1- and 2- cohomology for algebraic groups and finite groups of Lie type

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1. Notation and preliminaries
2. Summary of some earlier work and calculations
   - Results that do not attempt a direct comparison to G-cohomology
   - Results that attempt a comparison to G-cohomology
3. Understanding restriction from G to $G(\mathbb{F}_q)$
   - Induction functors, long exact sequence for restriction, filtrations
   - Results for 1- and 2-cohomology
4. Some examples and open questions for symplectic groups

- G - simple, simply-connected algebraic group scheme over $k = \overline{\mathbb{F}}_p$
- $B = T \ltimes U$ - "Borus"
- $\Phi \supset \Phi^+ \supset \Delta$ - root system, positive subsystem, simple roots
- W - Weyl group
- $X(T) \supset X(T)_+$ - weight lattice, subset of dominant weights
- $F : G \to G$ - Frobenius morphism
- $G(\mathbb{F}_q) = G^{F^r}$ - finite subgroup of $\mathbb{F}_q$-rational points in G, $q = p^r$
- $B(\mathbb{F}_q)$, $T(\mathbb{F}_q)$, $U(\mathbb{F}_q)$ - finite subgroups of B, T, U
- $G_r = \ker(F^r : G \to G)$ - r-th Frobenius kernel of G

e.g.,

- $G = SL_n$
- $G(\mathbb{F}_q) = SL_n(\mathbb{F}_q)$
- B, T - lower triangular, diagonal matrices in G
- $F((a_{ij})) = (a_{ij}^p)$

We have various rational G-modules associated to each $\lambda \in X(T)_+$:

- $H^0(\lambda) = \operatorname{ind}_B^G(\lambda)$ - induced module
- $V(\lambda) = H^0(-w_0\lambda)^*$ - Weyl module
- $L(\lambda) = \operatorname{soc}_G H^0(\lambda) = V(\lambda)/\operatorname{rad}_G V(\lambda)$ - irreducible module

Facts:

- For all $n \geq 0$, $\mathrm{H}^n(G(\mathbb{F}_q), V) \hookrightarrow \mathrm{H}^n(B(\mathbb{F}_q), V) = \mathrm{H}^n(U(\mathbb{F}_q), V)^{T(\mathbb{F}_q)}$.
- Set $X_r(T) = \{\lambda \in X(T)_+ : 0 \leq (\lambda, \alpha^\vee) < p^r \text{ for all } \alpha \in \Delta\}$.
  The $L(\lambda)$ for $\lambda \in X_r(T)$ form a complete set of pairwise nonisomorphic irreducible $G(\mathbb{F}_q)$-modules (and similarly for G_r).
- $\operatorname{Ext}^i_G(V(\lambda), H^0(\mu)) \neq 0$ only if $i = 0$ and $\lambda = \mu$.

Goal

Given $\lambda \in X_r(T)$, compute $\mathrm{H}^1(G(\mathbb{F}_q), L(\lambda))$ and $\mathrm{H}^2(G(\mathbb{F}_q), L(\lambda))$.

Subgoals (i.e., what people have actually managed to do):

- Compute for $L(\lambda)$ in various classes of modules.
- Determine sufficient conditions for the cohomology groups to vanish.
- Compute under restrictions on p and q (specific small values, or $\gg 0$).

Cline, Parshall, Scott (1975, 1977), Jones (1975)

Computed $\mathrm{H}^1(G(\mathbb{F}_q), L(\lambda))$ for λ a minimal nonzero dominant weight, i.e., for λ a minuscule weight or a maximal short root.

- No restrictions on p or q.
- Included the twisted groups of Steinberg, Ree, and Suzuki.
- Lower bound: $\dim \operatorname{rad}_G V(\lambda) \leq \dim \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda))$
- Upper bound:

$$\sum_{\alpha \in \Delta} \dim Z^1(U_\alpha(\mathbb{F}_q), V)^{T(\mathbb{F}_q)} - \dim V^{T(\mathbb{F}_q)} + \dim V^{B(\mathbb{F}_q)}.$$

- Requires analyzing whether weights of V are Galois equivalent to roots, i.e., whether $\sigma \circ \omega|_{T(\mathbb{F}_q)} = \beta|_{T(\mathbb{F}_q)}$ for some $\sigma \in \mathrm{Gal}(\mathbb{F}_q)$.

