Some $\mathbb{Z}/2\mathbb{Z}$-graded analogues of one-parameter subgroups and applications to the cohomology of $GL_{m|n(r)}$

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This should appear on the arXiv in the very near future...

Throughout, we'll work over an algebraically closed field k of characteristic $p \geq 3$.

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Exposition: Finite Groups

Suppose G is a finite group.

Problem in modular representation theory

Describe the structure of the cohomology ring $H^\bullet(G, k) = \mathrm{Ext}_G^\bullet(k, k)$.

More generally, given a kG-module M, describe $\mathrm{Ext}_G^\bullet(M, M)$.

Cohomology groups encode representation-theoretic information.

Geometric interpretation of the problem

Describe the affine variety $|G| := \mathrm{MaxSpec}\,(H^\bullet(G, k))$.

More generally, for each kG-module M, describe the subvariety

$$|G|_M := \mathrm{MaxSpec}\,(H^\bullet(G, k)/\ker(\Phi_M))\,,$$

where $\Phi_M : \mathrm{Ext}_G^\bullet(k, k) \to \mathrm{Ext}_G^\bullet(M, M)$ is the algebra homomorphism induced by $- \otimes M$.

If $H \leq G$, then get a restriction map $\mathrm{res}_{G,H} : \mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(H,k)$, hence a map of varieties $\mathrm{res}^*_{G,H} : |H| \to |G|$.

Theorem (Quillen, 1971)

$$|G| = \bigcup_E \mathrm{res}^*_{G,E}(|E|)$$

union is taken over the elementary abelian p-subgroups of G.

More precisely, $|G| = \bigsqcup V^+_{G,E}$, where the union is taken over the conjugacy classes of elementary abelian p-subgroups of G.

$V^+_{G,E}$ identifies up to inseparable isogeny with $|E|\,/W_G(E)$, where $W_G(E) = N_G(E)/C_G(E)$.

Theorem (Avrunin and Scott, 1982)

There exists an analogous stratification of $|G|_M$ for each f.d. kG-module M. If E is an elementary abelian p-group, then $|E|_M$ identifies with **Carlson's rank variety** $V_E(M)$.

Cohomology of elementary abelian p-groups

$E = (\mathbb{Z}/p\mathbb{Z})^{\times r}$ elementary abelian p-group of rank r

$$kE = k[g_1, \ldots, g_r]/\langle g_1^p - 1, \ldots, g_r^p - 1\rangle \cong k[z_1, \ldots, z_r]/\langle z_1^p, \ldots, z_r^p\rangle$$

Isomorphism of algebras via the identification $z_i = g_i - 1$.

Cohomology of E

$$\mathrm{H}^\bullet(E, k) \cong k[x_1, \ldots, x_r] \otimes \Lambda(\lambda_1, \ldots, \lambda_r)$$

where $\deg(x_i) = 2$, and $\deg(\lambda_i) = 1$. In particular,

$$|E| = \mathrm{MaxSpec}\left(\mathrm{H}^\bullet(E, k)\right) \cong \mathbb{A}^r.$$

Carlson's Rank Variety

$$V_E(M) = \left\{v = \textstyle\sum_{i=1}^r a_i(g_i - 1) : M|_{\langle 1+v\rangle} \text{ is not free}\right\} \cup \{0\}$$

Rising Action: Generalizations

Restricted Lie algebras (RLAs)

$\mathfrak{g}$: finite-dimensional restricted Lie algebra with p-map $x \mapsto x^{[p]}$

$V(\mathfrak{g})$: restricted enveloping algebra (f.d. cocommutative Hopf algebra)

Friedlander–Parshall (1980s), Suslin–Friedlander–Bendel (1997)

There exist homeomorphisms

$$|V(\mathfrak{g})| \cong \left\{X \in \mathfrak{g} : X^{[p]} = 0\right\}$$

$$|V(\mathfrak{g})|_M \cong \left\{X \in \mathfrak{g} : X^{[p]} = 0 \text{ and } M|_{\langle X\rangle} \text{ is not free}\right\} \cup \{0\}\,.$$

Varieties again determined by restrictions to certain simpler (cyclic) sub-objects.

