r-1}(\mathfrak{u}_J^*)^{[1]} \subset {}_{II}E_2^{\ell(w)+2r,0}$ identifies with the space of all degree-r homogeneous polynomials in the x_j ($1 \leq j \leq N$) containing at least one factor of x_N. By induction, $S^{r-1}(\mathfrak{u}_J^*)^{[1]} \subset \mathrm{im}(\psi)$. It follows that the subspace of $E_2^{\ell(w)+2r,0}$ corresponding to all degree-r homogeneous polynomials containing at least one factor of x_N is in the image of ψ.

Now, the chosen order $\{f_1, \ldots, f_N\}$ for the root vectors in $\mathcal{U}_\zeta(\mathfrak{u}_J)$ was arbitrary. Selecting a different ordering, we may assume that any particular root vector is specified by f_N. Since every element of $S^r(V_N)$ is a linear combination of monomials of total degree r, and since every such monomial contains at least one x_i as a factor, it follows that the map $\psi : S^r(\mathfrak{u}_J^*)^{[1]} \to E_2^{\ell(w)+2r,0}$ is surjective. This proves the claim for ψ, which completes the proof of the proposition. □

Remark 4.19. It follows from the proof of Proposition 4.18 that the subalgebra of $H^\bullet(u_\zeta(\mathfrak{u}_J), k)$ generated by $\{d_2(z_1), \ldots, d_2(z_N)\}$ is isomorphic as a $U_\zeta(\mathfrak{p}_J)$-module algebra to $S^\bullet(\mathfrak{u}_J^*)^{[1]}$. This observation improves the results of [9, Lemma 5.4.1].

Corollary 4.20. Suppose $\ell \geq h$, so that $J = \emptyset$, $w = 1$, $\mathfrak{p}_J = \mathfrak{b}$, $\mathfrak{u}_J = \mathfrak{u}$, and $\mathfrak{l}_J = \mathfrak{h}$. Then the $U_\zeta(\mathfrak{b})$-module isomorphism $H^{2\bullet}(u_\zeta(\mathfrak{b}), k) \cong S^\bullet(\mathfrak{u}^*)^{[1]}$ of Proposition 4.18 is also an isomorphism of algebras.

Proof. We have $\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 = \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{b})} 0 = k$. Then $H^\bullet(u_\zeta(\mathfrak{b}), k) \cong (H^\bullet(u_\zeta(\mathfrak{u}), k))^{u_\zeta^0}$ as $U_\zeta(\mathfrak{b})$-module algebras by Proposition 4.16. (The isomorphism is an isomorphism of algebras because (4.2.4) is a spectral sequence of algebras if $\ell \geq h$.)

Since $\ell \geq h$, the spectral sequence (4.2.13) becomes

$$E_2^{a,b} = H^b(Z, k) \otimes (H^a(u_\zeta(\mathfrak{u}), k))^{u_\zeta^0} \Rightarrow \left(H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}), k)\right)^{u_\zeta^0}. \tag{4.2.15}$$

It inherits from (4.2.12) the structure of a spectral sequence of algebras. Then the proof of Proposition 4.18 shows that $E_2^{\bullet,0} = (H^\bullet(u_\zeta(\mathfrak{u}), k))^{u_\zeta^0}$ is generated as an algebra by $\{d_2(z_1), \ldots, d_2(z_N)\}$, and that this algebra is isomorphic as a $U_\zeta(\mathfrak{b})$-module algebra to $S^\bullet(\mathfrak{u}^*)^{[1]}$. □

Theorem 4.21. (cf. [9, Theorem 1.3.3]) Let ℓ be as in Assumption 1.5, and let $w \in W$ be such that $w(\Phi_0) = \Phi_J$. Suppose $\ell \nmid n+1$ when Φ is of type A_n, and $\ell \neq 9$ when Φ is of type E_6. Assume $R^i \mathrm{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) = 0$ for all $i > 0$. Then $H^{\mathrm{odd}}(u_\zeta(\mathfrak{g}), k) = 0$, and there exists a G-module isomorphism $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong \mathrm{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$.

Proof. According to Theorem 4.14 and Proposition 4.18, there exists a first quadrant G-module spectral sequence satisfying

$$E_2^{i,j} = R^i \mathrm{ind}_{P_J}^G S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}_J^*) \Rightarrow H^{i+j-\ell(w)}(u_\zeta(\mathfrak{g}), k). \tag{4.2.16}$$

By the assumptions, this spectral sequence collapses to yield a G-module isomorphism $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong \mathrm{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$ and $H^{\mathrm{odd}}(u_\zeta(\mathfrak{g}), k) = 0$. □

Remark 4.22. The condition $R^i \operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) = 0$ for all $i > 0$ is satisfied, for example, if $p := \operatorname{char}(k)$ is good for G and if $J \subseteq \Pi$ consists of a set of pairwise orthogonal short roots [61, §8]. Brion and Kumar speculate that this cohomological vanishing property should remain true for arbitrary subsets $J \subseteq \Pi$ provided p is good for G [12, §5.C]. For fixed $J \subseteq \Pi$, the vanishing property is also known to hold if p is sufficiently large, albeit with no known lower bound on p [33]. Christopher Bendel, working with undergraduate student researchers at the University of Wisconsin–Stout, has produced a computer program based on the work of A.L. Christophersen [17] that can potentially compute lower bounds for p for any given root system Φ and any given subset $J \subseteq \Pi$ [8].

Corollary 4.23. Suppose $\operatorname{char}(k)$ is good for G and $\ell \geq h$, h the Coxeter number of Φ. Then the G-module isomorphism $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong \operatorname{ind}_B^G S^\bullet(\mathfrak{u}^*)$ extends to an isomorphism of algebras. In particular, if k is algebraically closed, then $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong k[\mathcal{N}]$ as graded G-algebras.

Proof. According to Corollary 4.20, there exists an isomorphism of $U_\zeta(\mathfrak{b})$-module algebras $H^{2\bullet}(u_\zeta(\mathfrak{b}), k) \cong S^\bullet(\mathfrak{u}^*)^{[1]}$. Since the action of $u_\zeta(\mathfrak{b})$ is trivial, this may be viewed as an isomorphism of $\operatorname{hy}(B) = U_\zeta(\mathfrak{b})//u_\zeta(\mathfrak{b})$-module algebras, or, equivalently, as an isomorphism of B-module algebras.

We have $R^i \operatorname{ind}_B^G S^\bullet(\mathfrak{u}^*) = 0$ for all $i > 0$ by [43, Theorem 2]. Then the spectral sequence (4.2.16) collapses to yield the G-module isomorphism $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong \operatorname{ind}_B^G S^\bullet(\mathfrak{u}^*)$. That this isomorphism is also an isomorphism of algebras follows by the same reasoning as in the classical case; see [2, §3.2] for details. The last statement of the corollary is immediate from Corollary A.8. ☐

The last theorem of this chapter has been observed previously when $\operatorname{char}(k) = 0$. The same proofs work for $u_\zeta(\mathfrak{g})$ under our more general assumptions, and are omitted.

Theorem 4.24. [9, Theorem 1.3.4], [51, Corollary 6.5] Assume $\operatorname{char}(k)$ is good for G and $\ell \geq h$, h the Coxeter number of Φ. Then for any finite-dimensional $u_\zeta(\mathfrak{g})$-module M, $H^\bullet(u_\zeta(\mathfrak{g}), M)$ is a finite module for the Noetherian ring $H^\bullet(u_\zeta(\mathfrak{g}), k)$.

Mastnak, Pevtsova, Schauenburg, and Witherspoon [51] obtain the above theorem as a corollary to their study of cohomology for finite-dimensional pointed Hopf algebras. Specifically, they show that if u belongs to a certain class of finite-dimensional pointed Hopf algebras over an algebraically closed field k of characteristic zero, then Theorem 4.24 holds with the Frobenius–Lusztig kernel $u_\zeta(\mathfrak{g})$ replaced by u [51, Theorem 6.5].

Chapter 5

Cohomology of higher Frobenius–Lusztig kernels

The goal of this chapter is to investigate cohomological finiteness properties for the higher Frobenius–Lusztig kernels of $U_\zeta(\mathfrak{g})$. We begin by proving a generalization of Theorem 4.24 for the subalgebras $U_\zeta(B_r)$ and $U_\zeta(U_r)$ of $U_\zeta(G_r)$ corresponding to the Borel subgroup B of G and the unipotent radical U of B. Then, after establishing certain preliminary results on cohomological restriction maps, we are able to show (under certain restrictions on Φ, ℓ and p) that Theorem 4.24 also generalizes to the higher Frobenius–Lusztig kernel $U_\zeta(G_1)$.

5.1 Cohomological finite generation for Borel subalgebras

Fix $r \in \mathbb{N}$, and let B_r and U_r denote the r-th Frobenius kernels of the Borel subgroup B of G and of the unipotent radical U of B. Set H equal to either U_r or B_r. Friedlander and Parshall have shown that if M is a finite-dimensinoal rational H-module, then the rational cohomology $H^\bullet(H,M)$ is a finite module for the Noetherian algebra $H^\bullet(H,k)$ [27, Proposition 1.12]. Our goal in this section is to prove an analogous cohomological finite generation result for the finite-dimensional subalgebras $U_\zeta(U_r)$ and $U_\zeta(B_r)$ of the higher Frobenius–Lusztig kernel $U_\zeta(G_r)$ of $U_\zeta(\mathfrak{g})$.

In this section we do not require the field k to be algebraically closed, though we do require $p := \mathrm{char}(k)$ to be odd if $r > 0$ (and $p \neq 3$ if Φ has type G_2), so that the higher Frobenius–Lusztig kernels $U_\zeta(G_r)$ of $U_\zeta(\mathfrak{g})$ exist. If $r = 0$, then the possibility $p = 0$ is admitted, and we set $U_\zeta(U_r) = u_\zeta(\mathfrak{u})$ and $U_\zeta(B_r) = u_\zeta(\mathfrak{b})$.

Recall the multiplicative filtration indexed by $\Lambda = \mathbb{N}^{2N+1}$ on $\mathbb{U}_\mathbb{Q}(\mathfrak{g})$ defined in §3.1. This filtration restricts to a filtration on the divided power integral form $U_\mathsf{A} \subset \mathbb{U}_\mathbb{Q}(\mathfrak{g})$, hence induces a multiplicative filtration on the algebra $U_\zeta(\mathfrak{g})$. By restriction, we obtain a multiplicative Λ-filtration on $U_\zeta(G_r)$.

We reindex the Λ-filtration on $U_\zeta(G_r)$ to a multiplicative $\mathbb{N}$-filtration as follows. First, for $\mathbf{r}, \mathbf{s} \in \mathbb{N}^N$ ($N = |\Phi^+|$) and $u \in U_\zeta^0$, define the divided power monomial $M_{\mathbf{r},\mathbf{s},u} = F^{(\mathbf{r})} u E^{(\mathbf{s})} \in U_\zeta(\mathfrak{g})$, and define $d(M_{\mathbf{r},\mathbf{s},u}) \in \Lambda$ as in (3.1.1). Next, for $m \in \mathbb{N}$, define $\Lambda_m \subset \Lambda$ by

$$\Lambda_m = \left\{ d(M_{\mathbf{r},\mathbf{s},1}) : \mathbf{r}, \mathbf{s} \in \mathbb{N}^N, 0 \leq r_i, s_i < m \ \forall\ 1 \leq i \leq N \right\}.$$

Then $M_{\mathbf{r},\mathbf{s},u} \in U_\zeta(G_r)$ if and only if $u \in U_\zeta(T_r) = U_\zeta^0 \cap U_\zeta(G_r)$ and $d(M_{\mathbf{r},\mathbf{s},u}) \in \Lambda_{p^r\ell}$. The following facts are immediate:

(1) Λ_m is a finite (totally ordered) subset of Λ.

(2) If $M_{\mathbf{r},\mathbf{s},u} \in U_\zeta(G_r)$, then $d(M_{\mathbf{r},\mathbf{s},u}) \in \Lambda_{p^r\ell} \subset \Lambda_{2p^r\ell-1}$.

(3) If $M_{\mathbf{r},\mathbf{s},u}, M_{\mathbf{r}',\mathbf{s}',u'} \in U_\zeta(G_r)$, then $d(M_{\mathbf{r},\mathbf{s},u}) + d(M_{\mathbf{r}',\mathbf{s}',u'}) \in \Lambda_{2p^r\ell-1}$.

(4) If $M_{\mathbf{r},\mathbf{s},u}, M_{\mathbf{r}',\mathbf{s}',u'} \in U_\zeta(G_r)$, then $M_{\mathbf{r},\mathbf{s},u} M_{\mathbf{r}',\mathbf{s}',u'} = \sum_{\mathbf{a},\mathbf{b},u''} M_{\mathbf{a},\mathbf{b},u''}$ with

$$d(M_{\mathbf{a},\mathbf{b},u''}) \in \{\eta \in \Lambda_{p^r\ell} \subset \Lambda_{2p^r\ell-1} : \eta \leq d(M_{\mathbf{r},\mathbf{s},u}) + d(M_{\mathbf{r}',\mathbf{s}',u'})\}.$$

Set $\theta = 2p^r\ell - 1$. Given $M_{\mathbf{r},\mathbf{s},u} \in U_\zeta(\mathfrak{g})$, define $\deg(M_{\mathbf{r},\mathbf{s},u}) \in \mathbb{N}$ by

$$\deg(M_{\mathbf{r},\mathbf{s},u}) = r_N + \theta r_{N-1} + \theta^2 r_{N-2} + \cdots + \theta^{2N-1} s_N + \theta^{2N} \operatorname{ht}(M_{\mathbf{r},\mathbf{s},u}). \tag{5.1.1}$$

If $d(M_{\mathbf{r},\mathbf{s},u}), d(M_{\mathbf{r}',\mathbf{s}',u'}) \in \Lambda_\theta$, then $d(M_{\mathbf{r},\mathbf{s},u}) < d(M_{\mathbf{r}',\mathbf{s}',u'})$ if and only if $\deg(M_{\mathbf{r},\mathbf{s},u}) < \deg(M_{\mathbf{r}',\mathbf{s}',u'})$. This, together with the four facts listed above, implies the following:

Proposition 5.1. For $n \in \mathbb{N}$, define $U_\zeta(G_r)_n$ to be the k-subspace of $U_\zeta(G_r)$ spanned by all monomials $M_{\mathbf{r},\mathbf{s},u} \in U_\zeta(G_r)$ such that $\deg(M_{\mathbf{r},\mathbf{s},u}) \leq n$. Then the subspaces $U_\zeta(G_r)_n$ for $n \in \mathbb{N}$ define a multiplicative $\mathbb{N}$-filtration on $U_\zeta(G_r)$. The graded algebra $\operatorname{gr}^\Lambda U_\zeta(G_r)$ associated to the Λ-filtration on $U_\zeta(G_r)$ is canonically isomorphic to the graded algebra $\operatorname{gr}^{\mathbb{N}} U_\zeta(G_r)$ associated to the $\mathbb{N}$-filtration on $U_\zeta(G_r)$.

By restriction, we obtain from Proposition 5.1 an $\mathbb{N}$-filtration on $U_\zeta(U_r)$. Write $\operatorname{gr} U_\zeta(U_r) = \operatorname{gr}^{\mathbb{N}} U_\zeta(U_r) = \operatorname{gr}^\Lambda U_\zeta(U_r)$ to denote the associated graded algebra.

Let $\mathscr{A}$ denote the twisted polynomial algebra generated by indeterminates

$$\left\{ X_\alpha, X_{p^i\ell\alpha} : \alpha \in \Phi^+, 0 \leq i \leq r-1 \right\}, \tag{5.1.2}$$

and satisfying the following relations:

$$\begin{aligned} X_\alpha X_\beta &= \zeta^{(\alpha,\beta)} X_\beta X_\alpha \quad \text{if} \quad \alpha \prec \beta, \\ X_{p^i\ell\alpha} X_\beta &= X_\beta X_{p^i\ell\alpha}, \\ X_{p^i\ell\alpha} X_{p^j\ell\beta} &= X_{p^j\ell\beta} X_{p^i\ell\alpha}. \end{aligned} \tag{5.1.3}$$

Then the associated graded algebra $\operatorname{gr} U_\zeta(U_r)$ is generated by (5.1.2) subject to the relations (5.1.3), as well as the following additional relations:

$$X_\alpha^\ell = X_{p^i\ell\alpha}^p = 0 \quad \text{for each } \alpha \in \Phi^+. \tag{5.1.4}$$

Under the canonical vector space isomorphism $U_\zeta(U_r) \to \operatorname{gr} U_\zeta(U_r)$, the generator X_α of $\operatorname{gr} U_\zeta(U_r)$ corresponds to the root vector $F_\alpha \in U_\zeta(U_r)$, and the generator $X_{p^i\ell\alpha}$ of $\operatorname{gr} U_\zeta(U_r)$ corresponds to the root vector $F_\alpha^{(p^i\ell)} \in U_\zeta(U_r)$. As in §4.1 for $\mathcal{U}_\zeta(\mathfrak{u})$, the Ad_r-action of U_ζ^0 on $U_\zeta(U_r)$ makes $\operatorname{gr} U_\zeta(U_r)$ a right U_ζ^0-module algebra such that X_α is of weight α for U_ζ^0 and $X_{p^i\ell\alpha}$ is of weight $p^i\ell\alpha$ for U_ζ^0.

Define the algebra $\Lambda_{\zeta,r}^\bullet$ to be the graded algebra with generators

$$\left\{x_\alpha, x_{p^i\ell\alpha} : \alpha \in \Phi^+, 0 \leq i \leq r-1\right\},$$

each of graded degree 1, subject to the following relations:

$$x_\alpha x_\beta + \zeta^{-(\alpha,\beta)} x_\beta x_\alpha = 0 \quad \text{if} \quad \alpha \prec \beta, \tag{5.1.5}$$

$$x_{p^i\ell\alpha} x_\beta + x_\beta x_{p^i\ell\alpha} = 0, \tag{5.1.6}$$

$$x_{p^i\ell\alpha} x_{p^j\ell\beta} + x_{p^j\ell\beta} x_{p^i\ell\alpha} = 0, \tag{5.1.7}$$

$$\text{and} \quad x_\alpha^2 = x_{p^i\ell\alpha}^2 = 0. \tag{5.1.8}$$

The algebra $\Lambda_{\zeta,r}^\bullet$ admits the structure of a left U_ζ^0-module algebra if we assign the generator x_α to have weight α and the generator $x_{p^i\ell\alpha}$ to have weight $p^i\ell\alpha$.