G_σ	char k = p	Dominant Weight	$\dim_{\bar{k}} V$	$\dim_{\bar{k}} H^1(G_\sigma, V)$
$A_n(q)$	arbitrary	λ_i, $1 \leqslant i \leqslant n$	$\binom{n+1}{i}$	0 (*)
	$p \nmid n+1$	$\lambda_1 + \lambda_n$	$n(n+2)$	0 (*)
	$p \mid n+1$	$\lambda_1 + \lambda_n$	$n(n+2)-1$	1 (*)
$^2A_n(q)$ $q>3$	arbitrary	λ_i, $1 \leqslant i \leqslant n$	$\binom{n+1}{i}$	0
	$p \nmid n+1$	$\lambda_1 + \lambda_n$	$n(n+2)$	0
	$p \mid n+1$	$\lambda_1 + \lambda_n$	$n(n+2)-1$	1
$B_n(q)$ $n \geqslant 3$	arbitrary	λ_1	2^n	0
	2	λ_n	$2n$	1
	$\neq 2$	λ_n	$2n+1$	0
$C_n(q)$ $n \geqslant 2$	2	λ_n	$2n$	1
	$\neq 2$	λ_n	$2n$	0
	$p \mid n$	λ_{n-1}	$(n-1)(2n+1)-1$	1
	$p \nmid n$	λ_{n-1}	$(n-1)(2n+1)$	0 (*)
$^2C_2(2^{2n+1})$	2	λ_1, λ_2	4	1 (*)

(Cont.)

$D_n(q)$ $n>3$	arbitrary	$\lambda_i, i=1,n-1,n$	$2n$ if $i=1$; 2^{n-1} if $i \neq 1$	0
	$\neq 2$	λ_2	$(2n-1)n$	0
$D_{2n}(q)$	2	λ_2	$2n(4n-1)-2$	2
$D_{2n+1}(q)$	2	λ_2	$(2n+1)(4n+1)-1$	0
$^2D_n(q)$ $n>3$, $q>3$	arbitrary	$\lambda_i, i=1,n-1,n$	$2n$ if $i=1$; 2^{n-1} if $i \neq 1$	0
$^2D_{2n}(q)$ $n>1$, $q>3$	2	λ_2	$2n(4n-1)-2$	2
	$\neq 2$	λ_2	$2n(4n-1)$	0
$^2D_{2n+1}(q)$ $q>3$	2	λ_2	$(2n+1)(4n+1)-1$	1
	$\neq 2$	λ_2	$(2n+1)(4n+1)$	0
$^3D_4(q)$ $q>3$	arbitrary	$\lambda_i, i=1,2,4$	8	0
	2	λ_2	26	2
	$\neq 2$	λ_2	28	0
$E_6(q)$	arbitrary	λ_1, λ_6	27	0
	3	λ_2	77	1
	$\neq 3$	λ_2	78	0

(Cont.)

$^2E_6(q)$ $q>3$	arbitrary	λ_1, λ_6	27	0
	3	$\lambda_{[illegible]}$	77	1
	≠3	λ_2	78	0
$E_7(q)$	arbitrary	λ_7	56	0
	2	λ_1	132	1
	≠2	λ_1	133	0
$E_8(q)$	arbitrary	λ_8	248	0
$F_4(q)$	3	λ_4	25	1
	≠3	λ_4	26	0
$^2F_4(2^{2n+1})$, $n \neq 0, 1$	2	λ_4	26	0
$G_2(q)$	2	λ_2	6	1
	≠2	λ_2	7	0
$^2G_2(3^{2n+1})$	3	λ_2	7	0 (*)

C. Exceptions to the above table.

(a) $\dim_k H^1(A_1(2^n), V(\lambda_1)) = 1$ for $n > 1$

(b) $\dim_k H^1(A_2(2), V(\lambda_i)) = 1$ for $i = 1,2$

(c) $\dim_k H^1(A_3(2), V(\lambda_2)) = 1$

(d) $\dim_k H^1(A_1(5), V(2\lambda_1)) = 1$

(e) $\dim_k H^1(A_1(2), V(2\lambda_1)) = 0$

(f) $\dim_k H^1(C_2(3), V(\lambda_1)) = 1$

(g) $\dim_k H^1(^2C_2(2), V(\lambda_i)) = 0$, $i = 1,2$

(h) $\dim_k H^1(^2G_2(3), V(\lambda_2)) = 1$

Avrunin (1978)

Suppose for all weights μ of $T(\mathbb{F}_q)$ in V and for all $\alpha, \beta \in \Phi$ that $\alpha \not\equiv \mu$ and $(\alpha, \beta) \not\equiv \mu$ mod $\mathrm{Gal}(\mathbb{F}_q)$. Then $\mathrm{H}^2(G(\mathbb{F}_q), V) = 0$.