Group schemes

Antiequivalence

finite group scheme $G \leftrightarrow$ f.d. commutative Hopf algebra $k[G]$

A finite group scheme G is **infinitesimal** if the augmentation ideal of $k[G]$ is nilpotent.

If G is a finite group scheme, then the dual Hopf algebra $k[G]^*$ is denoted kG, and called the **group algebra**.

Examples

- If G is an ordinary finite group, then the group algebra kG is the group algebra of a finite group scheme.
- If $\mathfrak{g}$ is a finite-dimensional restricted Lie algebra, then $V(\mathfrak{g})$ is the group algebra of an infinitesimal (height one) group scheme.

Examples of infinitesimal group schemes

$GL_{n(r)}$, r-th Frobenius kernel of GL_n

Given a commutative algebra A,

$$GL_{n(r)}(A) = \left\{ (a_{ij}) \in GL_n(A) : a_{ij}^{p^r} = \delta_{ij} \right\}$$

$\mathbb{G}_{a(r)}$, r-th Frobenius kernel of the additive group scheme $\mathbb{G}_a$

$$k[\mathbb{G}_a] = k[T]$$

$$k[\mathbb{G}_{a(r)}] = k[T]/\langle T^{p^r} \rangle$$

So given a commutative algebra A, $\mathbb{G}_{a(r)}(A) = \left\{ a \in A : a^{p^r} = 0 \right\}$.

$$k\mathbb{G}_{a(r)} = k[u_0, \ldots, u_{r-1}]/\langle u_0^p, \ldots, u_{r-1}^p \rangle$$

Infinitesimal one-parameter subgroups

Given an affine group scheme G, the scheme $V_r(G)$ of infinitesimal one-parameter subgroups $\nu : \mathbb{G}_{a(r)} \to G$ is defined by

$$V_r(G)(A) = \mathrm{Hom}_{Grp/A}(\mathbb{G}_{a(r)} \otimes_k A, G \otimes_k A).$$

Theorem (Suslin–Friedlander–Bendel, 1997)

If G is infinitesimal of height $\leq r$, then there is a homeomorphism

$$|G| \cong V_r(G)(k) = \mathrm{Hom}_{Grp/k}(\mathbb{G}_{a(r)}, G).$$

In particular,

$$\left|GL_{n(r)}\right| \cong \left\{(\alpha_0, \ldots, \alpha_{r-1}) \in M_n(k)^{\times r} : \alpha_i^p = 0 \text{ and } [\alpha_i, \alpha_j] = 0\right\}.$$

More generally, SFB describe $|G|_M$ in terms of restriction of M to $k[u_{r-1}]/\langle u_{r-1}^p\rangle \subset k[\mathbb{G}_{a(r)}]$ along homomorphisms $\nu : \mathbb{G}_{a(r)} \to G$ (must also consider scalar extensions).

Can this be generalized to supergroup schemes?

Something is "super" if it has a compatible $\mathbb{Z}/2\mathbb{Z}$-grading.

$V \otimes W \cong W \otimes V$ via the **supertwist** $v \otimes w \mapsto (-1)^{\overline{v} \cdot \overline{w}} w \otimes v$.

Define (Hopf) superalgebras and 'super' (co)commutativity in terms of the "usual diagrams," but use the supertwist when objects pass.

Super correspondences

finite supergroup scheme G

$$\updownarrow$$

f.d. (super)commutative Hopf superalgebra $k[G]$

$$\updownarrow$$

f.d. (super)cocommutative Hopf superalgebra $kG = (k[G])^*$

Hopf superalgebras

Examples of Hopf superalgebras

- Ordinary Hopf algebras (as purely even superalgebras).
- $\mathbb{Z}$-graded Hopf algebras in the sense of Milnor and Moore
- Enveloping superalgebras of (restricted) Lie superalgebras

An exterior algebra $\Lambda(V)$ is a (super)commutative superalgebra:

$$ab = (-1)^{\bar{a}\cdot\bar{b}} ba \quad \text{in } \Lambda(V) \text{ if } a, b \in V.$$

It is also a (super)cocommutative Hopf superalgebra.

