The Sulfuric Stink of Super

An homage to work of Brian Parshall via Lie superalgebras

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Dramatis Personae

Lie superalgebras

Let k be a field of (any) characteristic p. A **Lie superalgebra** over k is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ over k equipped with an even bilinear map $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and a quadratic operator $q : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ such that

(L0) $q(\lambda \cdot y) = \lambda^2 \cdot q(y)$ for all $\lambda \in k$ and $y \in \mathfrak{g}_{\bar{1}}$,

(L1) $[x,y] = -(-1)^{\bar{x}\cdot\bar{y}}[y,x]$

(L2) $[x,[y,z]] = [[x,y],z] + (-1)^{\bar{x}\cdot\bar{y}}[y,[x,z]]$ (Jacobi identity)

(L3) $[x,x] = 0$ for all $x \in \mathfrak{g}_{\bar{0}}$,

(L4) $[y,[y,y]] = 0$ for all $y \in \mathfrak{g}_{\bar{1}}$,

(L5) $[x,y] = q(x+y) - q(x) - q(y)$ for all $x, y \in \mathfrak{g}_{\bar{1}}$,

(L6) $[y,[y,z]] = [q(y),z]$ for all $y \in \mathfrak{g}_{\bar{1}}$ and $z \in \mathfrak{g}$.

So in particular, $\mathfrak{g}_{\bar{0}}$ is an ordinary Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module.

Lie superalgebras

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(L2) $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x} \cdot \bar{y}}[y, [x, z]]$ (Jacobi identity)

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(L5) $[x, y] = q(x + y) - q(x) - q(y)$ for all $x, y \in \mathfrak{g}_{\bar{1}}$,

(L6) $[y, [y, z]] = [q(y), z]$ for all $y \in \mathfrak{g}_{\bar{1}}$ and $z \in \mathfrak{g}$.

Redundancy of $q : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ when $\mathrm{char}(k) \neq 2$

(L5) implies for all $y \in \mathfrak{g}_{\bar{1}}$ that $q(y) = \frac{1}{2}[y, y]$.

Underlying ordinary Lie algebra $\bar{\mathfrak{g}}$ when $\mathrm{char}(k) = 2$

(L5) implies for all $y \in \mathfrak{g}_{\bar{1}}$ that $[y, y] = 0$. Then forgetting q, the Lie bracket makes $\mathfrak{g}$ into an ordinary Lie algebra, which we denote $\bar{\mathfrak{g}}$.

So Lie superalgebras in char. 2 are just $\mathbb{Z}_2$-graded Lie algebras?

Lie superalgebras

Let $T(\mathfrak{g}) = \bigoplus_{i \geq 0} \mathfrak{g}^{\otimes i}$ be the tensor (super)algebra on $\mathfrak{g}$. The **universal enveloping algebra** of $\mathfrak{g}$ is the k-superalgebra $U(\mathfrak{g}) = T(\mathfrak{g})/I$, where I is the two-sided ideal generated by

- $x \otimes y - (-1)^{\bar{x} \cdot \bar{y}} y \otimes x - [x, y]$ for $x, y \in \mathfrak{g}$ homogeneous,
- $y \otimes y - q(y)$ for $y \in \mathfrak{g}_{\bar{1}}$.

'Superness' detected by the module theory in characteristic 2

There is a canonical quotient $U(\overline{\mathfrak{g}}) \twoheadrightarrow U(\mathfrak{g})$ of Hopf (super)algebras.

$\mathfrak{g}$-supermodules are thus $\mathbb{Z}_2$-graded modules for the ordinary Lie algebra $\overline{\mathfrak{g}}$ on which the identity $y^2 = q(y)$ holds for all $y \in \mathfrak{g}_{\bar{1}}$.

Restricted Lie superalgebras

Suppose $\mathrm{char}(k) = p > 0$. Say that $\mathfrak{g}$ is a **restricted Lie superalgebra** if there exists a p-map $x \mapsto x^{[p]}$ on $\mathfrak{g}_{\bar{0}}$ such that, for all $x, y \in \mathfrak{g}_{\bar{0}}$:

(R1) $(\alpha \cdot x)^{[p]} = \alpha^p \cdot x^{[p]}$ for all $\alpha \in k$,

(R2) $\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)^p$ where $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}(x)(z) = [x, z]$,

(R3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, where $i \cdot s_i(x, y)$ is the coefficient of t^{i-1} in the formal expression $\mathrm{ad}(t \cdot x + y)^{p-1}(x)$.