- Look at a central series for $U(\mathbb{F}_q)$ where the factors are products of root subgroups to analyze the weights of $T(\mathbb{F}_q)$ in $\mathrm{H}^2(U(\mathbb{F}_q), V)$.
- Use this to deduce that $\mathrm{H}^2(U(\mathbb{F}_q), V)^{T(\mathbb{F}_q)} = 0$.
- Now use the fact that $\mathrm{H}^2(G(\mathbb{F}_q), V) \hookrightarrow \mathrm{H}^2(B(\mathbb{F}_q), V)$.

Corollary (Avrunin)

Suppose $q > 3$. Let $\lambda \in X(T)_+$ be minuscule. Then $\mathrm{H}^2(G(\mathbb{F}_q), L(\lambda)) = 0$, except possibly in the cases shown on the next slide.

G	q	λ
$A_1(q)$	2^k	λ_1
$A_2(q)$	$5, 3^k$	λ_1, λ_2
$A_3(q)$	2^k	λ_2
$A_n(q), n \geqslant 3$	4	λ_3, λ_{n-2}
$B_3(q)$	2^k	λ_1
$B_4(q)$	2^k	λ_1
$C_n(q)$	2^k	λ_n
$D_3(q)$	4	λ_1, λ_2
$D_4(q)$	2^k	λ_1, λ_2
$D_n(q), n \geqslant 3$	2^k	λ_n
${}^2A_2(q^2)$	$4, 3^k$	λ_1, λ_2
${}^2A_3(q^2)$	2^k	λ_2
${}^2A_3(q^2)$	4	λ_1, λ_3
${}^2C_2(q^2)$	$2^{2k+1/2}$	λ_2
${}^2D_3(q^2)$	4	λ_1, λ_2
${}^2D_4(q^2)$	2^k	λ_1, λ_2
${}^2D_n(q^2), n \geqslant 3$	2^k	λ_n
${}^3D_4(q^3)$	2^k	$\lambda_1, \lambda_2, \lambda_4$
${}^2E_6(q^2)$	4	λ_1, λ_6

A few of these possibilities are, in fact, not exceptions. In unpublished work, McLaughlin has shown that the cohomology groups vanish in the cases above where G is $A_1(4)$, $A_3(4)$ with $\lambda = \lambda_1$ or λ_3, or $D_3(4)$ with $\lambda = \lambda_1$ or λ_2. Landázuri [9] has shown that $H^2(A_n(4), V(\lambda)) = 0$ for $\lambda = \lambda_3$ or λ_{n-2}, and the author [2] has shown that $H^2({}^2A_3(16), V(\lambda)) = 0$ for $\lambda = \lambda_1$ or λ_3.

Nonzero cohomology is known in some of these cases. McLaughlin has shown that the cohomology groups are nontrivial when G is $A_1(2^k)$ with $k > 2$, $A_2(3^k)$ with $k > 1$ or $A_2(5)$. Landázuri proved in [9] that $H^2(B_n(2^k), V(\lambda_1)) \neq 0$ for $n = 3, 4$ and $k \geqslant 2$. Bell has computed the second degree cohomology of the Suzuki groups on $V(\lambda_2)$ in [3]; this is nonzero if $q^2 > 8$. Also, it follows from work of Griess [8] that the second degree cohomology groups of $C_n(2^k)$, $D_n(2^k)$, and ${}^2D_n(2^{2k})$ on $V(\lambda_n)$ are nonzero. Finally, the author [2] has shown that $H^2({}^2A_2(q^2), V(\lambda_i)) \neq 0$ for $i = 1, 2$ and $q^2 = 16$ or 3^{2k}.

Bell (1978)

Computed $\text{Ext}^1_{SL_{n+1}(\mathbb{F}_q)}(V_i^\sigma, V_j^\tau)$ for all $1 \leq i, j \leq n$ and $\sigma, \tau \in \text{Gal}(\mathbb{F}_q)$. Here $V_i = \Lambda^i(V)$ where V is the natural representation. Have $V = L(\omega_i)$.