Conflict: A confounding example

Things seem to be more complicated, even if you only care about restricted Lie superalgebras (height-one infinitesimal supergroups).

Cautionary example

$\mathfrak{g}$ restricted Lie superalgebra (RLSA) generated by even element u and odd element v such that $V(\mathfrak{g}) = k[u, v]/\langle u^p, v^2\rangle$.

The sub-RLSAs of $\mathfrak{g}$ are k, $k.u$, $k.v$, and $\mathfrak{g}$.

Define M to be the $\mathfrak{g}$-supermodule with homogeneous basis

$$\{x_0, \ldots, x_{p-1}, y_0, \ldots, x_{p-1}\} \quad x_i \text{ even}, \quad y_i \text{ odd},$$

such that $u.x_i = x_{i+1}$, $u.y_i = y_{i+1}$, $v.x_i = y_{i+1}$, and $v.y_i = x_{i+p-1}$.

Then M is projective over all proper RLSAs of $\mathfrak{g}$, but not over $\mathfrak{g}$.

Need more than just cyclic subalgebras to detect projectivity ...

Multiparameter supergroups

Let $f = T^{p^t} + \sum_{i=1}^{t-1} a_i T^{p^i} \in k[T]$ be a p-polynomial (no linear term).

Let $\eta \in k$ be a scalar.

Definition of the multiparameter infinitesimal supergroup $\mathbb{M}_{r;f,\eta}$

Define $\mathbb{M}_{r;f,\eta}$ by specifying its group algebra.

$$k\mathbb{M}_{r;f,\eta} = k[u_0, \ldots, u_{r-1}, v]/\langle u_0^p, \ldots, u_{r-2}^p, u_{r-1}^p + v^2, f(u_{r-1}) + \eta u_0 \rangle$$

$u_0, \ldots, u_{r-1}$ are even; their coproducts look like they do in $k\mathbb{G}_{a(r)}$.

v is an odd primitive generator.

Our multiparameter supergroups are a **family** of potential replacements for $\mathbb{G}_{a(r)}$ when trying to apply the SFB setup to infinitesimal supergroup schemes.

Cohomology algebras

$$\mathrm{H}^\bullet(\mathbb{M}_{r;T^p,0}, k) \cong k[x_1, \ldots, x_r, y] \,{}^g\!\otimes \Lambda(\lambda_1, \ldots, \lambda_r),$$

with $\deg(x_i) = 2$, $\deg(y) = \deg(\lambda_i) = 1$, $\overline{x_i} = \overline{\lambda_i} = \overline{0}$, and $\overline{y} = \overline{1}$.

If $s \geq 2$, then

$$\mathrm{H}^\bullet(\mathbb{M}_{r;T^{p^s},0}, k) \cong k[x_1, \ldots, x_r, y, w_s]/\langle x_r - y^2\rangle \,{}^g\!\otimes \Lambda(\lambda_1, \ldots, \lambda_r).$$

where $\deg(w_s) = 2$ and $\overline{w_s} = \overline{0}$.

Representations of $\mathbb{M}_{r;f,\eta}$

Let $A = A_{\overline{0}} \oplus A_{\overline{1}}$ be a commutative superalgebra.

$\mathrm{Mat}_{m|n}(A) = \mathrm{Mat}_{m|n}(A)_{\overline{0}} \oplus \mathrm{Mat}_{m|n}(A)_{\overline{1}}$

$\mathrm{Mat}_{m|n}(A)_{\overline{0}}$ identifies with the set of all block matrices

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

$T_1 \in M_{m\times m}(A_{\overline{0}})$, $T_2 \in M_{m\times n}(A_{\overline{1}})$, $T_3 \in M_{n\times m}(A_{\overline{1}})$, and $T_4 \in M_{n\times n}(A_{\overline{0}})$.

$\mathrm{Mat}_{m|n}(A)_{\overline{1}}$ identifies with block matrices with parities reversed.