The proof of the following lemma is essentially the same as that of Lemma 4.4.

Lemma 5.2. There exists a graded U_ζ^0-module algebra isomorphism $H^\bullet(\mathscr{A}, k) \cong \Lambda_{\zeta,r}^\bullet$.

We would like to compute the structure of the cohomology ring $H^\bullet(\operatorname{gr} U_\zeta(U_r), k)$. We follow the strategy of [30, §2.4]. Enumerate the indeterminates in (5.1.2) as $X_1, X_2, \ldots, X_m$. If $X_j = X_\alpha$ for some $\alpha \in \Phi^+$, set $X_j^\epsilon = X_j^\ell$; otherwise, set $X_j^\epsilon = X_j^p$. For $1 \leq j \leq m$, let R_j denote the vector subspace of $\mathscr{A}$ spanned by the elements $X_1^\epsilon, X_2^\epsilon, \ldots, X_j^\epsilon$, and let $\mathscr{Z}_j$ denote the central subalgebra of $\mathscr{A}$ generated by R_j. Set $\mathfrak{u}_j = \mathscr{A}//\mathscr{Z}_j$. Then $\mathfrak{u}_0 = \mathscr{A}$, $\mathfrak{u}_{j+1} = \mathfrak{u}_j//(X_{j+1}^\epsilon)$, and $\mathfrak{u}_m = \operatorname{gr} U_\zeta(U_r)$.

Proposition 5.3. (cf. [30, Proposition 2.3.2]) For each $0 \leq j \leq m$, there is a natural isomorphism of graded algebras and of U_ζ^0-modules

$$H^\bullet(\mathfrak{u}_j, k) \cong \Lambda_{\zeta,r}^\bullet \otimes S^\bullet(R_j^*),$$

where the elements of $S^1(R_j^*) = R_j^* = \operatorname{Hom}_k(R_j, k)$ are assigned graded degree two.

Proof. We proceed by induction on j, following the strategy of [30, §2.4]. In fact, with the exception of the proof of Step 4, the proposition follows formally from the same arguments as those given by Ginzburg and Kumar in the case $r = 0$. For the convenience of the reader we summarize the full argument here.

For $j = 0$, the proposition reduces to Lemma 5.2. Now, by way of induction, assume the validity of the proposition for $0 \leq i \leq j$, and let $A = A_{j+1}$ denote the central subalgebra of $\mathfrak{u}_j$ generated by X_{j+1}^{ϵ}. Then $\mathfrak{u}_j//A \cong \mathfrak{u}_{j+1}$, and $\mathfrak{u}_j$ is free (in particular, flat) over A, hence there exists by Theorem 2.24 a spectral sequence of U_ζ^0-modules satisfying

$$E_2^{a,b} = H^a(\mathfrak{u}_{j+1}, H^b(A,k)) \Rightarrow H^{a+b}(\mathfrak{u}_j, k). \tag{5.1.9}$$

Step 1. For each $a \geq 0$, the canonical restriction homomorphism $r_a : H^a(\mathfrak{u}_{j+1}, k) \rightarrow H^a(\mathfrak{u}_j, k)$ induced by the algebra homomorphism $\mathfrak{u}_j \twoheadrightarrow \mathfrak{u}_{j+1}$ is surjective.

Proof. By the induction hypothesis, $H^\bullet(\mathfrak{u}_j, k)$ is generated by elements of degree ≤ 2, so it suffices to prove the surjectivity of the maps r_1 and r_2. The map r_1 is an isomorphism because, for any algebra B with augmentation $\varepsilon : B \rightarrow k$, there exists a natural isomorphism $H^1(B,k) \cong (B_\varepsilon/B_\varepsilon^2)^*$ [31, Lemma 2.2]. The surjectivity of the map r_2 follows from [31, Lemma 2.10]. □

Step 2. In the spectral sequence (5.1.9), $E_\infty^{a,b} = 0$ for all $b > 0$.

Proof. There exists a natural commutative diagram

$$\begin{array}{ccccc} E_2^{a-2,1} & \xrightarrow{d_2} & E_2^{a,0} & \twoheadrightarrow & E_\infty^{a,0} \hookrightarrow H^a(\mathfrak{u}_j, k) \\ & & \| & \nearrow_{r_a} & \\ & & H^a(\mathfrak{u}_{j+1}, k) & & \end{array} \tag{5.1.10}$$

By Step 1, the map r_a is surjective. The commutativity of the diagram then implies that the inclusion $E_\infty^{a,0} \hookrightarrow H^a(\mathfrak{u}_j, k)$ is an isomorphism, hence that $E_\infty^{a,b} = 0$ for all $b > 0$ (because the spectral sequence converges to $H^\bullet(\mathfrak{u}_j, k)$). □

Since A is central in $\mathfrak{u}_j$, and since $\mathfrak{u}_j$ is free (in particular, flat) as a right A-module, the action of $\mathfrak{u}_{j+1}$ on $H^\bullet(A,k)$ is trivial by Corollary 2.20. Then the differential $d_2^{0,1}$ of (5.1.9) is a map $H^1(A,k) \rightarrow H^2(\mathfrak{u}_{j+1}, k)$, and has image in the center of $H^2(\mathfrak{u}_{j+1}, k)$ by Lemma 2.25. Observe that $d_2^{0,1}$ must have trivial kernel because $\dim H^1(A,k) = 1$ and because $E_\infty^{0,1} = 0$ by Step 2. Choose $0 \neq v \in \operatorname{im}(d_2^{0,1}) \subseteq H^2(\mathfrak{u}_{j+1}, k)$.

Step 3. The kernel of the homomorphism $r : H^\bullet(\mathfrak{u}_{j+1}, k) \rightarrow H^\bullet(\mathfrak{u}_j, k)$ is generated as a two-sided ideal by v.

Proof. From the commutative diagram (5.1.10) and the isomorphism $E_\infty^{a,0} \cong H^a(\mathfrak{u}_j, k)$, we conclude that the kernel of the restriction map $r : H^\bullet(\mathfrak{u}_{j+1}, k) \rightarrow H^\bullet(\mathfrak{u}_j, k)$ is equal to the image of $E_2^{\bullet,1}$ under the differential d_2. Since $\mathfrak{u}_{j+1}$ acts trivially on $H^\bullet(A,k)$, we have $E_2^{\bullet,1} = H^\bullet(\mathfrak{u}_{j+1}, H^1(A,k)) \cong H^\bullet(\mathfrak{u}_{j+1}, k) \otimes H^1(A,k)$. Choose $y \in H^1(A,k)$

with $d_2^{0,1}(y) = v$. Now an arbitrary element of $E_2^{n,1}$ can be written in the form $x \otimes y$ for some $x \in H^n(\mathfrak{u}_{j+1}, k)$. Then

$$d_2(x \otimes y) = d_2(x) \otimes y + (-1)^n x \otimes d_2(y) = (-1)^n x \otimes v$$

is an element of the two-sided ideal in $H^\bullet(\mathfrak{u}_{j+1}, k)$ generated by v. □

Step 4. The algebra homomorphism $r : H^\bullet(\mathfrak{u}_{j+1}, k) \to H^\bullet(\mathfrak{u}_j, k)$ admits a graded algebra splitting that commutes with the action of U_ζ^0.

Proof. By induction, there exists a U_ζ^0-module algebra isomorphism

$$H^\bullet(\mathfrak{u}_j, k) \cong \Lambda_{\zeta,r}^\bullet \otimes S^\bullet(R_j^*).$$

To prove the claim, we must lift the generators of the algebras $\Lambda_{\zeta,r}^\bullet$ and $S^\bullet(R_j^*)$ to $H^\bullet(\mathfrak{u}_{j+1}, k)$ and check that the relations among them are preserved.

Observe that there exists a commutative diagram of U_ζ^0-modules

$$\begin{array}{ccccccc} R_{j+1}^* & = & H^1(\mathscr{Z}_{j+1}, k) & \xrightarrow{d'} & H^2(\mathscr{A}//\mathscr{Z}_{j+1}, k) & = & H^2(\mathfrak{u}_{j+1}, k) \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow r & & \downarrow\!\!\!\downarrow r \\ R_j^* & = & H^1(\mathscr{Z}_j, k) & \xrightarrow{d''} & H^2(\mathscr{A}//\mathscr{Z}_j, k) & = & H^2(\mathfrak{u}_j, k) \end{array}$$

where the horizontal maps d', d'' are the transgression maps of Lemma 2.25, and the projection $R_{j+1}^* \twoheadrightarrow R_j^*$ is induced by the inclusion $R_j \hookrightarrow R_{j+1}$. Also, the image of $R_j^* = H^1(\mathscr{Z}_j, k)$ in $H^2(\mathfrak{u}_j, k)$ under d'' identifies with the subspace $S^1(R_j^*) \subset S^\bullet(R_j^*) \subset H^\bullet(\mathfrak{u}_j, k)$. Now any U_ζ^0-module splitting of the projection $R_{j+1}^* \twoheadrightarrow R_j^*$ provides a lifting of the generators of $S^\bullet(R_j^*)$ to $H^\bullet(\mathfrak{u}_{j+1}, k)$. Lemma 2.25 guarantees that the lifted generators have central image in $H^\bullet(\mathfrak{u}_{j+1}, k)$.

Next, consider the generators $x_1, \ldots, x_m$ of $\Lambda_{\zeta,r}^\bullet$ as elements of $H^1(\mathfrak{u}_j, k)$. We have already seen in Step 1 that the restriction map $r_1 : H^1(\mathfrak{u}_{j+1}, k) \to H^1(\mathfrak{u}_j, k)$ is a U_ζ^0-equivariant isomorphism. We use the inverse map $(r_1)^{-1} : H^1(\mathfrak{u}_j, k) \to H^1(\mathfrak{u}_{j+1}, k)$ to transfer the generators of $H^1(\mathfrak{u}_j, k)$ to $H^1(\mathfrak{u}_{j+1}, k)$. Set $\widetilde{x}_i = (r_1)^{-1}(x_i)$. To prove the claim of Step 4, it now suffices to show that the elements $\widetilde{x}_1, \ldots, \widetilde{x}_m \in H^\bullet(\mathfrak{u}_{j+1}, k)$ satisfy the relations (5.1.5–5.1.8).

Consider, for example, the relation (5.1.5). If $\alpha \prec \beta$, then we have

$$r\left(\widetilde{x}_\alpha \widetilde{x}_\beta + \zeta^{-(\alpha,\beta)} \widetilde{x}_\beta \widetilde{x}_\alpha\right) = x_\alpha x_\beta + \zeta^{-(\alpha,\beta)} x_\beta x_\alpha = 0,$$

hence

$$\widetilde{x}_\alpha \widetilde{x}_\beta + \zeta^{-(\alpha,\beta)} \widetilde{x}_\beta \widetilde{x}_\alpha = cv \tag{5.1.11}$$

for some $c \in k$ by Step 3. So we must show $c = 0$. Here is where our argument deviates from that of [30, §2.4]. Ginzburg and Kumar conclud that $c = 0$ when

$r = 0$ by arguing that the weights of v and of $x_\alpha x_\beta + \zeta^{-(\alpha,\beta)} x_\beta x_\alpha$ for u_ζ^0 could not be equal. (To do this, they also had to assume $\ell > 3$ whenever Φ has rank > 1.) Their precise argument fails under our more general setup, but observe that the value of c in (5.1.11) is independent of the action of U_ζ^0 on $H^\bullet(\mathfrak{u}_{j+1}, k)$. Since the algebra $\mathscr{A}$ is defined in terms of homogeneous relations on the independent generators $X_1, \ldots, X_m$, we can define an action of any m-dimensional algebraic torus $T^m = (k^\times)^m$ on $\mathscr{A}$ by declaring the generator X_i to have weight $-\chi_i$ for T^m, where $\chi_i : T^m \to k^\times$ denotes the i-th coordinate function. This induces an action of T^m on $\Lambda^\bullet_{\zeta,r} \cong H^\bullet(\mathscr{A}, k)$ such that x_i has weight χ_i for T^m.

Now suppose $x_a = x_\alpha$ and $x_b = x_\beta$. Then the left side of (5.1.11) has weight $\chi_a + \chi_b$ for T^m, while the right side has weight $\epsilon \cdot \chi_{j+1}$ for T^m, where as before we set $\epsilon = \ell$ if $X_{j+1} = X_\gamma$ for some $\gamma \in \Phi^+$, and $\epsilon = p$ otherwise. Since $a \neq b$ and $\ell, p > 2$, we must have $\chi_a + \chi_b \neq \epsilon \cdot \chi_{j+1}$. This forces $c = 0$. The other relations among the $\widetilde{x}_i$ are proved in a similar manner. The details are left to the reader. □

Step 5. The element v introduced in Step 3 is not a zero-divisor in $H^\bullet(\mathfrak{u}_{j+1}, k)$.

Proof. By Step 2, $E_3^{a,1} \cong E_\infty^{a,1} = 0$, hence the differential $d_2^{a,1}$ in the spectral sequence (5.1.9) must be injective. Now for $0 \neq x \otimes y \in H^a(\mathfrak{u}_{j+1}, k) \otimes H^1(A, k) = E_2^{a,1}$, we have $0 \neq d_2(x \otimes y) = d_2(x) \cdot y + (-1)^a x \cdot d_2(y) = (-1)^a x \cdot v$. This proves the claim. □

The results of Steps 3–5 complete the proof of the proposition. □

Set $A = U_\zeta(U_r)$, and let $\mathsf{B}^\bullet = \mathsf{B}^\bullet(A) = A_\varepsilon^{\otimes \bullet}$ be the complex with differential $d : \mathsf{B}^n \to \mathsf{B}^{n-1}$ defined as in (4.1.2). Then $C = C^\bullet(A) = \operatorname{Hom}_k(\mathsf{B}^\bullet(A), k)$ is the cobar complex computing $H^\bullet(A, k)$. Given $i \in \mathbb{N}$, write A_i to denote the i-th filtered part of the $\mathbb{N}$-filtration on A from Proposition 5.1, and set $A_{\varepsilon,i} = A_\varepsilon \cap A_i$. Now define an $\mathbb{N}$-filtration on $\mathsf{B}^\bullet(A)$ by setting

$$F^i \mathsf{B}^n = \sum_{\sum i_j < i} A_{\varepsilon,i_1} \otimes \cdots \otimes A_{\varepsilon,i_n},$$

and set $F^i C = \operatorname{Hom}_k(A_\varepsilon^{\otimes \bullet} / F^i \mathsf{B}^\bullet, k)$. This makes $C^\bullet(A)$ a filtered differential graded algebra with product defined by the cup product. Then by Theorem 2.21 there exists a multiplicative spectral sequence satisfying

$$E_1^{i,j} = H^{i+j}(F^i C / F^{i+1} C) \Rightarrow H^{i+j}(A, k). \tag{5.1.12}$$

It is a spectral sequence of U_ζ^0-modules.

The $\mathbb{N}$-grading on $\operatorname{gr} A$ induces an $\mathbb{N}$-grading on the complex $\mathsf{B}^\bullet(\operatorname{gr} A)$ via

$$\mathsf{B}^n(\operatorname{gr} A)_i = \sum_{\sum i_j = i} (\operatorname{gr} A)_{\varepsilon,i_1} \otimes \cdots \otimes (\operatorname{gr} A)_{\varepsilon,i_n},$$

hence an $\mathbb{N}$-grading on the cobar complex $C^\bullet(\operatorname{gr} A)$ via

$$C^\bullet(\operatorname{gr} A)_i = \operatorname{Hom}_k(\mathsf{B}^\bullet(\operatorname{gr} A)_i, k).$$

It follows that the cohomlogy ring $H^\bullet(\operatorname{gr} A, k)$ inherits an $\mathbb{N}$-grading; we denote the i-th graded part of $H^\bullet(\operatorname{gr} A, k)$ by $H^\bullet(\operatorname{gr} A, k)_{(i)}$. Then the spectral sequence (5.1.12) may be rewritten as follows (cf. [30, §5.5]):

$$E_1^{i,j} = H^{i+j}(\operatorname{gr} A, k)_{(i)} \Rightarrow H^{i+j}(A, k). \tag{5.1.13}$$

In general, (5.1.13) is not a first quadrant spectral sequence, though for fixed $n \in \mathbb{N}$ we do have $H^n(\operatorname{gr} A, k)_{(i)} = 0$ for $i \gg 0$.

Recall the notation introduced before Proposition 5.3. Set $R = R_m$ and $\mathscr{Z} = \mathscr{Z}_m$. Proposition 5.3 asserts the existence of a graded algebra isomorphism $H^\bullet(\operatorname{gr} A, k) \cong \Lambda^\bullet_{\zeta,r} \otimes S^\bullet(R^*)$, where the elements of $R^* = S^1(R^*)$ are assigned graded degree two.

Proposition 5.4. In the spectral sequence (5.1.13), the subspace $R^* = S^1(R^*)$ of $H^2(\operatorname{gr} A, k)$ consists of permanent cycles.

Proof. We begin by determining explicit cocycle representatives in the cobar complex $C^\bullet(\operatorname{gr} U_\zeta(U_r))$ for generators of the subalgebra $S^\bullet(R^*) \subset H^\bullet(\operatorname{gr} U_\zeta(U_r), k)$.