- Rank one calculations preformed by hand.
- For higher ranks, use an LHS spectral sequence and induction on the rank to compute for an appropriate maximal parabolic subgroup.
- When nonzero for the parabolic, explicitly construct a cocycle, and then determine whether it can be extended to all of $SL_{n+1}(\mathbb{F}_q)$.

$\dim_K H^n(SL_{l+1}(q), M)$

$M = V_i$, $1 \leqslant i \leqslant l$			
n	$\dim_K$	Exceptional (l, q, i)	Exceptional $\dim_K$
0	0	None	—
1	0	$(1, 2^s > 2, 1)$, $(2, 2, 1)$, $(2, 2, 2)$, $(3, 2, 2)$	1
2	0	$(1, 2^s > 4, 1)$, $(2, 3^s > 3, 1)$, $(2, 3^s > 3, 2)$, $(2, 5, 1)$, $(2, 5, 2)$, $(2, 2, 1)$, $(2, 2, 2)$, $(3, 2, 1)$, $(3, 2, 3)$, $(3, 2^s > 2, 2)$, $(4, 2, 1)$, $(4, 2, 4)$	1

$M = H_\sigma(V_i, V_j)$, $1 \leqslant i, j \leqslant l$, $\sigma \in \Gamma$			
n	$\dim_K$	Exceptional (l, q, σ, i, j)	Exceptional $\dim_K$
0	0	$(l, q, 1, i, i)$	1
1	0	$q = 2$ with $\{i, j\} \cap \{1, l\} = \varnothing$	?
		$(1, 3^s, \frac{1}{3}, 1, 1)$, $(1, 5, 1, 1, 1)$, $(2, 2^s \neq 4, \frac{1}{2}, 2, 1)$, $(2, 2^s \neq 4, 2, 1, 2)$, $(2, 2^s \neq 4, 2, 2, 1)$, $(2, 2^s \neq 4, \frac{1}{2}, 1, 2)$ $(l > 2, 2^s, \frac{1}{2}, 2, 1)$, $(l > 2, 2^s, 2, 1, 2)$, $(l > 2, 2^s, 2, l, l-1)$ $(l > 2, 2^s, \frac{1}{2}, l-1, l)$, $(3, 2, 1, 1, 3)$, $(3, 2, 1, 3, 1)$	1
		$(2, 4, 2, 1, 2)$, $(2, 4, 2, 2, 1)$	2

$M = H_\sigma(V_i, H_\tau(V_j, V_k))$, $1 \leqslant i, j, k \leqslant l$, $\sigma, \tau \in \Gamma$			
n	$\dim_K$	Exceptional $(l, q, \sigma, \tau, i, j, k)$	Exceptional $\dim_K$
0	0	$(1, 2, 1, 1, 1, 1, 1)$, $(l, q, 1, 1, i, j, i + j (\mathrm{mod}\ l + 1))$	1

Kleshchev (1994)

Let $\lambda \in X_r(T)$. Suppose that all weights spaces of $L(\lambda)$ are 1-dimensional. Then $\mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) = 0$ except for the cases on the next slide. In the exceptional cases, one has

$$\dim \mathrm{H}^1(A_2(4), L(3\omega_1)) = \dim \mathrm{H}^1(A_2(4), L(3\omega_2)) = 2,$$

but in all other exceptional cases $\dim \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) = 1$.

Obtains upper bound estimates depending on the composition factors of $L(\lambda)$ restricted to a a suitable parabolic subgroup. Are 1-dimensional weight spaces essential, or just a convenient class of $L(\lambda)$??

Compare with work of Bell: Dimension of H^1 can grow as λ gets large.