Ambient scheme

$$V_r(GL_{m|n})(A) = \Big\{(\alpha_0, \ldots, \alpha_{r-1}, \beta) \in (\mathrm{Mat}_{m|n}(A)_{\bar{0}})^{\times r} \times \mathrm{Mat}_{m|n}(A)_{\bar{1}} : [\alpha_i, \alpha_j] = [\alpha_i, \beta] = 0 \text{ for all } 0 \le i, j \le r-1, \alpha_i^p = 0 \text{ for all } 0 \le i \le r-2, \text{ and } \alpha_{r-1}^p + \beta^2 = 0\Big\}.$$

Homomorphisms $\rho : \mathbb{M}_{r;f,\eta} \otimes_k A \to GL_{m|n} \otimes_k A$, or equivalently, representations of $k\mathbb{M}_{r;f,\eta} \otimes_K A$, correspond to points in

$$V_{r;f,\eta}(GL_{m|n})(A) = \left\{(\alpha_0, \ldots, \alpha_{r-1}, \beta) \in V_r(GL_{m|n})(A) : f(\alpha_{r-1}) + \eta\alpha_0 = 0\right\}.$$

Note that $V_r(GL_{m|n})(k) = \bigcup_{f,\eta} V_{r;f,\eta}(GL_{m|n})(k)$.

Climax: Applications to the cohomology of $GL_{m|n(r)}$

Following the approach of SFB

1. Explicitly calculated the images of certain "universal extension classes" under the maps in cohomology

$$\rho^*_{(\underline{\alpha}|\beta)} : \mathrm{H}^\bullet(GL_{m|n(r)}, k) \to \mathrm{H}^\bullet(\mathbb{M}_{r;f,\eta}, k)$$

   corresponding to homomorphisms $\rho_{(\underline{\alpha}|\beta)} : \mathbb{M}_{r;f,\eta} \to GL_{m|n(r)}$.
2. Then able to construct algebra homomorphisms

$$k[V_r(GL_{m|n})] \xrightarrow{\overline{\phi}} H(GL_{m|n(r)}, k) \xrightarrow{\psi_{r;f,\eta}} k[V_{r;f,\eta}(GL_{m|n})]$$

   such that $H(GL_{m|n}, k) := \mathrm{H}^{even}(G, k)_{\overline{0}} \oplus \mathrm{H}^{odd}(G, k)_{\overline{1}}$ is finite over $\overline{\phi}$.
3. Get induced morphisms of varieties

$$\Theta_{r;f,\eta} : V_{r;f,\eta}(GL_{m|n})(k) \xrightarrow{\Psi_{r;f,\eta}} \left|GL_{m|n(r)}\right| \xrightarrow{\Phi} V_r(GL_{m|n})(k)$$

   Showed that $\Theta_{r;f,\eta}$ is the Frobenius morphism composed with the natural inclusion.

$$\Theta_{r;f,\eta} : V_{r;f,\eta}(GL_{m|n})(k) \xrightarrow{\Psi_{r;f,\eta}} \left|GL_{m|n(r)}\right| \xrightarrow{\Phi} V_r(GL_{m|n})(k)$$

Since the $V_{r;f,\eta}(GL_{m|n})(k)$ cover $V_r(GL_{m|n})(k)$, we get

Corollary

$$V_r(GL_{m|n})(k) = \bigcup \mathrm{im}(\Theta_{r;f,\eta}) = \bigcup V_{r;f,\eta}(GL_{m|n})(k)$$

In particular Φ is a finite surjective map.

Some kind of analogue for $GL_{m|n}$ of the Quillen stratification.

Falling Action and Denouement: Open Problems

Open Problems

Problem

What is the correct coordinate-free approach?

— SFB looked at $\mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, G)$

— $\mathrm{Hom}_{Grp}(\mathbb{M}_{r;T^{p^s},0}, \mathbb{M}_{r;T^{p^s},0})$ already seems too big

Problem

Can you detect projectivity of modules, or nilpotence of cohomology classes, by looking at restrictions to multiparameter supergroups...