Consider the Lyndon–Hochschild–Serre spectral sequence from Theorem 2.23 associated to the twisted polynomial algebra $\mathscr{A}$, its normal (central) subalgebra $\mathscr{Z}$, and the quotient $\mathscr{A}//\mathscr{Z} \cong \operatorname{gr} U_\zeta(U_r)$:

$$E_2^{i,j} = H^i(\operatorname{gr} U_\zeta(U_r), H^j(\mathscr{Z}, k)) \Rightarrow H^{i+j}(\mathscr{A}, k). \tag{5.1.14}$$

Recall the relevant notation from the proof of Theorem 2.23: $P^\bullet = \mathsf{B}_\bullet(\operatorname{gr} U_\zeta(U_r))$ is the left bar complex of $\operatorname{gr} U_\zeta(U_r)$, a $\operatorname{gr} U_\zeta(U_r)$-projective resolution of k; $Q_\bullet = Q_\bullet(k) = \operatorname{Hom}_{\mathscr{A}}(\mathsf{B}_\bullet(\mathscr{A}, \mathscr{A}), k)$ is the $\mathscr{A}$-coinduced resolution of k defined in §2.1; and C is the double complex $C^{\bullet,\bullet} = \operatorname{Hom}_{\mathscr{A}}(P^\bullet, Q_\bullet)$. The horizontal differentials of C are induced by d_h, the differential of the complex $P^\bullet$, while the vertical differential along the i-th column of C is given by $(-1)^i d_v$, where d_v denotes the differential of the complex $Q_\bullet$. The total complex $\operatorname{Tot}(C)$ has total differential $d = d_h + (-1)^i d_v$, and is filtered via the column-wise filtration: $F^i \operatorname{Tot}(C)_n = \bigoplus_{r \geq i} C^{r,n-r}$. Then (5.1.14) is the spectral sequence determined by this filtration of $\operatorname{Tot}(C)$; see §2.2 for details.

By the proof of Proposition 5.3, the subspace $R^* \subset H^2(\operatorname{gr} U_\zeta(U_r), k)$ identifies with the image of $E_2^{0,1} = H^1(\mathscr{Z}, k) \cong \Lambda^1(R^*)$ under the differential $d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}$. Choose $j \in \{1, \dots, m\}$, and consider the vector $x_j \in R^* = \Lambda^1(R^*) \cong H^1(\mathscr{Z}, k)$ that is dual to the basis vector $X_j^\epsilon \in R$. (As usual, set $\epsilon = \ell$ if $X_j = X_\alpha$ for some $\alpha \in \Phi^+$, and set $\epsilon = p := \operatorname{char}(k)$ otherwise.) We will determine a cocycle representative for $d_2^{0,1}(x_j) \in H^2(\operatorname{gr} U_\zeta(U_r), k)$ in $C^2(\operatorname{gr} U_\zeta(U_r))$ by examining in detail the low degree terms of the spectral sequence (5.1.14).

According to §2.2, the $E_2^{0,1}$ term of (5.1.14) is represented by elements $(x,y) \in C^{0,1} \oplus C^{1,0}$ such that $d_v(x) = 0$ and $d_h(x) - d_v(y) = 0$, while $E_2^{2,0}$ is represented by elements of $(\ker d) \cap C^{2,0}$. The differential $d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}$ is then induced by the total differential of $\mathrm{Tot}(C)$. We claim that $x_j \in \Lambda^1(R^*) \cong H^1(\mathscr{Z}, k) = E_2^{0,1}$ is represented by the element $(f_{0,1}, f_{1,0}) \in C^{0,1} \oplus C^{1,0}$, and that $d_2^{0,1}(x_j) \in H^2(\mathrm{gr}\, U_\zeta(U_r), k) = E_2^{2,0}$ is represented by $f_{2,0} \in C^{2,0}$, where the elements $f_{0,1}, f_{1,0}, f_{2,0}$ are defined as follows:

- $f_{0,1} \in \mathrm{Hom}_k(\mathscr{A}_\varepsilon \otimes \mathscr{A}//\mathscr{Z}, k) \cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}//\mathscr{Z}, \mathrm{Hom}_{\mathscr{A}}(\mathscr{A} \otimes \mathscr{A}_\varepsilon \otimes \mathscr{A}, k)) = C^{0,1}$ evaluates to 1 on the monomial $X_j^a \otimes X_j^b$ if $a \geq 1$ and $a + b = \epsilon$, and evaluates to zero on all other monomial basis elements of $\mathscr{A}_\varepsilon \otimes \mathscr{A}//\mathscr{Z}$.

- $f_{1,0} \in \mathrm{Hom}_k((\mathscr{A}//\mathscr{Z})_\varepsilon, \mathrm{Hom}_k(\mathscr{A}//\mathscr{Z}, k))$
  $\cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}//\mathscr{Z} \otimes (\mathscr{A}//\mathscr{Z})_\varepsilon, \mathrm{Hom}_{\mathscr{A}}(\mathscr{A} \otimes \mathscr{A}, k)) = C^{1,0}$
  sends the monomial X_j^a $(1 \leq a < \epsilon)$ to the linear map $g_{1,0,a} \in \mathrm{Hom}_k(\mathscr{A}//\mathscr{Z}, k)$, and evaluates to zero on all other monomial basis elements of $(\mathscr{A}//\mathscr{Z})_\varepsilon$. For $1 \leq a \leq \epsilon$, the linear map $g_{1,0,a}$ evaluates to 1 on the monomial $X_j^{\epsilon - a}$, and evaluates to zero on all other monomial basis elements of $\mathscr{A}//\mathscr{Z}$.

- $f_{2,0} = d_h(f_{1,0})$. Specifically,

  $$\begin{aligned} f_{2,0} \in & \mathrm{Hom}_k((\mathscr{A}//\mathscr{Z})_\varepsilon \otimes (\mathscr{A}//\mathscr{Z})_\varepsilon, \mathrm{Hom}_k(\mathscr{A}//\mathscr{Z}, k)) \\ & \cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}//\mathscr{Z} \otimes (\mathscr{A}//\mathscr{Z})_\varepsilon \otimes (\mathscr{A}//\mathscr{Z})_\varepsilon, \mathrm{Hom}_{\mathscr{A}}(\mathscr{A} \otimes \mathscr{A}, k)) = C^{2,0} \end{aligned}$$

  evaluates to zero on the vector $X_j^a \otimes X_j^b$ $(1 \leq a, b < \epsilon)$ if $a + b \neq \epsilon$, evaluates to the counit $\varepsilon : \mathscr{A}//\mathscr{Z} \to k$ on the vector $X_j^a \otimes X_j^b$ if $a + b = \epsilon$, and evaluates to zero on all other basis elements of $(\mathscr{A}//\mathscr{Z})_\varepsilon \otimes (\mathscr{A}//\mathscr{Z})_\varepsilon$.

It is a straight-forward calculation to check that the relations $d_v(f_{0,1}) = 0$, $d_h(f_{0,1}) - d_v(f_{1,0}) = 0$ and $f_{2,0} = d_h(f_{1,0})$ are satisfied. The equality $f_{2,0} = d(f_{0,1} \oplus f_{1,0})$ implies $f_{2,0} \in (\ker d) \cap C^{2,0}$. In particular, $f_{2,0} \in \ker d_v|_{C^{2,0}}$. Also, the projectivity of P^2 as a module for $\mathscr{A}//\mathscr{Z}$ implies that

$$\begin{aligned} \ker d_v|_{C^{2,0}} &\cong \ker\left(d_v : \mathrm{Hom}_{\mathscr{A}//\mathscr{Z}}(P^2, Q_0^{\mathscr{Z}}) \to \mathrm{Hom}_{\mathscr{A}//\mathscr{Z}}(P^2, Q_1^{\mathscr{Z}})\right) \\ &= \mathrm{Hom}_{\mathscr{A}//\mathscr{Z}}\left(P^2, \ker\left(d_v : Q_0^{\mathscr{Z}} \to Q_1^{\mathscr{Z}}\right)\right) \\ &\cong \mathrm{Hom}_{\mathscr{A}//\mathscr{Z}}(P^2, k), \end{aligned}$$

since the kernel of the map $d_v : Q_0^{\mathscr{Z}} \to Q_1^{\mathscr{Z}}$ is the one-dimensional subspace of $\mathrm{Hom}_k(\mathscr{A}, k) \cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{A} \otimes \mathscr{A}, k) = Q_0(k)$ spanned by the counit $\varepsilon : \mathscr{A} \to k$.

We now check that the vector $f_{0,1} \oplus f_{1,0} \in \mathrm{Tot}(C)_1$ is a representative for $x_j \in H^1(\mathscr{Z}, k) = E_2^{0,1}$. The cohomology groups $H^\bullet(\mathscr{Z}, k)$ can be computed by applying the functor $-^{\mathscr{Z}}$ to an $\mathscr{Z}$ injective resolution of k, and then computing the cohomology of the resulting complex. Since $\mathscr{A}$ is flat (in fact, free) as a right $\mathscr{Z}$-module, we can by Lemma 2.1 choose for the $\mathscr{Z}$-injective resolution of k the $\mathscr{A}$-coinduced resolution of

k: $Q_\bullet(k) = \mathrm{Hom}_{\mathscr{A}}(\mathscr{A} \otimes \mathscr{A}_\varepsilon^{\otimes \bullet} \otimes \mathscr{A}, k)$. Another choice is the $\mathscr{Z}$-coinduced resolution of k: $\widetilde{Q}_\bullet = \mathrm{Hom}_{\mathscr{Z}}(\mathscr{Z} \otimes \mathscr{Z}_\varepsilon^{\otimes \bullet} \otimes \mathscr{Z}, k)$. In the latter case,

$$(\widetilde{Q}_\bullet)^{\mathscr{Z}} \cong \mathrm{Hom}_k(\mathscr{Z}_\varepsilon^{\otimes \bullet} \otimes \mathscr{Z}, k)^{\mathscr{Z}} \cong \mathrm{Hom}_k(\mathscr{Z}_\varepsilon^{\otimes \bullet}, k)$$

reduces to the usual cobar complex computing $H^\bullet(\mathscr{Z}, k)$. Computing $H^1(\mathscr{Z}, k)$ via the cobar complex, we have $x_j = \mathrm{cls}(f)$, where $f \in \mathrm{Hom}_k(\mathscr{Z}_\varepsilon, k)$ evaluates to 1 on the vector X_j^ϵ, and evaluates to zero on all other monomials in $\mathscr{Z}_\varepsilon$ (cf. the proof of Lemma 4.4). Now, restriction of functions defines a chain map $\varphi : Q_\bullet(k) \to \widetilde{Q}_\bullet(k)$ that induces the identity in cohomology. The reader may easily observe that, under the identification $(\widetilde{Q}_1)^{\mathscr{Z}} = \mathrm{Hom}_k(\mathscr{Z}_\varepsilon, k)$, we have $\varphi(f_{0,1}) = f$. Since (in the notation of §2.2) $E_2^{0,1} = Z_2^{0,1}/(\ker d_v|_{C^{1,0}} + d(C^{0,0}))$, this implies the claim for $f_{0,1} \oplus f_{1,0}$.

We now return to the spectral sequence (5.1.13). If $X_j = X_\alpha$ for some $\alpha \in \Phi^+$, set $F_j = F_\alpha$. If $X_j = X_{p^i\ell\alpha}$ for some $\alpha \in \Phi^+$ and some $i \geq 0$, set $F_j = F_\alpha^{(p^i\ell)}$. Now define $f_2 \in C^2(U_\zeta(U_r)) \cong \mathrm{Hom}_k(U_\zeta(U_r)_\varepsilon^{\otimes 2}, k)$ to be the linear map that evaluates to 1 on the monomial $F_j^a \otimes F_j^b$ if $a, b \geq 1$ and $a + b = \epsilon$, and evaluates to zero on all other monomial basis elements of $U_\zeta(U_r)_\varepsilon^{\otimes 2}$. We claim that f_2 is a cocycle in $C^\bullet(U_\zeta(U_r))$. Indeed, let δ denote the differential of the cobar complex $C^\bullet(U_\zeta(U_r))$, and let $F^{(\mathbf{a})}, F^{(\mathbf{b})}, F^{(\mathbf{c})} \in U_\zeta(U_r)_\varepsilon$ be monomial basis vectors for $U_\zeta(U_r)$. Consider $c := (\delta f_2)(F^{(\mathbf{a})} \otimes F^{(\mathbf{b})} \otimes F^{(\mathbf{c})}) = f_2(F^{(\mathbf{a})} \otimes F^{(\mathbf{b})}F^{(\mathbf{c})} - F^{(\mathbf{a})}F^{(\mathbf{b})} \otimes F^{(\mathbf{c})})$. If $c \neq 0$, then by the definition of f_2 we must have (up to a unit in k) $F^{(\mathbf{a})} = F_j^a$ and $F^{(\mathbf{c})} = F_j^c$ for some $a, c \geq 1$. Lemma 3.1 further implies that if $c \neq 0$, then we must have (up to a unit in k) $F^{(\mathbf{b})} = F_j^b$ for some $b \geq 1$. Now $(\delta f_2)(F_j^a \otimes F_j^b \otimes F_j^c) = f_2(F_j^a \otimes F_j^{b+c} - F_j^{a+b} \otimes F_j^c)$, and this evaluates to zero for all possible combinations of a, b, c. Thus $f_2 \in \ker \delta$.

From equations (2.2.1) and (2.2.5) we conclude that any cocycle in $C^\bullet(U_\zeta(U_r))$ determines a permanent cycle in (5.1.13). More explicitly, given a cocycle $f \in C^n(U_\zeta(U_r))$, choose p maximal such that $f \notin F^{p+1}C^n(U_\zeta(U_r))$. Then f determines an element of $Z_r^{p,n-p}$, and hence an element of $E_r^{p,n-p}$ for all $1 \leq r \leq \infty$. In the particular case $f = f_2 \in C^2(U_\zeta(U_r))$, the image of f_2 in $E_1^{\bullet,2-\bullet} \cong H^2(\mathrm{gr}\, U_\zeta(U_r), k)$ identifies with the cohomology class determined by $f_{2,0}$, hence $\mathrm{cls}(f_{2,0}) = d_2^{0,1}(x_j)$ is a permanent cycle in (5.1.13). □

Remark 5.5. Bendel et al. prove Proposition 5.4 in the special case $r = 0$ by a weight argument, though they must assume $\ell > 3$ whenever Φ has type B or C; see [9, Proposition 6.2.2]. Our proof does not require the extra restriction on ℓ.

Theorem 5.6. *Let ℓ be an odd positive integer that is coprime to 3 if Φ has type G_2, and let $r \geq 0$. Set A equal to either $U_\zeta(U_r)$ or $U_\zeta(B_r)$, and let M be a finite-dimensional A-module. Then $H^\bullet(A, M)$ is a finite module for the Noetherian algebra $H^\bullet(A, k)$.*

Proof. We first prove the result for $A = U_\zeta(U_r)$. Consider the multiplicative spectral sequence

$$E_1^{i,j} = H^{i+j}(\mathrm{gr}\, A, k)_{(i)} \Rightarrow H^{i+j}(A, k).$$

According to Proposition 5.4, the Noetherian subalgebra $S^\bullet(R^*) \subset H^\bullet(\operatorname{gr} A, k)$ is generated by permanent cycles. Since the E_1 term of the spectral sequence is a finite module over $S^\bullet(R^*)$, this implies by a standard argument (cf. [25, Lemmas 7.4.4, 7.4.5]) that $H^\bullet(A, k)$ is a Noetherian algebra.

Let M be a finite-dimensional A-module. Up to isomorphism, there exists a unique simple module for $U_\zeta(U_r)$: the one-dimensional trivial module k. Then there exists a filtration of M by submodules $M = M_0 \supset M_1 \supset \cdots \supset M_r \supset M_{r+1} = 0$ such that, for each $0 \leq i \leq r$, $M_i/M_{i+1} \cong k$. Now by a standard argument using induction on the dimension of M and the long exact sequence in cohomology, $H^\bullet(A, M)$ is a finite module over the Noetherian algebra $H^\bullet(A, k)$.

Now set $A = U_\zeta(B_r)$, and let M be a finite-dimensional $U_\zeta(B_r)$-module. Since $H^\bullet(U_\zeta(B_r), M) = H^\bullet(U_\zeta(U_r), M)^{U_\zeta(T_r)}$ and $H^\bullet(U_\zeta(B_r), k) = H^\bullet(U_\zeta(U_r), k)^{U_\zeta(T_r)}$, the theorem follows from [27, Lemma 1.13] and the fact that $U_\zeta(T_r)$ acts compatibly and completely reducibly on $H^\bullet(U_\zeta(U_r), k)$ and $H^\bullet(U_\zeta(U_r), M)$. □

5.2 Restriction maps for Borel subalgebras

Let $\Phi_J \subseteq \Phi$ be an indecomposable root subsystem of Φ corresponding to a subset of simple roots $J \subseteq \Pi$. Let $\mathfrak{g}' \subset \mathfrak{g}$ denote the Lie subalgebra of $\mathfrak{g}$ generated by the root vectors in $\mathfrak{g}$ corresponding to the elements of Φ_J. Let $\mathfrak{b}' \subset \mathfrak{b}$ denote the corresponding Borel subalgebra, and $\mathfrak{u}' \subset \mathfrak{u}$ the corresponding nilpotent radical. In this section we study the restriction homomorphism $H^{2\bullet}(u_\zeta(\mathfrak{b}), k) \to H^{2\bullet}(u_\zeta(\mathfrak{b}'), k)$ induced by the canonical injective algebra homomorphism $u_\zeta(\mathfrak{b}') \hookrightarrow u_\zeta(\mathfrak{b})$ (i.e., the algebra homomorphism mapping $F_\alpha \mapsto F_\alpha$ and $K_\alpha \mapsto K_\alpha$ for all $\alpha \in J$). If $\ell \geq h$, so that the identifications $H^{2\bullet}(u_\zeta(\mathfrak{b}), k) \cong S^\bullet(\mathfrak{u}^*)^{[1]}$ and $H^{2\bullet}(u_\zeta(\mathfrak{b}'), k) \cong S^\bullet(\mathfrak{u}'^*)^{[1]}$ of Corollary 4.20 hold, we show that the restriction map $S^\bullet(\mathfrak{u}^*)^{[1]} \to S^\bullet(\mathfrak{u}'^*)^{[1]}$ is simply the restriction of functions (or, equivalently, it is the projection map induced by the vector space inclusion $\mathfrak{u}' \subset \mathfrak{u}$).