Group	Highest weight λ
$A_l(p^n), n>1$	$p^j(\omega_{l-1}+(p-2)\omega_l)+p^{j+1}\omega_1$, $p^j((p-2)\omega_1+\omega_2)+p^{j+1}\omega_l$, $j=0,\dots,n-1$; $\omega_1+p^{n-1}(\omega_{l-1}+(p-2)\omega_l)$, $\omega_l+p^{n-1}((p-2)\omega_1+\omega_2)$
$A_2(3^n)$	$3^j(\omega_1+\omega_2)$, $j=0,\dots,n-1$
$A_3(3^n)$	$3^j(2\omega_2)$, $j=0,\dots,n-1$
$A_2(2)$	ω_1, ω_2
$A_3(2)$	ω_2
$C_2(5^n)$	$5^j(2\omega_2)$, $j=0,\dots,n-1$
$C_2(3)$	ω_2
$C_3(3^n)$	$3^j\omega_2$, $j=0,\dots,n-1$
$C_4(3^n)$	$3^j\omega_4$, $j=0,\dots,n-1$
$C_l(2^n)$	$2^j\omega_1$, $j=0,\dots,n-1$
$C_2(2^n)$	$2^j\omega_2$, $j=0,\dots,n-1$
$F_4(3^n)$	$3^j\omega_4$, $j=0,\dots,n-1$
$G_2(3^n)$	$\omega_1+3^{n-1}\omega_2$
$G_2(2^n)$	$2^j\omega_1$, $j=0,\dots,n-1$

Want more direct comparisons between cohomology for G and $G(\mathbb{F}_q)$.

Cline, Parshall, Scott, van der Kallen (1977)

Let V be a finite-dimensional rational G-module, and let $i \in \mathbb{N}$. Then for all sufficiently large e and q, the restriction map is an isomorphism

$$\mathrm{H}^i(G, V^{(e)}) \xrightarrow{\sim} \mathrm{H}^i(G(\mathbb{F}_q), V^{(e)}).$$

Stable value of $\mathrm{H}^i(G(\mathbb{F}_q), V)$ when $q \gg 0$ is denoted $\mathrm{H}^i_{gen}(G, V)$.

$$\begin{array}{ccc} \mathrm{H}^i(G, V) & \xrightarrow{\sim} & \mathrm{H}^i(B, V) \\ \downarrow & & \downarrow \\ \mathrm{H}^i(G(\mathbb{F}_q), V) & \hookrightarrow & \mathrm{H}^i(B(\mathbb{F}_q), V). \end{array}$$

Some sharper statements for 1- and 2-cohomology:

- If $p \neq 2$, then

$$\mathrm{H}^1(G,V) \cong \mathrm{H}^1_{gen}(G,V) \quad \text{and} \quad \mathrm{H}^2(G,V^{(1)}) \cong \mathrm{H}^2_{gen}(G,V).$$

- If $p \neq 2,3$ and if no root is a weight of V, then

$$\mathrm{H}^2(G,V) \cong \mathrm{H}^2_{gen}(G,V).$$

- If $V^T = V^{T(\mathbb{F}_q)}$, then $\mathrm{H}^1(G,V) \hookrightarrow \mathrm{H}^1(G(\mathbb{F}_q),V)$.
- If U, W are finite-dimensional G-modules, and if every composition factor of U and W have q-restricted highest weights, then

$$\mathrm{H}^1(G,\mathrm{Hom}_k(U,W)) \hookrightarrow \mathrm{H}^1(G(\mathbb{F}_q),\mathrm{Hom}_k(U,W)).$$

So for H^1 and H^2, we can get answers in terms of G if we take $q \gg 0$, and if we sometimes also replace V by $V^{(1)}$ or $V^{(2)}$.

Consider the (exact!) induction functor $\mathrm{ind}_{G(\mathbb{F}_q)}^G(-) = (- \otimes k[G])^{G(\mathbb{F}_q)}$.

Generalized Frobenius Reciprocity: $\mathrm{H}^n(G, \mathrm{ind}_{G(\mathbb{F}_q)}^G(N)) \cong \mathrm{H}^n(G(\mathbb{F}_q), N)$.

Bendel, Nakano, Pillen (2001)

Let $\pi \subset X(T)$ be a saturated set of weights, and let C_π be the category of G-modules all of whose highest weights lie in π. Let N be a $G(\mathbb{F}_q)$-module and let M be a G-module. Then there exists a spectral sequence

$$E_2^{i,j} = \mathrm{Ext}_G^i(M, R^j(\mathcal{O}_\pi \circ \mathrm{ind}_{G(\mathbb{F}_q)}^G)(N)) \Rightarrow \mathrm{Ext}_{G(\mathbb{F}_q)}^{i+j}(M, N).$$

Using this and related ideas, BNP have in a series of papers obtained many results relating cohomology for G and $G(\mathbb{F}_q)$, e.g., for $p > 3(h-1)$, describe $\mathrm{Ext}_{G(\mathbb{F}_q)}^1$ between simple modules as Ext_G^1 plus a remainder term.