In other words, $\mathfrak{g}$ is a restricted Lie superalgebra if $\mathfrak{g}_{\bar{0}}$ is an ordinary restricted Lie algebra, and $\mathfrak{g}_{\bar{1}}$ is a restricted $\mathfrak{g}_{\bar{0}}$-module.

Restricted enveloping algebra

$$V(\mathfrak{g}) = U(\mathfrak{g})/\langle x^p - x^{[p]} : x \in \mathfrak{g}_{\bar{0}} \rangle$$

For arbitrary $\mathfrak{g}$, the map $q : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ is like a 2-map defined only on $\mathfrak{g}_{\bar{1}}$.

Restricted Lie superalgebras

Let $\mathfrak{g}$ be a restricted Lie superalgebra over a field k of characteristic $p = 2$, with p-operation $x \mapsto x^{[2]}$ on $\mathfrak{g}_{\bar{0}}$, and let $\overline{\mathfrak{g}}$ be the ordinary Lie algebra obtained from $\mathfrak{g}$ by forgetting the operator $q : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$.

Lifting the p-map from $\mathfrak{g}_{\bar{0}}$ to $\overline{\mathfrak{g}}$ when $p = 2$

Given $z \in \mathfrak{g}$, let $z = z_{\bar{0}} + z_{\bar{1}}$ be its decomposition into even and odd parts. Then the map

$$z \mapsto z^{\{2\}} := (z_{\bar{0}})^{[2]} + q(z_{\bar{1}}) + [z_{\bar{1}}, z_{\bar{0}}]$$

defines a p-map on $\overline{\mathfrak{g}}$ that makes $\overline{\mathfrak{g}}$ into a ($\mathbb{Z}_2$-graded) ordinary restricted Lie algebra, and the quotient map $U(\overline{\mathfrak{g}}) \twoheadrightarrow U(\mathfrak{g})$ then induces an isomorphism of Hopf (super)algebras $V(\overline{\mathfrak{g}}) \cong V(\mathfrak{g})$.

So in characteristic 2, restricted $\mathfrak{g}$-supermodules are $\mathbb{Z}_2$-graded restricted $\overline{\mathfrak{g}}$-modules.

Scenes

Motivating Question

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k.

What does the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, k) = \mathrm{Ext}^\bullet_{\mathfrak{g}}(k, k)$ look like?

What does its maximal ideal spectrum $\mathrm{Max}(\mathrm{H}^\bullet(\mathfrak{g}, k))$ look like?

Elementary result

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k. Then $\mathrm{H}^\bullet(\mathfrak{g}, k)$ is finite-dimensional, and $\mathrm{H}^i(\mathfrak{g}, k) = 0$ for $i > \dim_k(\mathfrak{g})$, e.g., because $\mathrm{H}^\bullet(\mathfrak{g}, k) = \mathrm{H}^\bullet(\Lambda(\mathfrak{g}^*), \partial)$ and $\Lambda^i(\mathfrak{g}^*) = 0$ for $i > \dim_k(\mathfrak{g})$.

So $\mathrm{Max}(\mathrm{H}^\bullet(\mathfrak{g}, k))$ is not very interesting in this situation.

Different Source of Motivation

Let $k = \overline{k}$ of characteristic $p > 0$.

Friedlander–Parshall (1980s), Suslin–Friedlander–Bendel (1997)

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra over k. Then

$$\mathrm{Max}\,(\mathrm{H}^\bullet(V(\mathfrak{g}), \mathbb{C})) \simeq \mathcal{N}_p(\mathfrak{g}) = \left\{x \in \mathfrak{g} : x^{[p]} = 0\right\}.$$

$\mathcal{N}_p(\mathfrak{g})$ is the **restricted nullcone** of $\mathfrak{g}$.

If $\mathfrak{g} = \mathfrak{gl}_n(k)$, then $x^{[p]} = x^p$, and $\mathcal{N}_p(\mathfrak{g})$ is the variety of p-nilpotent matrices. If $p > n$, then $\mathcal{N}_p(\mathfrak{g})$ is all nilpotent matrices in $\mathfrak{g}$.

Support varieties

Let A be a Hopf algebra over k. Then $\mathrm{H}^\bullet(A,k)$ is graded-commutative. Suppose $\mathrm{H}^\bullet(A,k)$ is finitely-generated as a k-algebra.