Assume $\ell \geq h$ (hence also $\ell \geq h'$, h' the Coxeter number of Φ_J). Then the spectral sequence (4.2.4) takes the form

$$E_2^{i,j} = H^i(u_\zeta^0, H^j(u_\zeta(\mathfrak{u}), k)) \Rightarrow H^{i+j}(u_\zeta(\mathfrak{b}), k).$$

It is a spectral sequence of algebras. We have $E_2^{i,j} = 0$ for all $i \geq 0$ by (the proof of) Proposition 4.16, so the spectral sequence collapses, yielding the algebra isomorphism $H^\bullet(u_\zeta(\mathfrak{b}), k) \cong (H^\bullet(u_\zeta(\mathfrak{u}), k))^{u_\zeta^0}$. An analogous result holds for $H^\bullet(u_\zeta(\mathfrak{b}'), k)$. Now, the inclusion of algebra $u_\zeta(\mathfrak{b}') \hookrightarrow u_\zeta(\mathfrak{b})$ induces a morphism of spectral sequences

$$\begin{array}{ccc} E_2^{a,b} = H^i(u_\zeta^0, H^j(u_\zeta(\mathfrak{u}), k)) & \Longrightarrow & H^{a+b}(u_\zeta(\mathfrak{b}), k) \\ \downarrow & & \downarrow \\ {}'E_2^{a,b} = H^i(u_\zeta'^0, H^j(u_\zeta(\mathfrak{u}'), k) & \Longrightarrow & H^{i+j}(u_\zeta(\mathfrak{b}'), k), \end{array}$$

such that the vertical maps are the obvious cohomological restriction maps. Then, identifying $H^\bullet(u_\zeta(\mathfrak{b}),k) = (H^\bullet(u_\zeta(\mathfrak{u}),k))^{u_\zeta^0}$ and $H^\bullet(u_\zeta(\mathfrak{b}'),k) = (H^\bullet(u_\zeta(\mathfrak{u}'),k))^{u_\zeta'^0}$, this shows that the cohomological restriction map $H^\bullet(u_\zeta(\mathfrak{b}),k) \to H^\bullet(u_\zeta(\mathfrak{b}'),k)$ is induced by the corresponding restriction map $H^\bullet(u_\zeta(\mathfrak{u}),k) \to H^\bullet(u_\zeta(\mathfrak{u}'),k)$. We now prove the main result of this section.

Proposition 5.7. Suppose $\ell \geq h$, h the Coxeter number of Φ. Under the algebra isomorphisms $H^{2\bullet}(u_\zeta(\mathfrak{b}),k) \cong S^\bullet(\mathfrak{u}^*)$ and $H^{2\bullet}(u_\zeta(\mathfrak{b}'),k) \cong S^\bullet(\mathfrak{u}'^*)$ of Corollary 4.20, the restriction homomorphism $r : H^{2\bullet}(u_\zeta(\mathfrak{b}),k) \to H^{2\bullet}(u_\zeta(\mathfrak{b}'),k)$ is simply the restriction of functions.

Proof. Let $Z' \subset \mathcal{U}_\zeta(\mathfrak{u}')$ be the central subalgebra generated by the ℓ-th powers of the root vectors in $\mathcal{U}_\zeta(\mathfrak{u}')$. The algebra homomorphism $\mathcal{U}_\zeta(\mathfrak{u}') \hookrightarrow \mathcal{U}_\zeta(\mathfrak{u})$ maps Z' into Z, and induces a morphism η' of spectral sequences

$$\begin{array}{ccc} {}_{I'}E_2^{a,b} := H^b(Z,k) \otimes H^a(u_\zeta(\mathfrak{u}),k) & \Longrightarrow & H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}),k) \\ \downarrow \eta' & & \downarrow \eta' \\ {}_{II'}E_2^{a,b} := H^b(Z',k) \otimes H^a(u_\zeta(\mathfrak{u}'),k) & \Longrightarrow & H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}'),k), \end{array}$$

such that the vertical maps are induced by the obvious cohomological restriction morhpisms. Identifying $H^\bullet(Z,k) = \Lambda^\bullet(\mathfrak{u}^*)^{[1]}$ and $H^\bullet(Z',k) = \Lambda^\bullet(\mathfrak{u}'^*)^{[1]}$, the restriction map $\eta' : H^\bullet(Z,k) \to H^\bullet(Z',k)$ is induced by the vector space inclusion $\mathfrak{u}' \subset \mathfrak{u}$; cf. the proof of Lemma 4.4.

The algebra u_ζ^0 acts on $\mathcal{U}_\zeta(\mathfrak{u}')$ and $\mathcal{U}_\zeta(\mathfrak{u})$ via the adjoint action, and the inclusion $\mathcal{U}_\zeta(\mathfrak{u}') \hookrightarrow \mathcal{U}_\zeta(\mathfrak{u})$ is a homomorphism of u_ζ^0-modules. Then η' is a u_ζ^0-module homomorphism. It follows that η' induces a morphism η between the following spectral sequences that were studied in §4.2:

$$\begin{array}{ccc} {}_{I}E_2^{a,b} := H^b(Z,k) \otimes (H^a(u_\zeta(\mathfrak{u}),k))^{u_\zeta^0} & \Longrightarrow & \left(H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}),k)\right)^{u_\zeta^0} \\ \downarrow \eta & & \downarrow \eta \\ {}_{II}E_2^{a,b} := H^b(Z',k) \otimes (H^a(u_\zeta(\mathfrak{u}'),k))^{u_\zeta'^0} & \Longrightarrow & \left(H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}'),k)\right)^{u_\zeta'^0}, \end{array}$$

The proposition now follows from the given description for the effect of η' on $H^\bullet(Z,k)$, the fact that η' is an algebra homomorphism, and the fact that $(H^a(u_\zeta(\mathfrak{u}),k))^{u_\zeta^0}$ is generated as an algebra by $\{d_2(z_1),\dots,d_2(z_N)\}$ (resp. $(H^a(u_\zeta(\mathfrak{u}'),k))^{u_\zeta'^0}$ is generated as an algebra by $\{d_2(z_1'),\dots,d_2(z_{N'}')\}$); cf. the proof of Corollary 4.20. □

5.3 Restriction maps for reductive subalgebras

Now we study the restriction homomorphism $H^\bullet(u_\zeta(\mathfrak{g}),k) \to H^\bullet(u_\zeta(\mathfrak{g}'),k)$ induced by the injective algebra homomorphism $u_\zeta(\mathfrak{g}') \hookrightarrow u_\zeta(\mathfrak{g})$. Let $\mathcal{N}'$ denote the variety of nilpotent elements in $\mathfrak{g}'$. Our goal in this section is to prove (in analogy to the classical situation, cf. [28, Corollary 1.6]), under the identifications $H^{2\bullet}(u_\zeta(\mathfrak{g}),k) \cong k[\mathcal{N}]$ and $H^{2\bullet}(u_\zeta(\mathfrak{g}'),k) \cong k[\mathcal{N}']$ of Corollary 4.23, that the restriction map $k[\mathcal{N}] \to k[\mathcal{N}']$ is simply the restriction of functions.

Recall the functors $\mathcal{F}_1, \mathcal{F}_2$ defined (for $J = \emptyset$) in the proof of Theorem 4.14:

$$\mathcal{F}_1(-) = (-)^{u_\zeta(\mathfrak{g})} \circ H^0(U_\zeta/U_\zeta(\mathfrak{b}), -), \tag{5.3.1}$$

$$\mathcal{F}_2(-) = \operatorname{ind}_B^G(-) \circ (-)^{u_\zeta(\mathfrak{b})}. \tag{5.3.2}$$

The functors $\mathcal{F}_1$ and $\mathcal{F}_2$ are naturally equivalent because they are both right adjoint to the functor $\mathcal{G}(-) = (-)^{[1]}|_{U_\zeta(\mathfrak{b})}$ from the category of rational G-modules to the category of integrable $U_\zeta(\mathfrak{b})$-modules. Let M be an integrable $U_\zeta(\mathfrak{b})$-module. We can realize the equivalences $\theta_M : \mathcal{F}_1(M) \xrightarrow{\sim} \mathcal{F}_2(M)$ and $\theta_M^{-1} : \mathcal{F}_2(M) \xrightarrow{\sim} \mathcal{F}_1(M)$ as follows:

- The evaluation map $H^0(U_\zeta/U_\zeta(\mathfrak{b}), M) \to M$ is a $U_\zeta(\mathfrak{b})$-module homomorphism. The induced map $\mathcal{F}_1(M) \to M$ has image in $M^{u_\zeta(\mathfrak{b})}$, and is a homomorphism of rational B-modules. Then, by Frobenius reciprocity, there exists a G-module homomorphism $\theta_M : \mathcal{F}_1(M) \to \operatorname{ind}_B^G M^{u_\zeta(\mathfrak{b})} = \mathcal{F}_2(M)$.

- The evaluation map $\operatorname{ind}_B^G(M^{u_\zeta(\mathfrak{b})}) \to M^{u_\zeta(\mathfrak{b})} \subset M$ is a $U_\zeta(\mathfrak{b})$-module homomorphism. By Frobenius reciprocity, there exists a $U_\zeta(\mathfrak{g})$-module homomorphism $\mathcal{F}_2(M) \to H^0(U_\zeta/U_\zeta(\mathfrak{b}), M)$. Since $u_\zeta(\mathfrak{g})$ acts trivially on $\mathcal{F}_2(M)$, the map has image in $\mathcal{F}_1(M)$. Then θ_M^{-1} is the induced map $\mathcal{F}_2(M) \to \mathcal{F}_1(M)$.

The natural equivalence $\theta : \mathcal{F}_1 \xrightarrow{\sim} \mathcal{F}_2$ induces a natural equivalence of right derived functors $\theta^\bullet : R^\bullet\mathcal{F}_1 \xrightarrow{\sim} R^\bullet\mathcal{F}_2$.

Now define functors $\mathcal{F}_1', \mathcal{F}_2'$ from the category of integrable $U_\zeta(\mathfrak{b}')$-modules to the category of rational G'-modules by substituting the symbols $\mathfrak{g}', \mathfrak{b}', B', G'$ for the symbols $\mathfrak{g}, \mathfrak{b}, B, G$ in equations (5.3.1) and (5.3.2). Then $\mathcal{F}_1'$ and $\mathcal{F}_2'$ are naturally equivalent via a natural transformation $\theta' : \mathcal{F}_1' \xrightarrow{\sim} \mathcal{F}_2'$ admitting a description similar to that of $\theta : \mathcal{F}_1 \xrightarrow{\sim} \mathcal{F}_2$.

Let M be an integrable $U_\zeta(\mathfrak{b})$-module, and let N be an integrable $U_\zeta(\mathfrak{b}')$-module. Suppose $\eta : M \to N$ is a homomorphism of $U_\zeta(\mathfrak{b}')$-modules. Then η induces morphisms $\eta_1 : \mathcal{F}_1(M) \to \mathcal{F}_1'(N)$ and $\eta_2 : \mathcal{F}_2(M) \to \mathcal{F}_2'(N)$ as follows:

- The evaluation map $\varepsilon : H^0(U_\zeta/U_\zeta(\mathfrak{b}), M) \to M$ is a homomorphism of $U_\zeta(\mathfrak{b}')$-modules. By Frobenius reciprocity, the composition $\eta \circ \varepsilon$ corresponds to a $U_\zeta(\mathfrak{g}')$-module homomorphism $H^0(U_\zeta/U_\zeta(\mathfrak{b}), M) \to H^0(U_\zeta(\mathfrak{g}')/U_\zeta(\mathfrak{b}'), N)$. Call this map $\operatorname{ind}(\eta)$. Restricting to the subspace $\mathcal{F}_1(M) \subset H^0(U_\zeta/U_\zeta(\mathfrak{b}), M)$, we obtain the G'-module homomorphism $\eta_1 : \mathcal{F}_1(M) \to \mathcal{F}_1'(M)$.

- The evaluation map $\varepsilon : \operatorname{ind}_B^G(M^{u_\zeta(\mathfrak{b})}) \to M^{u_\zeta(\mathfrak{b})} \subset M^{u_\zeta(\mathfrak{b}')}$ is a B'-module homomorphism. Since η is a homomorphism of $U_\zeta(\mathfrak{b}')$-modules, the composition $\eta \circ \varepsilon$ has image in $N^{u_\zeta(\mathfrak{b}')}$. Then, by Frobenius reciprocity, there exists a G'-module homomorphism $\eta_2 : \mathcal{F}_2(M) \to \mathcal{F}_2'(N)$ corresponding to the composition $\eta \circ \varepsilon$.

In particular, if $M = N$ and $\sigma = \mathrm{id}$, then we obtain morphisms of $U_\zeta(\mathfrak{g}')$-modules (equivalently, rational G'-modules) $\sigma_1 : \mathcal{F}_1(M) \to \mathcal{F}_1'(M)$ and $\sigma_2 : \mathcal{F}_2(M) \to \mathcal{F}_2'(M)$, which are natural in M, as well as morphisms of the higher derived functors $\sigma_1^\bullet : (R^\bullet\mathcal{F}_1)(M) \to (R^\bullet\mathcal{F}_1')(M)$ and $\sigma_2^\bullet : (R^\bullet\mathcal{F}_2)(M) \to (R^\bullet\mathcal{F}_2')(M)$.

The morphisms $\sigma_1^\bullet, \sigma_2^\bullet$ are defined as follows: Let $M \to I^\bullet$ be a resolution of M by injective integrable $U_\zeta(\mathfrak{b})$-modules, and let $M \to Q^\bullet$ be a resolution of M by injective integrable $U_\zeta(\mathfrak{b}')$-modules. Then there exists a chain map $\varphi : I^\bullet \to Q^\bullet$ (i.e., a collection of compatible $U_\zeta(\mathfrak{b}')$-module homomorphisms) lifting the identity $\mathrm{id} : M \to M$. It induces chain maps $\varphi_1 : \mathcal{F}_1(I^\bullet) \to \mathcal{F}_1'(Q^\bullet)$ and $\varphi_2 : \mathcal{F}_2(I^\bullet) \to \mathcal{F}_2'(Q^\bullet)$. Then $\sigma_1^\bullet = H(\varphi_1)$ and $\sigma_2^\bullet = H(\varphi_2)$, the induced maps in cohomology. From the definitions it is clear that $\theta'^\bullet \circ \sigma_1^\bullet = \sigma_2^\bullet \circ \theta^\bullet$.

Now let M be an integrable $U_\zeta(\mathfrak{g})$-module. By the tensor identity and Kempf's vanishing theorem, we have

$$\begin{aligned} &H^i(U_\zeta(\mathfrak{g})/U_\zeta(\mathfrak{b}), M) = 0 \quad \text{and} \\ &H^i(U_\zeta(\mathfrak{g}')/U_\zeta(\mathfrak{b}'), M) = 0. \end{aligned} \tag{5.3.3}$$

Suppose M also satisfies

$$\begin{aligned} &R^i \operatorname{ind}_B^G H^j(u_\zeta(\mathfrak{b}), M) = 0 \quad \forall\, i > 0 \quad \text{and} \\ &R^i \operatorname{ind}_{B'}^{G'} H^j(u_\zeta(\mathfrak{b}'), M) = 0 \quad \forall\, i > 0. \end{aligned} \tag{5.3.4}$$

For such M, we have by the proof of Theorem 4.14 natural identifications

$$\begin{aligned} &(R^i\mathcal{F}_1)(M) \cong H^i(u_\zeta(\mathfrak{g}), M) \\ &(R^i\mathcal{F}_1')(M) \cong H^i(u_\zeta(\mathfrak{g}'), M) \end{aligned} \tag{5.3.5}$$

and

$$\begin{aligned} &(R^i\mathcal{F}_2)(M) \cong \operatorname{ind}_B^G H^i(u_\zeta(\mathfrak{b}), M) \\ &(R^i\mathcal{F}_2')(M) \cong \operatorname{ind}_{B'}^{G'} H^i(u_\zeta(\mathfrak{b}'), M). \end{aligned} \tag{5.3.6}$$

We would like to obtain explicit descriptions of the morphisms $\sigma_1^\bullet : (R^\bullet\mathcal{F}_1)(M) \to (R^\bullet\mathcal{F}_1')(M)$ and $\sigma_2^\bullet : (R^\bullet\mathcal{F}_2)(M) \to (R^\bullet\mathcal{F}_2')(M)$ compatible with the identifications (5.3.5) and (5.3.6).

Lemma 5.8. Let M be an integrable $U_\zeta(\mathfrak{g})$-module.

(a) Under the identifications (5.3.5), the morphism $\sigma_1^\bullet : (R^\bullet\mathcal{F}_1)(M) \to (R^\bullet\mathcal{F}_1')(M)$ is the cohomological restriction map induced by the injective algebra homomorphism $u_\zeta(\mathfrak{g}') \hookrightarrow u_\zeta(\mathfrak{g})$.

(b) Suppose M satisfies (5.3.4). Under the identifications (5.3.6), the morphism $\sigma_2^\bullet : (R^\bullet\mathcal{F}_2)(M) \to (R^\bullet\mathcal{F}_2')(M)$ is the G'-module homomorphism induced by Frobenius reciprocity from the restriction map $H^\bullet(u_\zeta(\mathfrak{b}), M) \to H^\bullet(u_\zeta(\mathfrak{b}'), M)$.

Proof. Let $M \to I^\bullet$ be a resolution of M by injective integrable $U_\zeta(\mathfrak{b})$-modules, and let $M \to Q^\bullet$ be a resolution of M by injective integrable $U_\zeta(\mathfrak{b}')$-modules. Then there exists a chain map (a collection of $U_\zeta(\mathfrak{b}')$-module homomorphisms) $\varphi : I^\bullet \to Q^\bullet$ lifting the identity $\mathrm{id} : M \to M$.