There exists a short exact sequence

$$0 \to k \to \mathrm{ind}_{G(\mathbb{F}_q)}^{G}(k) \to N \to 0.$$

Let M be a rational G-module. From the tensor identity obtain

$$0 \to M \to \mathrm{ind}_{G(\mathbb{F}_q)}^{G}(M) \to M \otimes N \to 0.$$

Now using $\mathrm{Ext}_G^n(k, \mathrm{ind}_{G(\mathbb{F}_q)}^{G}(M)) \cong \mathrm{Ext}_{G(\mathbb{F}_q)}^n(k, M)$, we get:

Long exact sequence for restriction

$$\begin{array}{cccccc}
0 & \to & \mathrm{Hom}_G(k,M) & \stackrel{\text{res}}{\to} & \mathrm{Hom}_{G(\mathbb{F}_q)}(k,M) & \to & \mathrm{Hom}_G(k, M \otimes N) \\
 & \to & \mathrm{Ext}_G^1(k,M) & \stackrel{\text{res}}{\to} & \mathrm{Ext}_{G(\mathbb{F}_q)}^1(k,M) & \to & \mathrm{Ext}_G^1(k, M \otimes N) \\
 & \to & \mathrm{Ext}_G^2(k,M) & \stackrel{\text{res}}{\to} & \mathrm{Ext}_{G(\mathbb{F}_q)}^2(k,M) & \to & \mathrm{Ext}_G^2(k, M \otimes N) \\
 & \to & \cdots & & & &
\end{array}$$

Bendel, Nakano, Pillen (2010)

$\operatorname{ind}_{G(\mathbb{F}_q)}^{G}(k)$ admits a filtration by G-submodules with sections of the form

$$H^0(\mu) \otimes H^0(\mu^*)^{(r)} \quad \mu \in X(T)_+.$$

Corollary: $N = \operatorname{coker}(k \to \operatorname{ind}_{G(\mathbb{F}_q)}^{G}(k))$ admits such a filtration with $\mu \neq 0$.

Then $\operatorname{Ext}^i_G(k, L(\lambda) \otimes N) = 0$ if it is zero for each section, i.e., if for $\mu \neq 0$,

$$\operatorname{Ext}^i_G(V(\mu)^{(r)}, L(\lambda) \otimes H^0(\mu)) = 0.$$

Analyze the spectral sequences

$$\operatorname{Ext}^i_{G/G_r}(V(\mu)^{(r)}, \operatorname{Ext}^j_{G_r}(k, L(\lambda) \otimes H^0(\mu))) \Rightarrow \operatorname{Ext}^{i+j}_G(V(\mu)^{(r)}, L(\lambda) \otimes H^0(\mu))$$

and $$R^i \operatorname{ind}_{B/B_r}^{G/G_r} \operatorname{Ext}^j_{B_r}(k, L(\lambda) \otimes \mu) \Rightarrow \operatorname{Ext}^{i+j}_{G_r}(k, L(\lambda) \otimes H^0(\mu)).$$

The strategy is approximated by the following diagram stolen from BNP:

$$\mathrm{H}^i(G(\mathbb{F}_q), k) \overset{\text{Induction Functor}}{\Longrightarrow} \mathrm{H}^i(G, \mathcal{G}_r(k))$$

$$\Downarrow \text{ Filtrations}$$

$$\mathrm{H}^i(G, H^0(\lambda) \otimes H(\lambda^*)^{(r)}) \underset{\text{LHS Spectral Sequences}}{\Longrightarrow} \mathrm{H}^i(G_1, H^0(\lambda)) \underset{\text{Kostant Partition Functions}}{\Longrightarrow} \text{Root Combinatorics.}$$

Theorem (UGA VIGRE Algebra Group)

Let $\lambda \in X_r(T)$. Suppose $\operatorname{Ext}^1_{U_r}(k, L(\lambda))$ is semisimple as a B/U_r-module, and that $\operatorname{Ext}^1_{U_r}(k, L(\lambda))^{T(\mathbb{F}_q)} = \operatorname{Ext}^1_{U_r}(k, L(\lambda))^T$. Then

$$\mathrm{H}^1(G, L(\lambda)) \cong \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)).$$

Theorem (UGA VIGRE Algebra Group)