First we prove (a). Recall that $\mathcal{F}_1, \mathcal{F}_1'$ are defined as compositions of functors:

$$\begin{aligned}\mathcal{F}_1 &= (-)^{u_\zeta(\mathfrak{g})} \circ H^0(U_\zeta(\mathfrak{g})/U_\zeta(\mathfrak{b}), -)\\ \mathcal{F}_1' &= (-)^{u_\zeta(\mathfrak{g}')} \circ H^0(U_\zeta(\mathfrak{g}')/U_\zeta(\mathfrak{b}'), -).\end{aligned}$$

Since M satisfies (5.3.3), we conclude that

$$\begin{aligned}&M \to \overline{I}^\bullet := H^0(U_\zeta(\mathfrak{g})/U_\zeta(\mathfrak{b}), I^\bullet) \quad \text{and}\\ &M \to \overline{Q}^\bullet := H^0(U_\zeta(\mathfrak{g}')/U_\zeta(\mathfrak{b}'), Q^\bullet)\end{aligned}$$

are resolutions of M by injective integrable $U_\zeta(\mathfrak{g})$-modules and injective integrable $U_\zeta(\mathfrak{g}')$-modules, respectively. Also, $\mathrm{ind}(\varphi) : \overline{I}^\bullet \to \overline{Q}^\bullet$ is a chain map (a collection of $U_\zeta(\mathfrak{g}')$-module homomorphisms) lifting the identity $M \to M$. Now

$$\begin{aligned}(R^\bullet\mathcal{F}_1)(M) &= H(\mathcal{F}_1(I^\bullet)) \cong H((\overline{I}^\bullet)^{u_\zeta(\mathfrak{g})}) = H^\bullet(u_\zeta(\mathfrak{g}), M) \quad \text{and}\\ (R^\bullet\mathcal{F}_1')(M) &= H(\mathcal{F}_1'(Q^\bullet)) \cong H((\overline{Q}^\bullet)^{u_\zeta(\mathfrak{g}')}) = H^\bullet(u_\zeta(\mathfrak{g}'), M).\end{aligned} \tag{5.3.7}$$

(Recall the fact mentioned in §4.2 that injective integrable $U_\zeta(\mathfrak{g})$-modules are injective for $u_\zeta(\mathfrak{g})$, hence that $H^\bullet(u_\zeta(\mathfrak{g}), -)$ may be computed via resolutions by injective integrable $U_\zeta(\mathfrak{g})$-modules.) Under the identifications of (5.3.7), we have $\varphi_1 = \mathrm{Hom}_{u_\zeta(\mathfrak{g}')}(k, \mathrm{ind}(\varphi))$. Since $\sigma_1^\bullet$ is defined by $\sigma_1^\bullet = H(\varphi_1)$, we have $\sigma_1^\bullet = H(\varphi_1) = H(\mathrm{Hom}_{u_\zeta(\mathfrak{g}')}(k, \mathrm{ind}(\varphi))) = \mathrm{res} : H^\bullet(u_\zeta(\mathfrak{g}), M) \to H^\bullet(u_\zeta(\mathfrak{g}'), M)$, the restriction map in cohomology induced by the inclusion $u_\zeta(\mathfrak{g}') \hookrightarrow u_\zeta(\mathfrak{g})$. This proves (a).

Now we prove (b). Recall that $\mathcal{F}_2, \mathcal{F}_2'$ are defined as compositions of functors:

$$\begin{aligned}\mathcal{F}_2 &= \mathrm{ind}_B^G(-) \circ (-)^{u_\zeta(\mathfrak{b})}\\ \mathcal{F}_2' &= \mathrm{ind}_{B'}^{G'}(-) \circ (-)^{u_\zeta(\mathfrak{b}')}.\end{aligned}$$

Since M satisfies (5.3.4), we conclude that

$$\begin{aligned}(R^\bullet\mathcal{F}_2)(M) &= H(\mathcal{F}_2(I^\bullet)) = H(\mathrm{ind}_B^G((I^\bullet)^{u_\zeta(\mathfrak{b})}))\\ &\cong \mathrm{ind}_B^G H((I^\bullet)^{u_\zeta(\mathfrak{b})}) = \mathrm{ind}_B^G H^\bullet(u_\zeta(\mathfrak{b}), M) \quad \text{and}\\ (R^\bullet\mathcal{F}_2')(M) &= H(\mathcal{F}_2'(Q^\bullet)) = H(\mathrm{ind}_{B'}^{G'}((Q^\bullet)^{u_\zeta(\mathfrak{b}')}))\\ &\cong \mathrm{ind}_{B'}^{G'} H((Q^\bullet)^{u_\zeta(\mathfrak{b}')}) = \mathrm{ind}_{B'}^{G'} H^\bullet(u_\zeta(\mathfrak{b}'), M)\end{aligned}$$

Note that $\varphi_2 = \mathrm{ind}(\mathrm{Hom}_{u_\zeta(\mathfrak{b}')}(k, \varphi) : (I^\bullet)^{u_\zeta(\mathfrak{b})} \to (Q^\bullet)^{u_\zeta(\mathfrak{b}')})$, that is, φ_2 is the chain map induced by Frobenius reciprocity from the chain map

$$\mathrm{ind}_B^G(I^\bullet)^{u_\zeta(\mathfrak{b})} \xrightarrow{\varepsilon} (I^\bullet)^{u_\zeta(\mathfrak{b})} \xrightarrow{\varphi} (Q^\bullet)^{u_\zeta(\mathfrak{b}')}.$$

This implies that $\sigma_2^\bullet = H(\varphi_2) = \mathrm{ind}(\mathrm{res} : H^\bullet(u_\zeta(\mathfrak{b}), M) \to H^\bullet(u_\zeta(\mathfrak{b}'), M))$, that is, $\sigma_2^\bullet$ is the map induced by Frobenius reciprocity from the B'-module homomorphism $\mathrm{ind}_B^G H^\bullet(u_\zeta(\mathfrak{b}), M) \xrightarrow{\varepsilon} H^\bullet(u_\zeta(\mathfrak{b}), M) \xrightarrow{\mathrm{res}} H^\bullet(u_\zeta(\mathfrak{b}'), M)$. This proves (b). □

Applying Lemma 5.8 in the special case $M = k$, we obtain the following theorem.

Theorem 5.9. Assume k to be algebraically closed and of characteristic good for G, and assume that $\ell \geq h$, h the Coxeter number of Φ. Under the identifications $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong k[\mathcal{N}]$ and $H^{2\bullet}(u_\zeta(\mathfrak{g}'), k) \cong k[\mathcal{N}']$ of Corollary 4.23, the restriction homomorphism $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \to H^{2\bullet}(u_\zeta(\mathfrak{g}'), k)$ induced by the inclusion of algebras $u_\zeta(\mathfrak{g}') \hookrightarrow u_\zeta(\mathfrak{g})$ is the restriction of functions.

Proof. Consider the following diagram

$$\begin{array}{ccccccccc}
H^\bullet(u_\zeta(\mathfrak{g}), k) & = & (R^\bullet\mathcal{F}_1)(k) & \overset{\theta^\bullet}{=\!=} & (R^\bullet\mathcal{F}_2)(k) & = & \mathrm{ind}_B^G H^\bullet(u_\zeta(\mathfrak{b}), k) & = & k[\mathcal{N}] \\
\downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \sigma_1^\bullet} & & \downarrow{\scriptstyle \sigma_2^\bullet} & & \downarrow{\scriptstyle \mathrm{ind(res)}} & & \downarrow \\
H^\bullet(u_\zeta(\mathfrak{g}'), k) & = & (R^\bullet\mathcal{F}_1')(k) & \overset{\theta'^\bullet}{=\!=} & (R^\bullet\mathcal{F}_2')(k) & = & \mathrm{ind}_{B'}^{G'} H^\bullet(u_\zeta(\mathfrak{b}'), k) & = & k[\mathcal{N}']
\end{array}$$

in which all of the horizontal maps are isomorphisms. The diagram commutes by Lemma 5.8 and by remarks made earlier in this section. We wish to determine the induced map $k[\mathcal{N}] \dashrightarrow k[\mathcal{N}']$.

From Corollary 4.20 we obtain the algebra isomorphisms $H^{2\bullet}(u_\zeta(\mathfrak{b}), k) \cong S^\bullet(\mathfrak{u}^*)$ and $H^{2\bullet}(u_\zeta(\mathfrak{b}'), k) \cong S^\bullet(\mathfrak{u}'^*)$. Proposition 5.7 states that under this identification, the restriction homomorphism $H^{2\bullet}(u_\zeta(\mathfrak{b}), k) \xrightarrow{\mathrm{res}} H^{2\bullet}(u_\zeta(\mathfrak{b}'), k)$ is the restriction of functions from $\mathfrak{u}$ to $\mathfrak{u}'$. Then in the diagram, the vertical map $\mathrm{ind(res)} : \mathrm{ind}_B^G S^\bullet(\mathfrak{u}^*) \to \mathrm{ind}_{B'}^{G'} S^\bullet(\mathfrak{u}'^*)$ is the unique G'-module homomorphism obtained by Frobenius reciprocity from the composition $\mathrm{ind}_B^G S^\bullet(\mathfrak{u}^*) \xrightarrow{\varepsilon} S^\bullet(\mathfrak{u}^*) \xrightarrow{\mathrm{res}} S^\bullet(\mathfrak{u}'^*)$.

By the proof of Corollary A.8, the isomorphism $k[\mathcal{N}] \xrightarrow{\sim} \mathrm{ind}_B^G S^\bullet(\mathfrak{u}^*)$ is the unique G-module homomorphism such that the composition $k[\mathcal{N}] \to \mathrm{ind}_B^G S^\bullet(\mathfrak{u}^*) \xrightarrow{\varepsilon} S^\bullet(\mathfrak{u}^*)$ is the restriction of functions. Then the map

$$f : k[\mathcal{N}] \to \mathrm{ind}_B^G S^\bullet(\mathfrak{u}^*) \xrightarrow{\mathrm{ind(res)}} \mathrm{ind}_{B'}^{G'} S^\bullet(\mathfrak{u}'^*)$$

is the unique G'-module homomorphism such that the composition

$$\varepsilon \circ f : k[\mathcal{N}] \to \mathrm{ind}_B^G S^\bullet(\mathfrak{u}^*) \xrightarrow{\mathrm{ind(res)}} \mathrm{ind}_{B'}^{G'} S^\bullet(\mathfrak{u}'^*) \xrightarrow{\varepsilon} S^\bullet(\mathfrak{u}'^*)$$

is the restriction of functions from $\mathcal{N}$ to $\mathfrak{u}'$. But the G'-module homomorphism

$$g : k[\mathcal{N}] \xrightarrow{\mathrm{res}} k[\mathcal{N}'] \to \mathrm{ind}_{B'}^{G'} S^\bullet(\mathfrak{u}'^*)$$

also satisfies this property, that is, $\varepsilon \circ g : k[\mathcal{N}] \to S^\bullet(\mathfrak{u}'^*)$ is the restriction of functions from $\mathcal{N}$ to $\mathfrak{u}'$. Then $f = g$, hence the induced map $k[\mathcal{N}] \dashrightarrow k[\mathcal{N}']$ is the restriction homomorphism $\mathrm{res} : k[\mathcal{N}] \to k[\mathcal{N}']$ that restricts functions from $\mathcal{N}$ to $\mathcal{N}'$. □

5.4 The second Frobenius–Lusztig kernel

Throughout this section, assume k to be algebraically closed and $p := \mathrm{char}(k)$ to be odd and very good for G (i.e., p is good for G, and $p \nmid n+1$ if Φ has type A_n). Also assume $\ell \geq h$, h the Coxeter number of Φ, so that the algebra isomorphism $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong \mathrm{ind}_B^G S^\bullet(\mathfrak{u}^*)$ of Corollary 4.23 holds.

In this section we establish cohomological finite-generation results for the higher Frobenius–Lusztig kernel $U_\zeta(G_1)$ of $U_\zeta(\mathfrak{g})$ analogous to those proved by Friedlander and Parshall for the second Frobenius kernel G_2 of G [27, Theorem 1.11]. We achieve this for root systems of Lie type A or D by arguments similar to those employed in the classical situation. The classical arguments do not admit an obvious generalization to other root systems because, in contrast to the classical situation, an arbitrary quantized enveloping algebra may not embed as a Hopf-subalgebra of a quantized enveloping algebra of Lie type A.

Our first goal is to generalize Proposition 4.12 to the higher Frobenius–Lusztig kernels of $U_\zeta(\mathfrak{g})$, so that we may apply the theory of Chapter 2 to the study of cohomology for $U_\zeta(G_1)$. We begin by summarizing certain representation-theoretic data concerning the higher Frobenius–Lusztig kernels; this information is summarized in greater detail in Appendix B. Fix $r \in \mathbb{N}$. Define the set $X_{p^r\ell}$ of $p^r\ell$-restricted dominant weights in X^+ by $X_{p^r\ell} = \{\lambda \in X^+ : 0 \leq (\lambda, \alpha^\vee) < p^r\ell \;\forall\, \alpha \in \Pi\}$. For each $\lambda \in X_{p^r\ell}$, the $U_\zeta(\mathfrak{g})$-module $L_\zeta(\lambda)$ is irreducible for $U_\zeta(G_r)$, and every irreducible $U_\zeta(G_r)$-module is isomorphic to $L_\zeta(\lambda)$ for some $\lambda \in X_{p^r\ell}$. In particular, every irreducible $U_\zeta(G_r)$-module lifts to $U_\zeta(\mathfrak{g})$. Set $\mathrm{St}_{p^r\ell} = \nabla_\zeta((p^r\ell-1)\rho) = L_\zeta((p^r\ell-1)\rho)$. We call $\mathrm{St}_{p^r\ell}$ the $p^r\ell$-th Steinberg module for $U_\zeta(\mathfrak{g})$. It is simple, injective, and projective for $U_\zeta(G_r)$. As a module for $U_\zeta(U_r^+) = U_\zeta(G_r)^+$, $\mathrm{St}_{p^r\ell} \cong U_\zeta(U_r^+)$.

Lemma 5.10. Fix integers $0 \leq r \leq s \leq \infty$. The algebra $U_\zeta(G_s)$ is a smash product of the algebras $U_\zeta(G_r)$ and $U_\zeta(G_s)//U_\zeta(G_r)$. The algebra $U_\zeta(G_s)$ is a free (in particular, flat) module for both the left and right regular actions of $U_\zeta(G_r)$ on $U_\zeta(G_s)$.

Proof sketch. The proof is essentially the same as that of Proposition 4.12 in the case $J = \Pi$, except that the algebra $u_\zeta(\mathfrak{g})$ is replaced by $U_\zeta(G_r)$, the algebra $U_\zeta(\mathfrak{g})$ is replaced by $U_\zeta(G_s)$, and the module St_ℓ is replaced by $\mathrm{St}_{p^r\ell}$. □

Now, Theorem 2.23 implies the existence of a spectral sequence satisfying

$$E_2^{i,j}(\mathfrak{g}) = H^i(U_\zeta(G_1)//u_\zeta(\mathfrak{g}), H^j(u_\zeta(\mathfrak{g}), k)) \Rightarrow H^{i+j}(U_\zeta(G_1), k). \tag{5.4.1}$$

Using the isomorphism $U_\zeta(G_1)//u_\zeta(\mathfrak{g}) \cong \mathrm{hy}(G_1)$ and the results of §4.2, we rewrite (5.4.1) as

$$E_2^{i,j}(\mathfrak{g}) = H^i(G_1, \mathrm{ind}_B^G S^{j/2}(\mathfrak{u}^*)) \Rightarrow H^{i+j}(U_\zeta(G_1), k). \tag{5.4.2}$$

In particular, $E_2^{i,j}(\mathfrak{g}) = 0$ unless j is even.

Let ν denote the maximal root in Φ. If Φ has only one root length, then ν is the minimal element among the non-zero dominant weights lying in the root lattice (cf. [35, §13.4, Example 2]).

Lemma 5.11. [27, Lemma 1.5] Suppose Φ has rank n. Let $w \in W$ be such that $-w \cdot 0 = \rho - w\rho \geq s\nu$ for some positive integer s. Then $\ell(w) \geq n + s - 1$.

Proposition 5.12. In the spectral sequence (5.4.2), suppose $E_2^{i,j}(\mathfrak{g}) \neq 0$ with $i + j = 2p + 1$. Write $j = 2p - 2s$ for some $0 \leq s \leq p$.

(a) If Φ is of type A_n, then $n - 2 \leq s \leq n$.

(b) If Φ is of type D_n, then $n - 2 \leq s \leq 2(n - 1)$.

Proof. The proof here follows exactly the strategy of [27, Proposition 1.6]. We provide the details here in order to show that the argument extends to good characteristics. (The original result is proven under the assumption $p > h$.) Set $n = \operatorname{rank}(\Phi)$, and write $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, with the simple roots labeled in the standard way (as in, e.g., [15, Appendix]).

Since $\operatorname{char}(k)$ is good for G, the rational G-module $\operatorname{ind}_B^G S^{j/2}(\mathfrak{u}^*)$ admits a good filtration [37, Lemma II.12.12]. The non-vanishing of $E_2^{i,j}(\mathfrak{g})$ then implies that there exists a dominant weight $\mu \in X^+$ such that μ is a weight of $\operatorname{ind}_B^G S^{j/2}(\mathfrak{u}^*)$ and such that $H^i(G_1, H^0(\mu)) \neq 0$. If $j > 0$, then μ must be of the form $\mu = w \cdot 0 + p\lambda \neq 0$ for some $\lambda \in X^+$ and $w \in W$ with $\ell(w) \leq i$. Indeed, the proof of [2, Corollary 5.5] (which establishes the given form for μ in the classical $p > h$ case) remains valid for k of good characteristic if we apply the stronger form of [2, Proposition 5.4] proved in [43, Theorem 2].

It follows from Corollary A.8 that any weight μ of $\operatorname{ind}_B^G S^{p-s}(\mathfrak{u}^*)$ must satisfy $\mu \leq (p - s)\nu$ (because any weight of $S^\bullet(\mathfrak{g}^*)$ must be $\leq \nu$). Since p is very good for G (by assumption), we have $p \nmid [X : \mathbb{Z}\Phi]$, hence $\lambda = (\mu - w \cdot 0)/p \in X^+$ must belong to the root lattice. This implies that $\lambda \geq \nu$ by the comment immediately preceding Lemma 5.11. Now we get $-w \cdot 0 = p\lambda - \mu \geq s\nu + p(\lambda - \nu) \geq s\nu$, so Lemma 5.11 implies that $i \geq \ell(w) \geq n + s - 1$. The inequality $-w \cdot 0 \geq s\nu$ also implies $n \geq s$ (resp. $2(n - 1) \geq s$) if Φ has type A_n (resp. type D_n), because there are only n (resp. $2(n - 1)$) roots $\geq \alpha_1$ in Φ. Since $i = 2s + 1$, this proves the proposition. □

We can now prove the main theorem of this chapter.

Theorem 5.13. (cf. [27, Theorem 1.11]) Assume the field k to be algebraically closed and $p := \operatorname{char}(k)$ to be odd and very good for G. Assume also that ℓ is odd, $\ell \geq h$, and one of the following conditions is satisfied:

(1) Φ is either of type A_n or of type D_n, and $n > p + 2$,

(2) Φ is of type A_n and $\ell \geq n + 4$, or

(3) Φ is of type D_n and $\ell \geq 4n$.

Then for any finite-dimensional $U_\zeta(G_1)$-module M, $H^\bullet(U_\zeta(G_1), M)$ is a finite module for the Noetherian algebra $H^\bullet(U_\zeta(G_1), k)$.

Proof. By Theorem 2.23, there exists a spectral sequence satisfying

$$E_2^{i,j}(M) = H^i(U_\zeta(G_1)//u_\zeta(\mathfrak{g}), H^j(u_\zeta(\mathfrak{g}), M)) \Rightarrow H^{i+j}(U_\zeta(G_1), M). \tag{5.4.3}$$

Identifying $U_\zeta(G_1)//u_\zeta(\mathfrak{g})$ with $\mathrm{hy}(G_1)$, we may rewrite (5.4.3) as

$$E_2^{i,j}(M) = H^i(G_1, H^j(u_\zeta(\mathfrak{g}), M)) \Rightarrow H^{i+j}(U_\zeta(G_1), M). \tag{5.4.4}$$

Setting $M = k$, we recover (5.4.1). Also, the spectral sequence $E_r(M)$ is a module over $E_r(k)$ (i.e., for each $r \geq 2$, $E_r^{\bullet,\bullet}(M)$ is a bigraded module over $E_r^{\bullet,\bullet}(k)$, and the module structure is compatible with the differentials of the respective spectral sequences). Now, identifying the category of integrable $U_\zeta(G_1)//u_\zeta(\mathfrak{g}) = \mathrm{hy}(G_1)$-modules with the category of rational G_1-modules and applying Theorem 4.24, we obtain the following situation: $H^\bullet(u_\zeta(\mathfrak{g}), k)$ is a Noetherian k-algebra on which G_1 acts rationally by k-algebra automorphisms, $H^\bullet(u_\zeta(\mathfrak{g}), M)$ is a rational G_1-module on which $H^\bullet(u_\zeta(\mathfrak{g}), k)$ acts compatibly, and $H^\bullet(u_\zeta(\mathfrak{g}), M)$ is a finite module for $H^\bullet(u_\zeta(\mathfrak{g}), k)$. Then by [62, Theorem 3.5, Lemma 3.3], we conclude that $E_2^{\bullet,\bullet}(M) = H^\bullet(G_1, H^\bullet(u_\zeta(\mathfrak{g}), M))$ is a finite module for the Noetherian algebra $E_2^{\bullet,\bullet}(\mathfrak{g}) = H^\bullet(G_1, H^\bullet(u_\zeta(\mathfrak{g}), k))$. To prove the assertion of the theorem, it now suffices by a standard argument (cf. [25, Lemmas 7.4.4, 7.4.5]) to show that the E_2 page $E_2^{\bullet,\bullet}(\mathfrak{g}) = H^\bullet(G_1, H^\bullet(u_\zeta(\mathfrak{g}), k))$ of (5.4.1) is finitely-generated over a Noetherian subalgebra of permanent cycles.

Define $S \subset H^{2p}(u_\zeta(\mathfrak{g}), k)$ to be the vector subspace spanned by all p-th powers of elements of $H^2(u_\zeta(\mathfrak{g}), k)$, and let $R \subset H^\bullet(u_\zeta(\mathfrak{g}), k)$ be the subalgebra generated by S. Evidently, $R \subset H^0(G_1, H^\bullet(u_\zeta(\mathfrak{g}), k))$ (because G_1 acts trivially on all p-th powers of elements in $H^\bullet(u_\zeta(\mathfrak{g}), k)$), and $H^\bullet(u_\zeta(\mathfrak{g}), k)$ is finitely-generated over R. Applying Property CNoeth of [62, §3] once more, we conclude that $E_2^{\bullet,\bullet}(\mathfrak{g})$ is finitely-generated over the subalgebra $H^\bullet(G_1, R) = H^\bullet(G_1, k) \otimes R \subset E_2^{\bullet,\bullet}(\mathfrak{g})$. We claim that $H^\bullet(G_1, R)$ consists of permanent cycles. Since the differential of (5.4.1) is an algebra derivation, it suffices to show that the subspace $S \subset E_2^{0,2p}(\mathfrak{g})$ consists of permanent cycles.

Denote the differential $E_s^{i,j}(\mathfrak{g}) \to E_s^{i+s,j-s+1}(\mathfrak{g})$ of (5.4.1) by $d_s^{i,j}(\mathfrak{g})$. To prove the claim for S, it suffices to prove that $d_{2s+1}^{0,2p}(\mathfrak{g})(S) = 0$ for $1 \leq s \leq p$. (We have used the fact that $E_2^{i,j}(\mathfrak{g}) = 0$ unless j is even.) Suppose Φ is of type A_n or D_n. According to Proposition 5.12, if $d_{2s+1}^{0,2p}(\mathfrak{g}) \neq 0$, then $s \geq n-2$. If $n > p+2$, then $n-2 > p$, so $d_{2s+1}^{0,2p}(\mathfrak{g}) \equiv 0$ for all $1 \leq s \leq p$. This proves the claim in case condition (1) is satisfied.

By assumption, $\mathrm{rank}(\Phi) = n$. For each $m \geq n$, let Φ_m denote the rank m indecomposable root system of the same Lie type as Φ, and let $\mathfrak{g}_m$ denote the corresponding simple Lie algebra over k (i.e., the Lie algebra over k obtained via base-change from a Chevalley basis for the simple complex Lie algebra having root system Φ_m). Then the inclusion of root systems $\Phi \subseteq \Phi_m$ induces an inclusion of algebras $U_\zeta(\mathfrak{g}) \subset U_\zeta(\mathfrak{g}_m)$, hence a morphism of spectral sequences $f_s^{\bullet,\bullet} : E_s^{\bullet,\bullet}(\mathfrak{g}_m) \to E_s^{\bullet,\bullet}(\mathfrak{g})$, such that the map

$$f_2^{0,\bullet} : E_2^{0,\bullet}(\mathfrak{g}_m) \to E_2^{0,\bullet}(\mathfrak{g})$$

is induced by the restriction map $H^\bullet(u_\zeta(\mathfrak{g}_m), k) \to H^\bullet(u_\zeta(\mathfrak{g}), k)$ studied in §5.3. If ℓ is at least the Coxeter number of Φ_m, so that Corollary 4.23 holds for $u_\zeta(\mathfrak{g}_m)$ as well as for $u_\zeta(\mathfrak{g})$, then we can apply Theorem 5.9 to conclude that $S \subseteq \mathrm{im}(f_2^{0,2p})$.

Now suppose condition (2) is satisfied, so that Φ has type A_n and $\ell \geq n + 4$. Then $\Phi \subset \Phi_{n+3}$, and for each $1 \leq s \leq p$ we have the following commutative diagram (where $\mathfrak{g}' = \mathfrak{g}_{n+3}$):

$$\begin{array}{ccc} E_{2s+1}^{0,2p}(\mathfrak{g}') & \xrightarrow[d_{2s+1}^{0,2p}(\mathfrak{g}')]{} & E_{2s+1}^{2s+1,2p-2s}(\mathfrak{g}') \\ {\scriptstyle f_{2s+1}^{0,2p}}\downarrow & & \downarrow{\scriptstyle f_{2s+1}^{2s+1,2p-2s}} \\ E_{2s+1}^{0,2p}(\mathfrak{g}) & \xrightarrow[d_{2s+1}^{0,2p}(\mathfrak{g})]{} & E_{2s+1}^{2s+1,2p-2s}(\mathfrak{g}) \end{array}$$

According to Proposition 5.12, $d_{2s+1}^{0,2p}(\mathfrak{g}) \equiv 0$ if $1 \leq s \leq (n-3)$ or if $(n+1) \leq s \leq p$, and $d_{2s+1}^{0,2p}(\mathfrak{g}') \equiv 0$ if $1 \leq s \leq n$ (because $n = \mathrm{rank}(\Phi_{n+3})+3$). Since ℓ is at least the Coxeter number of Φ_{n+3}, we have $S \subseteq \mathrm{im}(f_2^{0,2p})$. It follows then from the commutativity of the above diagram that $d_{2s+1}^{0,2p}(\mathfrak{g})(S) = 0$ for $1 \leq s \leq n$, hence that $d_{2s+1}^{0,2p}(\mathfrak{g})(S) = 0$ for all $1 \leq s \leq p$. This proves that the set S consists of permanent cycles whenever condition (1) is satisfied. The proof that S consists of permanent cycles whenever condition (3) is satisfied is similar to that for condition (2), and the details are left to the reader. (Embed Φ in Φ_{2n+1}, which has Coxeter number $2(2n+1) - 2 = 4n$. Then argue as for type A, using part (b) of Proposition 5.12.) $\square$

Appendix A

The nilpotent variety

In this chapter, assume the field k to be algebraically closed. The following results are well-known if $p := \text{char}(k)$ is zero or if p is sufficiently large (a typical assumption is $p > h$, h the Coxeter number of Φ), though they hold more generally under weaker assumptions on p.

A.1 Coordinate ring of the nullcone

Let Φ be a finite, irreducible root system. Recall the dual root system $\Phi^\vee$ is defined by $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$, where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. A prime p is said to be bad for Φ if the group $\mathbb{Z}\Phi/\mathbb{Z}\Phi_1$ has p-torsion for some closed root subsystem Φ_1 of Φ. If p is not a bad prime for Φ, then it is called good for Φ. If, moreover, Φ has type A_n and if $p \nmid n+1$, then we call p a very good prime. The prime p is called a torsion prime for Φ if the group $\mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_1^\vee$ has p-torsion for some closed root subsystem Φ_1 of Φ. The bad primes and torsion primes for each indecomposable root system are as follows (cf. [58, §I.4.3]):

Root System	Bad Primes	Torsion Primes
A_n ($n \geq 1$)	none	none
B_n ($n \geq 2$)	2	2
C_n ($n \geq 3$)	2	none
D_n ($n \geq 4$)	2	2
E_6, E_7, F_4	$2, 3$	$2, 3$
E_8	$2, 3, 5$	$2, 3, 5$
G_2	$2, 3$	2

Let $\mathfrak{g}_\mathbb{C}$ be the simple, complex Lie algebra with root system Φ. Fix a Chevalley basis $\{X_\alpha, \alpha \in \Phi; h_i, 1 \leq i \leq n\}$ for $\mathfrak{g}_\mathbb{C}$, and let $\mathfrak{g} = \mathfrak{g}_k$ denote the Lie algebra obtained via base-change to k from the given Chevalley basis for $\mathfrak{g}_\mathbb{C}$. Let G denote the simple, simply-connected algebraic group over k of the same Lie type as $\mathfrak{g}$. Then $\mathfrak{g} \cong \mathrm{Lie}(G)$ (cf. [10, §3.3] or [59, Remark, p.50]).

We say that a prime p is good for $\mathfrak{g}$ (resp. G) if it is good for the root system Φ. It is well-known that if $p := \mathrm{char}(k)$ is good for $\mathfrak{g}$, then the set $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$ is a closed, irreducible subvariety of $\mathfrak{g}$ of codimension $n := \mathrm{rank}(\mathfrak{g})$ [58, Theorem III.3.3]. The set $\mathcal{N}$ is called the nullcone of $\mathfrak{g}$. Its coordinate ring is denoted by $k[\mathcal{N}]$. We would like to obtain an explicit description for $k[\mathcal{N}]$ whenever p is good for $\mathfrak{g}$. First we need some preliminary results.

Let X denote the weight lattice of Φ, and let $S(X)$ denote the (integral) symmetric algebra on the abelian group X. The Weyl group W of Φ acts naturally on X. This induces an action of W on $S(X)$. The following theorem describing the Weyl group invariants in $S(X)$ is due to Demazure:

Theorem A.1. [22] Let R be a ring in which $p \cdot 1_R$ is a unit for each torsion prime p of Φ. Then $S(X)^W \otimes k$ is a polynomial algebra on homogeneous generators of degrees $d_1, \ldots, d_n$, the degrees of the basic polynomial invariants of W. (See [15, Appendix] for a list of the d_i corresponding to each irreducible root system.) If, moreover, $2 \cdot 1_R$ is a unit in R whenever Φ has type C, then $S(X \otimes k)^W = S(X)^W \otimes k$.

Recall that G acts on $\mathfrak{g}$ via the adjoint action, and on $\mathfrak{g}^*$ via the coadjoint action. Then G acts naturally on $k[\mathfrak{g}] = S(\mathfrak{g}^*)$, the coordinate ring of the affine variety $\mathfrak{g}$.

Lemma A.2. Assume $\mathrm{char}(k)$ is good for $\mathfrak{g}$. Then the ring $S(\mathfrak{g}^*)^G$ of G-invariants in $S(\mathfrak{g}^*)$ is a polynomial algebra on homogeneous generators of degrees $d_1, \ldots, d_n$, the degrees of the basic polynomial invariants of W.

Proof. Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$ spanned by $h_1 \otimes 1, \ldots, h_n \otimes 1 \in \mathfrak{g}$. (We write $h_i \otimes 1$ to emphasize the fact that $\mathfrak{g}$ is obtained via base-change from the fixed Chevalley basis for $\mathfrak{g}_{\mathbb{C}}$.) The vector space $X \otimes k$ identifies naturally with the vector space $\mathfrak{h}^* = \mathrm{Hom}_k(\mathfrak{h}, k)$, and this identification is compatible with the actions of W on X and $\mathfrak{h}$. Then the symmetric algebra $S(X \otimes k)$ identifies naturally with the coordinate ring $k[\mathfrak{h}] = S(\mathfrak{h}^*)$ of $\mathfrak{h}$. Since $\mathrm{char}(k)$ is good for $\mathfrak{g}$, the hypotheses of Theorem A.1 are satisfied for $R = k$, hence $S(\mathfrak{h}^*)^W = S(X \otimes k)^W$ is a polynomial algebra on homogeneous generators of degrees $d_1, \ldots, d_n$. Now, according to Kac and Weisfeiler [39, Theorem 4], the embedding $\mathfrak{h} \subset \mathfrak{g}$ induces an isomorphism of algebras $S(\mathfrak{g}^*)^G \xrightarrow{\sim} S(\mathfrak{h}^*)^W$. Thus, the inverse image in $S(\mathfrak{g}^*)^G$ of the polynomial generators for $S(\mathfrak{h}^*)^W$ provides a set of generators for $S(\mathfrak{g}^*)^G$ satisfying the conditions of the lemma. □

We can now describe $k[\mathcal{N}]$ in terms of the G-invariant polynomials in $S(\mathfrak{g}^*)$.

Theorem A.3. (cf. [63, Proposition 6.9]) Let $S = S(\mathfrak{g}^*)$ denote the coordinate ring of the affine variety $\mathfrak{g}$. Let S^G denote the subring of all G-invariant polynomials, and let S_+^G denote the set of all homogeneous G-invariant polynomials of positive degree.

Assume that the characteristic p of k is good for G. Then the ideal in S generated by S_+^G is a prime ideal, and $k[\mathcal{N}] \cong S/S_+^G S$.

Proof sketch. We follow the strategy of Veldkamp [63, §§4–6], who proved the result for p satisfying $(p, |W|) = 1$; using Lemma A.2, we are able to extend the result to all good characteristics.

Let $J_1, \ldots, J_n \in S_+^G$ be the algebraically independent homogeneous generators identified in Lemma A.2. Define $\eta : \mathfrak{g} \to k^n$ by $\eta(X) = (J_1(X), \ldots, J_n(X))$. It suffices to show that $\eta^{-1}(0) = \mathcal{N}$, and that $S_+^G S = (J_1, \ldots, J_n)$ is a prime ideal in S.

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ defined in the proof of Lemma A.2, and let $\mathfrak{u}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\{X_\alpha : \alpha \in \Phi^-\}$. Then $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ is a Borel subalgebra in $\mathfrak{g}$. If $X \in \mathfrak{h}$, $Y \in \mathfrak{u}$, and $f \in S^G$, then $f(X+Y) = f(X)$ by [63, Lemma 4.4] (which is valid for all p). From this it follows that if $X \in \mathfrak{g}$ is arbitrary, and if $X = X_s + X_n$ is the Jordan–Chevalley decomposition for X, then

$$f(X) = f(X_s + X_n) = f(X_s). \tag{A.1.1}$$

Indeed, $f(\mathrm{Ad}(g)(X)) = f(X)$ for all $g \in G$, and after conjugation by G, we may assume $X_s \in \mathfrak{h}$ and $X_n \in \mathfrak{u}$.