Let $\lambda \in X_r(T)$. Suppose $\operatorname{Ext}^1_{U_r}(k, L(\lambda))$ is semisimple as a B/U_r-module, that $\operatorname{Ext}^i_{U_r}(k, L(\lambda))^{T(\mathbb{F}_q)} = \operatorname{Ext}^i_{U_r}(k, L(\lambda))^T$ for $i \in \{1, 2\}$, and that

$$p^r > \max\left\{-(\nu, \gamma^\vee) : \gamma \in \Delta, \nu \in X(T), \operatorname{Ext}^1_{U_r}(k, L(\lambda))_\nu \neq 0\right\}.$$

Then $\mathrm{H}^2(G, L(\lambda)) \cong \mathrm{H}^2(G(\mathbb{F}_q), L(\lambda))$.

Critical calculation using Andersen's results on B-cohomology and lots of weight combinatorics:

Theorem 3.2.4. *Suppose $\lambda \in X(T)_+$ is a dominant root or is less than or equal to a fundamental weight. Assume that $p > 5$ if Φ is of type E_8 or G_2, and $p > 3$ otherwise. Then as a B/U_r-module, $\operatorname{Ext}^1_{U_r}(L(\lambda), k) = \operatorname{soc}_{B/U_r} \operatorname{Ext}^1_{U_r}(L(\lambda), k)$, that is,*

$$\operatorname{Ext}^1_{U_r}\big(L(\lambda), k\big) \cong \bigoplus_{\alpha\in\Delta} -s_\alpha \cdot \lambda \oplus \bigoplus_{\substack{\alpha\in\Delta \\ 0<n<r}} -\big(\lambda - p^n\alpha\big) \oplus \bigoplus_{\substack{\sigma\in X(T)_+ \\ \sigma<\lambda}} (-\sigma)^{\oplus m_\sigma}$$

where $m_\sigma = \dim \operatorname{Ext}^1_G(L(\lambda), H^0(\sigma))$.

First Cohomology Main Theorem

Let $\lambda \in X(T)_+$ be a fundamental dominant weight. Assume $q > 3$ and

$$\begin{array}{ll} p > 2 & \text{if } \Phi \text{ has type } A_n,\ D_n; \\ p > 3 & \text{if } \Phi \text{ has type } B_n,\ C_n,\ E_6,\ E_7,\ F_4,\ G_2; \\ p > 5 & \text{if } \Phi \text{ has type } E_8. \end{array}$$

Then $\dim \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) = \dim \mathrm{H}^1(G, L(\lambda)) \leq 1$.

Space nonzero (and one-dimensional) in the following cases:

- Φ has type E_7, $p = 7$, and $\lambda = \omega_6$; and
- Φ has type C_n, $n \geq 3$, and $\lambda = \omega_j$ with $\frac{j}{2}$ a nonzero term in the p-adic expansion of $n + 1$, but not the last term in the expansion.

Reasons for vanishing: Linkage principle for G, $\mathrm{Ext}^1_G(V(0), H^0(\lambda)) = 0$.
Reasons for non-vanishing: Weyl module structure.

Second Cohomology Main Theorem A

Suppose $p > 3$ and $q > 5$. Let $\lambda \in X(T)_+$ be less than or equal to a fundamental dominant weight. Assume also that λ is not a dominant root. Then $\mathrm{H}^2(G, L(\lambda)) \cong \mathrm{H}^2(G(\mathbb{F}_q), L(\lambda))$.

Corollary

Suppose p, q, λ are as above. Then $\mathrm{H}^2(G(\mathbb{F}_q), L(\lambda)) = 0$ except possibly in a small number of explicit cases in exceptional types, and in type C_n when $\lambda = \omega_j$ with j even and $p \leq n$.

Don't know $\mathrm{H}^2(G, L(\omega_j))$ for all even j in type C_n when $p \leq n$.
Come back to this at the end . . .

Second Cohomology Main Theorem B

Let $p > 3$ and $q > 5$. Let $\lambda = \widetilde{\alpha}$ (highest root). Assume $p \nmid n+1$ in type A_n, and $p \nmid n-1$ in type B_n. Then

$$\mathrm{H}^2(G(\mathbb{F}_q), L(\widetilde{\alpha})) = k.$$

Also have $\mathrm{H}^2(A_2(5), L(\omega_1)) = \mathrm{H}^2(A_2(5), L(\omega_2)) = k$.