Let $\mathcal{O}$ be a semisimple G-orbit in $\mathfrak{g}$ (i.e., $\mathcal{O}$ is the G-orbit of a semisimple element in $\mathfrak{g}$). Then the intersection $\mathcal{O} \cap \mathfrak{h}$ is a W-orbit in $\mathfrak{h}$, and every semisimple orbit in $\mathfrak{g}$ intersects $\mathfrak{h}$ in precisely one W-oribt [58, II.3.16]. According to [39, Theorem 4], the embedding $\mathfrak{h}^* \subset \mathfrak{g}^*$ induces an isomorphism of algebras $S(\mathfrak{g}^*)^G \xrightarrow{\sim} S(\mathfrak{h}^*)^W = k[\mathfrak{h}^W]$. Then (A.1.1) implies $\eta(X) = 0$ if and only if X is nilpotent, hence $\eta^{-1}(0) = \mathcal{N}$.

We know that $\mathcal{N}$ is an irreducible subvariety of $\mathfrak{g}$ by [58, Theorem III.3.3]. Then, to show that $S_+^G S = (J_1, \ldots, J_n)$ is a prime ideal in S, it suffices by [42, Lemma 4] to show that there exists a point $y \in \mathcal{N}$ such that the set of derivatives $\{(dJ_1)_y, \ldots, (dJ_n)_y\}$ forms a linearly independent set. This is the content of [63, Corollary 6.6], the proof of which remains true when p is good for $\mathfrak{g}$ (and not just when $(p, |W|) = 1$). ☐

A.2 Characters of the nullcone

Now we would like to compute the formal characters of each homogenous component of the graded ring $k[\mathcal{N}]$. First we require a preliminary result from commutative algebra. Recall that a sequence $\{x_1, \ldots, x_n\}$ in a commutative ring R is called a regular sequence if, for all $1 \le i \le n$, multiplication by x_i is injective on $R/(x_1, \ldots, x_{i-1})$, and if $M/(x_1, \ldots, x_n) \neq 0$.

Theorem A.4 (Krull's Principal Ideal Theorem)**.** Let R be a commutative ring with identity. The height of every proper ideal $I \subset R$ generated by n elements $x_1, \ldots, x_n$ is at most n, with equality if and only if the elements $x_1, \ldots, x_n$ form a regular sequence in R.

Lemma A.5. (cf. [26, Proposition 4.2]) Assume that the characteristic of k is good for G. Then $S(\mathfrak{g}^*)$ is a free graded $S(\mathfrak{g}^*)^G$-module.

Proof. Our proof is essentially the same as that of [26, Proposition 4.2]. Set $S = S(\mathfrak{g}^*)$, and set $S' = S(\mathfrak{h}^*)$. Choose algebraically independent homogeneous generators $x_1, \ldots, x_n$, $n = \mathrm{rank}(\mathfrak{g})$, for S^G. By a freeness criterion of Bourbaki [11, V, §5.5, Lemma 5], to show that S is free over S^G, it suffices to show that $\{x_1, \ldots, x_n\}$ is a regular sequence in S.

Fix a root space decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{h} \oplus \mathfrak{u}^+$ for $\mathfrak{g}$, and choose elements $x_{n+1}, \ldots, x_m \in \mathfrak{g}^* = S^1(\mathfrak{g}^*)$, $m = \dim(\mathfrak{g})$, corresponding to basis elements in $\mathfrak{g}^*$ dual to the subspace $\mathfrak{u} \oplus \mathfrak{u}^+$. Now let $P \subset S$ be a prime ideal with $(x_1, \ldots, x_m) \subseteq P$. Recall that prime ideals of S containing $S_+^G S$ correspond to closed irreducible subvarieties of the nullcone $\mathcal{N}$. So let $V \subset \mathcal{N}$ be the closed irreducible subvariety corresponding to P, and choose $v \in V$. Then $v \in \mathcal{N}$ (obviously), and $v \in \mathfrak{h}$, because the coordinate functions $x_{n+1}, \ldots, x_m$ of $\mathfrak{u} \oplus \mathfrak{u}^+$ are all elements of P. But $\mathcal{N} \cap \mathfrak{h} = \{0\}$, hence $v = 0$. We conclude that the only prime ideal of S containing $(x_1, \ldots, x_m)$ is the maximal ideal $S_+ = \sum_{i \geq 1} S^i(\mathfrak{g}^*)$ corresponding to $0 \in \mathcal{N}$. Then $\mathrm{ht}(x_1, \ldots, x_m) = \mathrm{ht}\, S_+ = m$, whence $(x_1, \ldots, x_m)$ must be a regular sequence in S by Theorem A.4. In particular, $\{x_1, \ldots, x_n\}$ must be a regular sequence in S. This proves the lemma. □

Lemma A.6. Assume that the characteristic of k is good for G. Then there exists a graded T-module isomorphism $S \cong S^G \otimes k[\mathcal{N}]$.

Proof. By the complete reducibility of $S(\mathfrak{g}^*)$ as a T-module, we can choose a graded subspace $H \subset S$ such that $S = S_+^G S \oplus H$ is a T-module direct sum decomposition. Then by [42, Lemma 1.1], the natural map $S^G \otimes H \to S$ is a vector space isomorphism, and the elements of H form an S^G-module basis for S. From this it follows that there exists a graded T-module isomorphism $H \cong S/S_+^G S \cong k[\mathcal{N}]$, hence that there exists a graded T-module isomorphism $S \cong S^G \otimes k[\mathcal{N}]$. □

The next result is due to Hesselink in the case $\mathrm{char}(k) = 0$.

Theorem A.7. [34, Theorem] Retain the notations and assumptions of Theorem A.3. Let $k^n[\mathcal{N}]$ denote the image of $S^n(\mathfrak{g}^*)$ in $k[\mathcal{N}]$ under the natural projection map $S(\mathfrak{g}^*) \to S/S_+^G S \cong k[\mathcal{N}]$. Then the formal character of $k^n[\mathcal{N}]$ is given by the formula

$$\sum_{\lambda \in X^+} \sum_{w \in W} (-1)^{\ell(w)} p_n(w \cdot \lambda) \chi(\lambda),$$

where $w \cdot \lambda = w(\lambda + \rho) - \rho$ is the usual dot action of w on λ, $\chi(\lambda)$ is the formal character of the finite-dimensional irreducible complex $\mathfrak{g}_{\mathbb{C}}$-module of highest weight λ, and $p_n(\nu)$ is the dimension of the ν-weight space of $S^n(\mathfrak{u}^*)$ (i.e., $p_n(\nu)$ is the number Φ^+-tuples $(n(\alpha))_\alpha \in \mathbb{N}^N$ such that $\nu = \sum_{\alpha \in \Phi^+} n(\alpha)\alpha$ and $n = \sum_{\alpha \in \Phi^+} n(\alpha)$).

Proof. The proof of the theorem follows exactly as in [34]. □

Corollary A.8. (cf. [2, Lemma 3.9]) Assume that the characteristic of k is good for G. Then there exists an isomorphism of graded G-algebras $k[\mathcal{N}] \xrightarrow{\sim} \operatorname{ind}_B^G S^\bullet(\mathfrak{u}^*)$.

Proof. As in [2, §3.9], the natural restriction map $S(\mathfrak{g}^*) \to S(\mathfrak{u}^*)$ induces by Frobenius reciprocity an injective G-algebra homomorphism $\varphi : k[\mathcal{N}] \to \operatorname{ind}_B^G S^\bullet(\mathfrak{u}^*)$. To show that φ is surjective, it suffices to show that the formal character of each homogeneous component $k^n[\mathcal{N}]$ of $k[\mathcal{N}]$ is the same as that of $\operatorname{ind}_B^G S^n(\mathfrak{u}^*)$. But this follows from the calculations in [2, §3.8], since by [43, Theorem 2] we have $R^i \operatorname{ind}_B^G S^\bullet(\mathfrak{u}^*) = 0$ for all $i > 0$. □

Remark A.9. According to Kostant's formula [35, Theorem 24.2], the sum

$$\sum_{n=0}^{\infty} \sum_{w \in W} (-1)^{\ell(w)} p_n(w \cdot \lambda)$$

is the multiplicity with which the zero weight occurs in the weight space decomposition for the induced module $H^0(\lambda) = \operatorname{ind}_B^G \lambda$. Since $\operatorname{char}(k)$ is good for Φ, the induced module $\operatorname{ind}_B^G S^\bullet(\mathfrak{u}^*) \cong k[\mathcal{N}]$ admits a good filtration [37, II.12.12]. Then the multiplicity with which $H^0(\lambda)$ occurs in a good filtration for $k[\mathcal{N}]$ is equal to $H^0(\lambda)^T$. This observation is due to Kostant in the case $\operatorname{char}(k) = 0$ [42, Theorem 11].

Appendix B

Representations of higher Frobenius–Lusztig kernels

This appendix provides a brief summary of certain representation-theoretic data for the higher Frobenius–Lusztig kernels $U_\zeta(G_r)$ of $U_\zeta(\mathfrak{g})$ that is needed in §5.4. Since many of the results here follow, mutatis mutandis, from the same arguments as the corresponding classical results for the Frobenius kernels G_r of G, we do not always provide complete proofs, but instead provide references to the literature whenever the task of translating a proof is routine.

Let k be a field of positive characteristic $p \neq 2$, with $p \neq 3$ if Φ has type G_2. Let $\ell \in \mathbb{N}$ be an odd positive integer with ℓ coprime to 3 if Φ has type G_2, and let $\zeta \in k^\times$ be a primitive ℓ-th root of unity. Fix $r \in \mathbb{N}$, and let $U_\zeta(G_r)$ be the higher Frobenius–Lusztig kernel of $U_\zeta(\mathfrak{g})$ defined in §1.1. Let $U_\zeta(B_r)$, $U_\zeta(U_r)$, $U_\zeta(T_r)$, etc., denote the subalgebras of $U_\zeta(G_r)$ corresponding to the subgroups B_r, U_r, T_r, etc., of G_r.

Define the set $X_{p^r\ell}$ of $p^r\ell$-restricted dominant weights in X^+ by

$$X_{p^r\ell} = \left\{\lambda \in X^+ : 0 \leq (\lambda, \alpha^\vee) < p^r\ell \;\forall\, \alpha \in \Pi\right\}.$$

Fix $\lambda \in X_{p^r\ell}$, and write $\lambda = \lambda_0 + \ell\lambda_1$ with $\lambda_0 \in X_\ell$ and $\lambda_1 \in X^+$. Since $\lambda \in X_{p^r\ell}$, we have $\lambda_1 \in X_{p^r}$, the set of p^r-restricted dominant weights. According to [3, Theorem 1.10], the simple $U_\zeta(\mathfrak{g})$-module $L_\zeta(\lambda)$ is isomorphic to $L_\zeta(\lambda_0) \otimes L(\lambda_1)^{[1]}$. Here $L(\lambda_1)$ denotes the simple rational G-module of highest weight $\lambda_1 \in X^+$. Since $L_\zeta(\lambda_0)$ is simple as a module for $u_\zeta(\mathfrak{g})$ [3, Theorem 1.9], and since $L(\lambda_1)$ is simple as a module for G_r (equivalently, as a module for $\mathrm{hy}(G_r) = U_\zeta(G_r)//u_\zeta(\mathfrak{g})$), it follows that $L_\zeta(\lambda)$ is simple as a module for $U_\zeta(G_r)$. We will show that every simple $U_\zeta(G_r)$-module has the form $L_\zeta(\lambda)$ for some $\lambda \in X_{p^r\ell}$. We will also show that the simple module $\mathrm{St}_{p^r\ell} := L_\zeta((p^r\ell - 1)\rho)$, the $p^r\ell$-th Steinberg module for $U_\zeta(\mathfrak{g})$, is injective and projective as a module for $U_\zeta(G_r)$.

First some general facts on the representation theory of finite-dimensional Hopf algebras: Let H be a finite-dimensional Hopf algebra. The dual space $H^* = \mathrm{Hom}_k(H, k)$ is also a Hopf algebra in a natural way, and the functor $H \mapsto H^*$ is a self-duality on the category of all finite-dimensional Hopf algebras [37, I.8.3]. Then [44, Theorem] implies that there exists a non-degenerate associative bilinear form $b : H \times H \to k$, that is, a non-degenerate bilinear form $b : H \times H \to k$ such that $b(x, yz) = b(xy, z)$ for all $x, y, z \in H$.

Recall that a finite-dimensional k-algebra A is called a Frobenius algebra if the left modules ${}_AA$ and $(A_A)^*$ are isomorphic. Finite-dimensional modules for Frobenius algebras are projective if and only if they are injective [19, Theorem 62.11]. The property of being a Frobenius algebra is equivalent to the existence of a non-degenerate associative bilinear form $A \times A \to k$ [19, Theorem 61.3]. It follows that every finite-dimensional Hopf algebra H is a Frobenius algebra, and a finite-dimensional H-module is projective if and only if it is injective.

Let A be a k-algebra, M a left A-module, and $S : A \to A$ an anti-automorphism. Denote the left action of $a \in A$ on $m \in M$ by $a \cdot m = am$. Define ${}^{(S)}M$ to be the right A-module that coincides with M as a k-vector space, and with right A-action given by $m \cdot a = S(a)m$. Similarly, if N is a right A-module, define $N^{(S)}$ to be the left A-module that coincides with N as a k-vector space, and with left A-action given by $a \cdot n = nS(a)$. Of course, $\left({}^{(S)}M\right)^{(S^{-1})} \cong M$ and ${}^{(S^{-1})}\left(N^{(S)}\right) \cong N$. The following elementary lemmas are now easily verified:

Lemma B.1. Let A be a k-algebra and $S : A \to A$ an anti-automorphism. If M is a projective (resp. injective) left A-module, then ${}^{(S)}M$ is a projective (resp. injective) right A-module. If N is a projective (resp. injective) right A-module, then $N^{(S)}$ is a projective (resp. injective) left A-module.

Lemma B.2. Let A be a finite-dimensional k-algebra. If M is a finite-dimensional projective (resp. injective) right A-module, then $M^* = \operatorname{Hom}_k(M, k)$ is injective (resp. projective) in the category of finite-dimensional left A-modules.

Interchanging the words left and right in the statement of Lemma B.2 yields an analogous true statement. Combining Lemmas B.1 and B.2 gives the following result:

Lemma B.3. Let A be a finite-dimensional k-algebra, S an anti-automorphism of A, and M a finite-dimensional left A-module. Define a left action of A on M^* via $(a \cdot f)(m) = f(S(a)m)$. With this action, if M is a projective (resp. injective) left A-module, then M^* is injective (resp. projective) in the category of finite-dimensional left A-modules.

Next we make some comments concerning the algebras $U_\zeta(U_r)$ and $U_\zeta(B_r)$; analogous statements hold for the algebras $U_\zeta(U_r^+)$ and $U_\zeta(B_r^+)$.

Lemma B.4. Let V be a non-zero (left or right) $U_\zeta(U_r)$-module. Then $V^{U_\zeta(U_r)} \neq 0$.

Proof. The algebra $U_\zeta(U_r)$ is spanned by monomial basis vectors of the form $F^{(\mathbf{m})}$ with $0 \leq m_i < p^r\ell$ for all $1 \leq i \leq N$ (cf. §1.2). Let $0 \neq v \in V$, and choose $F^{(\mathbf{m})}$ with maximal m_i such that $w := F^{(\mathbf{m})}.v \neq 0$ (resp. $0 \neq w := v.F^{(\mathbf{m})}$). Then it follows from Lemma 3.1 that $0 \neq w \in V^{U_\zeta(U_r)}$. In particular, $V^{U_\zeta(U_r)} \neq 0$. □

Lemma B.4 implies that, up to isomorphism, there exists a unique irreducible (left or right) $U_\zeta(U_r)$-module, namely, the trivial module k. Since the space of invariants $U_\zeta(U_r)^{U_\zeta(U_r)}$ for the (left or right) regular action is one-dimensional, spanned by the vector $F^{(\mathbf{a})}$ with $a_i = p^r\ell - 1$ for all i, we conclude that the regular module $U_\zeta(U_r)$ is indecomposable, hence that it is the (left and right) projective cover for k. Since the dual of every irreducible left (resp. right) $U_\zeta(U_r)$-module is an irreducible right (resp. left) module, the regular module $U_\zeta(U_r)$ is also injective [19, Theorem 58.6], hence it is the (left and right) injective hull for k as well. (Consequently, $U_\zeta(U_r)$ is a Frobenius algebra by Lemma B.2.)

Lemma B.5. Let M be a finite-dimensional $U_\zeta(B_r)$-module. The following statements are equivalent:

(1) M is injective as a $U_\zeta(B_r)$-module.

(2) M is injective as a $U_\zeta(U_r)$-module.

(3) M is projective as a $U_\zeta(U_r)$-module.

(4) M is projective as a $U_\zeta(B_r)$-module.

Proof. Statements (1) and (4) are equivalent because $U_\zeta(B_r)$ is a finite-dimensional Hopf algebra, hence a Frobenius algebra. If M is projective for $U_\zeta(B_r)$, then it is projective for $U_\zeta(U_r)$ because $U_\zeta(B_r)$ is free as a left $U_\zeta(U_r)$-module, so (4) $\Rightarrow$ (3). A finite-dimensional $U_\zeta(U_r)$-module is injective if and only if it is projective, because the injective hull and the projective cover of the unique irreducible $U_\zeta(U_r)$-module k are isomorphic. So (2) is equivalent to (3). Finally, the implication (3) $\Rightarrow$ (4) follows by the same argument as in the classical case, cf. the proof of [37, Lemma II.9.4]: For any $U_\zeta(B_r)$-module N, we have

$$\mathrm{Hom}_{U_\zeta(U_r)}(M,N) = \bigoplus_\lambda \mathrm{Hom}_{U_\zeta(U_r)}(M,N)_\lambda = \bigoplus_\lambda \mathrm{Hom}_{U_\zeta(B_r)}(M\otimes\lambda, N).$$

If $\mathrm{Hom}_H(M,-)$ is exact, then so must be each $\mathrm{Hom}_{U_\zeta(B_r)}(M\otimes\lambda,-)$. In particular, $\mathrm{Hom}_{U_\zeta(B_r)}(M,-)$ must be exact. □

Lemma B.5 remains true if $U_\zeta(U_r), U_\zeta(B_r)$ are replaced by $U_\zeta(U_r^+), U_\zeta(B_r^+)$.