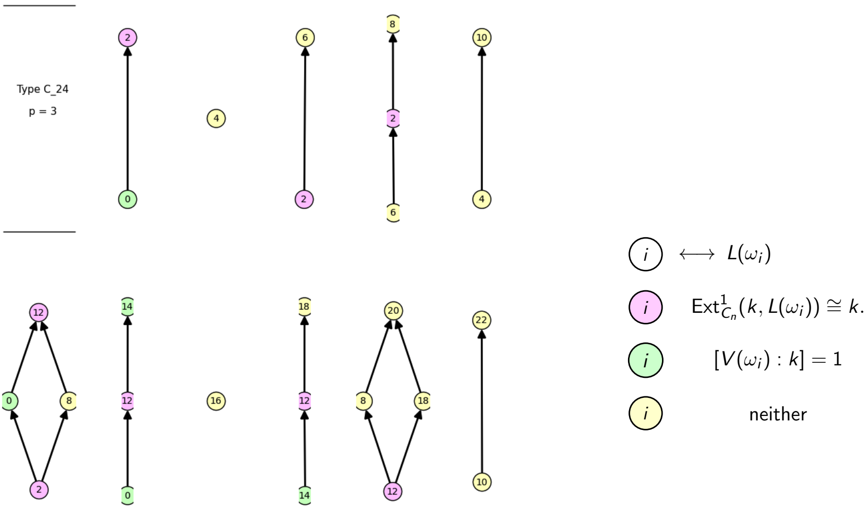
Different strategy for this case in analyzing the long exact sequence.

$$\to \mathrm{H}^2(G, L(\lambda)) \to \mathrm{H}^2(G(\mathbb{F}_q), L(\lambda)) \to \mathrm{H}^2(G, L(\lambda) \otimes N) \to \mathrm{H}^3(G, L(\lambda)) \to$$

Prove that $\mathrm{H}^2(G(\mathbb{F}_q), L(\lambda))$ is isomorphic to the cohomology of a single filtration layer in $\mathrm{H}^2(G, L(\lambda) \otimes N)$, i.e., layer for $H^0(\widetilde{\alpha}) \otimes H^0(\widetilde{\alpha})^{(r)}$.

cf. to CPSvdK: $\mathrm{H}^2(G, V^{(1)}) \cong \mathrm{H}^2_{gen}(G, V)$ if $p \neq 2$.

Adamovich described combinatorially the submodule structure of Weyl modules in Type C having fundamental highest weight. We use this and $\mathrm{Ext}^2_{C_n}(k, L(\omega_j)) \cong \mathrm{Ext}^1_{C_n}(\mathrm{rad}_G\, V(\omega_j), k)$ to make computations.



Values of n and j for which $\mathrm{H}^2(Sp_{2n}, L(\omega_j)) \neq 0$, $p = 3$.

In each case, H^2 is 1-dimensional.

n	j	n	j	n	j	n	j
6	6	15	6, 8	24	6, 8, 18	33	6, 8, 18
7	6	16	6, 10	25	6, 10, 18	34	6, 10, 18
8		17		26		35	
9	6	18	6, 14	27	6, 14	36	6, 14
10	6	19	6, 16	28	6, 16	37	6, 16
11		20	18	29	18	38	18
12	6	21	6, 18	30	6, 18	39	6, 18, 20
13	6	22	6, 18	31	6, 18	40	6, 18, 22
14		23	18	32	18		

For $n = 12$, we have also $\mathrm{H}^1(Sp_{2n}, L(\omega_6)) \neq 0$ (parity vanishing violated).

Values of n and j for which $\mathrm{H}^2(Sp_{2n}, L(\omega_j)) \neq 0$: $p = 5$.

In each case, H^2 is 1-dimensional.

n	j
10	10
11	10
12	10
13	10
14	
15	10
16	10
17	10
18	10
19	

n	j
20	10
21	10
22	10
23	10
24	
25	10
26	10
27	10
28	10
29	

n	j
30	10
31	10
32	10
33	10
34	
35	10, 12
36	10, 14
37	10, 16
38	10, 18
39	

n	j
40	10, 22
41	10, 24
42	10, 26
43	10, 28
44	
45	10, 32
46	10, 34
47	10, 36
48	10, 38
49	

n	j
50	10, 42
51	10, 44
52	10, 46
53	10, 48
54	50

For $n = 30$, we also have $\mathrm{H}^1(Sp_{2n}, L(\omega_{10})) \neq 0$ (parity vanishing violated).