It follows from [24, Lemma A3.4] that every $U_\zeta(T_r)$-module admits a weight space decomposition in the sense of [5, §1.2]. In particular, every irreducible $U_\zeta(T_r)$-module is one-dimensional. The set $\{k_\lambda : \lambda \in X_{p^r\ell}\}$ forms a complete set of non-isomorphic one-dimensional $U_\zeta(T_r)$-modules.

Write the identity $1 \in U_\zeta(T_r)$ as a sum of primitive orthogonal idempotents:

$$1 = \textstyle\sum_{\lambda\in X_{p^r\ell}} e_\lambda.$$

So, in the notation of [5, §1.2], $u.e_\lambda = \chi_\lambda(u).e_\lambda$ for all $u \in U_\zeta(T_r)$. Then $U_\zeta(B_r) = \bigoplus_{\lambda \in X_{p^r\ell}} U_\zeta(U_r)e_\lambda$ is a left $U_\zeta(B_r)$-module direct sum decomposition. Each $U_\zeta(U_r)e_\lambda$ is injective and projective for $U_\zeta(B_r)$ by Lemma B.5. The module $U_\zeta(U_r)e_\lambda$ has socle $\lambda - 2(p^r\ell - 1)\rho$ and head λ. We conclude that, as a $U_\zeta(B_r)$-module, $U_\zeta(U_r)e_\lambda$ is the projective cover of λ and the injective hull of $\lambda - 2(p^r\ell - 1)\rho$.

Now, borrowing notation from the classical case, define left $U_\zeta(G_r)$-modules by

$$Z_r(\lambda) = U_\zeta(G_r) \otimes_{U_\zeta(B_r^+)} \lambda$$
$$Z'_r(\lambda) = \operatorname{ind}_{U_\zeta(B_r)}^{U_\zeta(G_r)} \lambda = \operatorname{Hom}_{U_\zeta(B_r)}(U_\zeta(G_r), \lambda)$$

The left $U_\zeta(G_r)$-module structure of $Z'_r(\lambda)$ is induced by the right multiplication of $U_\zeta(G_r)$ on itself. As vector spaces, $Z_r(\lambda) \cong U_\zeta(U_r) \otimes \lambda$ and $Z'_r(\lambda) \cong \operatorname{Hom}_k(U_\zeta(U_r^+), \lambda)$. More generally, if M is a $U_\zeta(B_r)$-module, then

$$Z'_r(M) = \operatorname{ind}_{U_\zeta(B_r)}^{U_\zeta(G_r)}(M) = \operatorname{Hom}_{U_\zeta(B_r)}(U_\zeta(G_r), M) \cong \operatorname{Hom}_k(U_\zeta(U_r^+), M).$$

The last isomorphism is an isomorphism of vector spaces. In particular, $Z'_r(-)$ is exact. The discussion of the previous paragraph now implies the following lemma:

Proposition B.6. Let $\lambda \in X$.

(1) As a module for $U_\zeta(B_r)$, $Z_r(\lambda)$ is the projective cover of λ and the injective hull of $\lambda - 2(p^r\ell - 1)\rho$.

(2) As a module for $U_\zeta(B_r^+)$, $Z'_r(\lambda)$ is the injective hull of λ and the projective cover of $\lambda - 2(p^r\ell - 1)\rho$.

The module $Z_r(\lambda)^*$ has highest weight $2(p^r\ell - 1)\rho - \lambda$. Then there exists a $U_\zeta(B_r^+)$-module homomorphism $2(p^r\ell - 1)\rho - \lambda \to Z_r(\lambda)^*$, hence a $U_\zeta(G_r)$-module homomorphism $\varphi : Z_r(2(p^r\ell - 1)\rho - \lambda) \to Z_r(\lambda)^*$. Considered as a module for $U_\zeta(B_r)$, $Z_r(\lambda)^*$ is the projective cover of $2(p^r\ell - 1)\rho - \lambda)$ by Lemma B.3. Then the map φ must be surjective. Since the domain and range of φ are each of the same finite dimension $(p^r\ell)^N$, $N = |\Phi^+|$, the map φ must be an isomorphism of $U_\zeta(G_r)$-modules. Similarly, the natural $U_\zeta(B_r)$-module homomorphism $Z'_r(\lambda)^* \to 2(p^r\ell - 1)\rho - \lambda$ induces an isomorphism of $U_\zeta(G_r)$-modules $Z'_r(\lambda)^* \cong Z'_r(2(p^r\ell - 1)\rho - \lambda)$. We have proved:

Lemma B.7. Let $\lambda \in X$. Then there exist isomorphisms of $U_\zeta(G_r)$-modules

$$Z_r(\lambda)^* \cong Z_r(2(p^r\ell - 1)\rho - \lambda) \qquad \text{and}$$
$$Z'_r(\lambda)^* \cong Z'_r(2(p^r\ell - 1)\rho - \lambda).$$

Lemma B.4 and Proposition B.6 imply that $Z_r(\lambda)^{U_\zeta(U_r)}$ and $Z'_r(\lambda)^{U_\zeta(U_r^+)}$ are each one-dimensional, hence that $Z_r(\lambda)$ and $Z'_r(\lambda)$ each have simple socle when considered as a $U_\zeta(G_r)$-module. Dualizing using Lemma B.7, we conclude that $Z_r(\lambda)$ and $Z'_r(\lambda)$ each have simple head when considered as a $U_\zeta(G_r)$-module.

Set $L_{\zeta,r}(\lambda) = \operatorname{soc}_{U_\zeta(G_r)} Z'_r(\lambda)$.

Proposition B.8. (cf. [37, Proposition II.3.10]) Let $\lambda \in X$. Then

$$L_{\zeta,r}(\lambda)^{U_\zeta(U_r^+)} \cong \lambda, \tag{B.0.1}$$

$$Z_r(\lambda)/\operatorname{rad}_{U_\zeta(G_r)} Z_r(\lambda) \cong L_{\zeta,r}(\lambda), \tag{B.0.2}$$

$$\operatorname{End}_{U_\zeta(G_r)}(L_{\zeta,r}(\lambda)) \cong k. \tag{B.0.3}$$

Each simple $U_\zeta(G_r)$-module is isomorphic to exactly one $L_{\zeta,r}(\lambda)$ with $\lambda \in X_{p^r\ell}$.

Let $\lambda \in X_{p^r\ell}$. Since $L_\zeta(\lambda)^{U_\zeta(U^+)} = L_\zeta(\lambda)^{U_\zeta(U_r^+)} \cong \lambda$, and since $L_\zeta(\lambda)$ is simple as a module for $U_\zeta(G_r)$, we conclude that $L_{\zeta,r}(\lambda) \cong L_\zeta(\lambda)$ as $U_\zeta(G_r)$-modules. This proves the claim that every simple $U_\zeta(G_r)$-module lifts to a simple $U_\zeta(\mathfrak{g})$-module.

Given $\lambda \in X_{p^r\ell}$, let $Q_{\zeta,r}(\lambda)$ denote the injective hull of $L_{\zeta,r}(\lambda)$ as a $U_\zeta(G_r)$-module. It is also the projective cover of $L_{\zeta,r}(\lambda)$ as a $U_\zeta(G_r)$-module, cf. [37, II.11.5]. Arguing along the lines of [37, II.11.4], we get:

Proposition B.9. Let $\lambda \in X_{p^r\ell}$. The $U_\zeta(G_r)$-module $Q_{\zeta,r}(\lambda)$ admits a filtration of the form $0 = M_0 \subset M_1 \subset \cdots \subset M_s = Q_{\zeta,r}(\lambda)$ such that each factor has the form $M_i/M_{i-1} \cong Z_r(\lambda_i)$ for some $\lambda_i \in X_{p^r\ell}$. For each $\mu \in X_{p^r\ell}$, the number of i ($1 \le i \le s$) with $\lambda_i = \mu$ is equal to $[Z_r(\mu) : L_{\zeta,r}(\lambda)]$.

Recall $\mathrm{St}_{p^r\ell} = L_\zeta((p^r\ell - 1)\rho) \cong L_{\zeta,r}((p^r\ell - 1)\rho)$. We have

$$\mathrm{St}_{p^r\ell} = \operatorname{soc}_{U_\zeta(G_r)} Z_r'((p^r\ell-1)\rho) \cong Z_r((p^r\ell-1)\rho)/\operatorname{rad}_{U_\zeta(G_r)} Z_r((p^r\ell-1)\rho).$$

Again arguing as in the classical situation, we get:

Proposition B.10. (cf. [37, II.3.18]) As modules for $U_\zeta(G_r)$,

$$\mathrm{St}_{p^r\ell} \cong Z_r((p^r\ell-1)\rho) \cong Z_r'((p^r\ell-1)\rho).$$

Applying Proposition B.9, we immediately get:

Corollary B.11. $\mathrm{St}_{p^r\ell} \cong Q_{\zeta,r}((p^r\ell-1)\rho)$. It is injective and projective for $U_\zeta(G_r)$.

Bibliography

[1] Henning Haahr Andersen, *The strong linkage principle for quantum groups at roots of 1*, J. Algebra **260** (2003), no. 1, 2–15, Special issue celebrating the 80th birthday of Robert Steinberg.

[2] Henning Haahr Andersen and Jens Carsten Jantzen, *Cohomology of induced representations for algebraic groups*, Math. Ann. **269** (1984), no. 4, 487–525.

[3] Henning Haahr Andersen and Wen Kexin, *Representations of quantum algebras. The mixed case*, J. Reine Angew. Math. **427** (1992), 35–50.

[4] Henning Haahr Andersen, Patrick Polo, and Wen Kexin, *Injective modules for quantum algebras*, Amer. J. Math. **114** (1992), no. 3, 571–604.

[5] Henning Haahr Andersen, Patrick Polo, and Ke Xin Wen, *Representations of quantum algebras*, Invent. Math. **104** (1991), no. 1, 1–59.

[6] Sergey Arkhipov, Roman Bezrukavnikov, and Victor Ginzburg, *Quantum groups, the loop Grassmannian, and the Springer resolution*, J. Amer. Math. Soc. **17** (2004), no. 3, 595–678.

[7] D. W. Barnes, *Spectral sequence constructors in algebra and topology*, Mem. Amer. Math. Soc. **53** (1985), no. 317, viii+174.

[8] Christopher P. Bendel, *Project II: Vanishing of line bundle cohomology*, http://faculty.uwstout.edu/bendelc/research/student.html, April 2009.

[9] Christopher P. Bendel, Daniel K. Nakano, Brian J. Parshall, and Cornelius Pillen, *Cohomology of quantum groups via the geometry of the nullcone*, (submitted), 2009.

[10] Armand Borel, *Properties and linear representations of Chevalley groups*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Mathematics, Vol. 131, Springer, Berlin, 1970, pp. 1–55.

[11] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley.

[12] Michel Brion and Shrawan Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, vol. 231, Birkhäuser Boston Inc., Boston, MA, 2005.

[13] Jon F. Carlson, Zongzhu Lin, Daniel K. Nakano, and Brian J. Parshall, *The restricted nullcone*, Combinatorial and geometric representation theory (Seoul, 2001), Contemp. Math., vol. 325, Amer. Math. Soc., Providence, RI, 2003, pp. 51–75.

[14] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956.

[15] R. W. Carter, *Lie algebras of finite and affine type*, Cambridge Studies in Advanced Mathematics, vol. 96, Cambridge University Press, Cambridge, 2005.

[16] W. Chin and L. Krop, *Quantized hyperalgebras of rank 1*, Israel J. Math. **145** (2005), 193–219.

[17] A.L. Christophersen, *A classification of the normal nilpotent varieties for groups of type E_6*, Ph.D. thesis, University of Aarhus, 2006.

[18] Edward Cline, Brian Parshall, and Leonard Scott, *Induced modules and affine quotients*, Math. Ann. **230** (1977), no. 1, 1–14.

[19] Charles W. Curtis and Irving Reiner, *Representation theory of finite groups and associative algebras*, AMS Chelsea Publishing, Providence, RI, 2006, Reprint of the 1962 original.

[20] C. De Concini and C. Procesi, *Quantum groups*, D-modules, representation theory, and quantum groups (Venice, 1992), Lecture Notes in Math., vol. 1565, Springer, Berlin, 1993, pp. 31–140.

[21] Corrado De Concini and Victor G. Kac, *Representations of quantum groups at roots of* 1, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progr. Math., vol. 92, Birkhäuser Boston, Boston, MA, 1990, pp. 471–506.

[22] Michel Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287–301.

[23] V. G. Drinfel'd, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR **283** (1985), no. 5, 1060–1064.

[24] Jie Du and Leonard Scott, *Lusztig conjectures, old and new. I*, J. Reine Angew. Math. **455** (1994), 141–182.

[25] Leonard Evens, *The cohomology of groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1991, Oxford Science Publications.

[26] E. M. Friedlander and B. J. Parshall, *On the cohomology of algebraic and related finite groups*, Invent. Math. **74** (1983), no. 1, 85–117.

[27] Eric M. Friedlander and Brian J. Parshall, *Cohomology of infinitesimal and discrete groups*, Math. Ann. **273** (1986), no. 3, 353–374.

[28] ———, *Cohomology of Lie algebras and algebraic groups*, Amer. J. Math. **108** (1986), no. 1, 235–253 (1986).

[29] Eric M. Friedlander and Andrei Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), no. 2, 209–270.

[30] Victor Ginzburg and Shrawan Kumar, *Cohomology of quantum groups at roots of unity*, Duke Math. J. **69** (1993), no. 1, 179–198.

[31] I. G. Gordon, *Cohomology of quantized function algebras at roots of unity*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 337–359.

[32] W. J. Guerrier, *The factorization of the cyclotomic polynomials* mod p, Amer. Math. Monthly **75** (1968), 46.

[33] Charles Hague, private communication, February 2009.

[34] Wim H. Hesselink, *Characters of the nullcone*, Math. Ann. **252** (1980), no. 3, 179–182.

[35] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York, 1978, Second printing, revised.

[36] Jens Carsten Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, vol. 6, American Mathematical Society, Providence, RI, 1996.

[37] ———, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003.

[38] Michio Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), no. 1, 63–69.

[39] V. Kac and B. Weisfeiler, *Coadjoint action of a semi-simple algebraic group and the center of the enveloping algebra in characteristic* p, Nederl. Akad. Wetensch. Proc. Ser. A **79**=Indag. Math. **38** (1976), no. 2, 136–151.

[40] David Kazhdan and Mikhail Verbitsky, *Cohomology of restricted quantized universal enveloping algebras*, Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), Israel Math. Conf. Proc., vol. 7, Bar-Ilan Univ., Ramat Gan, 1993, pp. 107–115.

[41] Markku Koppinen, *Decomposing and lifting hyperalgebras*, Ann. Univ. Turku. Ser. A I (1983), no. 184, 97.

[42] Bertram Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[43] Shrawan Kumar, Niels Lauritzen, and Jesper Funch Thomsen, *Frobenius splitting of cotangent bundles of flag varieties*, Invent. Math. **136** (1999), no. 3, 603–621.

[44] Richard Gustavus Larson and Moss Eisenberg Sweedler, *An associative orthogonal bilinear form for Hopf algebras*, Amer. J. Math. **91** (1969), 75–94.

[45] Gail Letzter, *Coideal subalgebras and quantum symmetric pairs*, New directions in Hopf algebras, Math. Sci. Res. Inst. Publ., vol. 43, Cambridge Univ. Press, Cambridge, 2002, pp. 117–165.

[46] Serge Levendorskii and Yan Soibelman, *Algebras of functions on compact quantum groups, schubert cells and quantum tori*, Comm. Math. Phys. **139** (1991), no. 1, 141–170.

[47] G. Lusztig, *Modular representations and quantum groups*, Classical groups and related topics (Beijing, 1987), Contemp. Math., vol. 82, Amer. Math. Soc., Providence, RI, 1989, pp. 59–77. MR MR982278 (90a:16008)

[48] George Lusztig, *Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra*, J. Amer. Math. Soc. **3** (1990), no. 1, 257–296.

[49] ______, *Quantum groups at roots of* 1, Geom. Dedicata **35** (1990), no. 1-3, 89–113.

[50] Saunders Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press Inc., Publishers, New York, 1963.

[51] M. Mastnak, J. Pevtsova, P. Schauenburg, and S. Witherspoon, *Cohomology of finite dimensional pointed Hopf algebras*, http://arxiv.org/abs/0902.0801, January 2009.

[52] John McCleary, *A user's guide to spectral sequences*, second ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001.

[53] Susan Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.

[54] M. Scott Osborne, *Basic homological algebra*, Graduate Texts in Mathematics, vol. 196, Springer-Verlag, New York, 2000.

[55] Brian Parshall and Jian Pan Wang, *Quantum linear groups*, Mem. Amer. Math. Soc. **89** (1991), no. 439, vi+157.

[56] Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.

[57] Steen Ryom-Hansen, *A q-analogue of Kempf's vanishing theorem*, Mosc. Math. J. **3** (2003), no. 1, 173–187, 260.

[58] T. A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Mathematics, Vol. 131, Springer, Berlin, 1970, pp. 167–266.

[59] Robert Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968, Notes prepared by John Faulkner and Robert Wilson.

[60] John Brendan Sullivan, *Simply connected groups, the hyperalgebra, and Verma's conjecture*, Amer. J. Math. **100** (1978), no. 5, 1015–1019.

[61] Jesper Funch Thomsen, *Normality of certain nilpotent varieties in positive characteristic*, J. Algebra **227** (2000), no. 2, 595–613.

[62] Wilberd van der Kallen, *A reductive group with finitely generated cohomology algebras*, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007, pp. 301–314.

[63] F. D. Veldkamp, *The center of the universal enveloping algebra of a Lie algebra in characteristic p*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 217–240.

[64] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.