Cohomological spectrum and support varieties

The **cohomological spectrum** of A is the affine algebraic variety

$$|A| = \mathrm{Max}\left(\mathrm{H}^\bullet(A,k)\right).$$

Given an A-module M, let $I_A(M)$ be the kernel of the (k-algebra) map

$$\mathrm{H}^\bullet(A,k) = \mathrm{Ext}_A^\bullet(k,k) \xrightarrow{-\otimes M} \mathrm{Ext}_A^\bullet(M,M).$$

The **cohomological support variety** associated to M is

$$|A|_M = \mathrm{Max}\left(\mathrm{H}^\bullet(A,k)/I_A(M)\right),$$

which is a closed conical subvariety of $|A|$.

Friedlander–Parshall, Suslin–Friedlander–Bendel

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra over k, and let M be a finite-dimensional restricted $\mathfrak{g}$-module. Then

$$|V(\mathfrak{g})|_M \simeq \left\{x \in \mathcal{N}_p(\mathfrak{g}) : M|_{\langle x \rangle} \text{ is not free}\right\} \cup \{0\}\,.$$

Moreover, $|V(\mathfrak{g})|_M = \{0\}$ if and only if M is projective for $V(\mathfrak{g})$.

For $x \in \mathcal{N}_p(\mathfrak{g})$, $M|_{\langle x \rangle}$ is restriction to subalgebra of the form $k[x]/(x^p)$.

Superized Motivating Question

Let $\mathfrak{g}$ be a finite-dimensional Lie **super**algebra over a field k.

What does $|U(\mathfrak{g})| = \mathrm{Max}(\mathrm{H}^\bullet(\mathfrak{g}, k))$ look like?

Example

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, so that $\mathfrak{g}_{\overline{0}} = \mathfrak{gl}_m \oplus \mathfrak{gl}_n$. If $m \geq n$, then the inclusion $\mathfrak{gl}_m \subseteq \mathfrak{g}$ induces $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{gl}_m, \mathbb{C})$. In particular, $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C})$ is a finite-dimensional exterior algebra.

So in general in characteristic 0, the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, k)$ may not lead to an interesting support variety theory.

But see work of Boe, Kujawa, and Nakano for an extensive support variety theory based on relative cohomology for the pair $(\mathfrak{g}, \mathfrak{g}_{\overline{0}})$.

Support for Lie superalgebras in characteristic $p > 0$

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra over a field $k = \overline{k}$ of characteristic $p \geq 2$. Then $\mathrm{H}^\bullet(\mathfrak{g}, k)$ is finite over the image of a map of graded (super)algebras $\varphi : S(\mathfrak{g}_{\bar{1}}^*[2])^{(1)} \to \mathrm{H}^\bullet(\mathfrak{g}, k)$.

Theorem (Drupieski–Kujawa)

The map $\varphi : S(\mathfrak{g}_{\bar{1}}^*)^{(1)} \to \mathrm{H}^\bullet(\mathfrak{g}, k)$ induces a homeomorphism

$$|U(\mathfrak{g})| \simeq \mathcal{N}_{\mathrm{odd}}(\mathfrak{g}) := \{x \in \mathfrak{g}_{\bar{1}} : q(x) = 0\}\,.$$

For each finite-dimensional $\mathfrak{g}$-supermodule M, one gets

$$|U(\mathfrak{g})|_M \simeq \mathcal{X}_{\mathfrak{g}}(M) := \left\{x \in \mathcal{N}_{\mathrm{odd}}(\mathfrak{g}) : M|_{\langle x \rangle} \text{ is not free}\right\}.$$

The set $\mathcal{N}_{\mathrm{odd}}(\mathfrak{g})$ is the **odd nullcone** of $\mathfrak{g}$. $q(x) = \frac{1}{2}[x, x]$ if $p \neq 2$

Compare $\mathcal{X}_{\mathfrak{g}}(M)$ to the 'associated variety' of Duflo and Serganova.

Support for Lie superalgebras in characteristic $p > 0$

Theorem

Let M be a finite-dimensional $\mathfrak{g}$-supermodule. Then

$$|U(\mathfrak{g})|_M = \{0\} \iff \mathrm{projdim}_{U(\mathfrak{g})}(M) < \infty.$$

Corollary (cf. Bøgvad, 1984)

$$\mathcal{N}_{\mathrm{odd}}(\mathfrak{g}) = \{0\} \iff \mathrm{gldim}(U(\mathfrak{g})) < \infty.$$

Support for restricted Lie superalgebras

Now let $\mathfrak{g}$ be a finite-dimensional restricted Lie superalgebra over k.

What does $|V(\mathfrak{g})| = \mathrm{Max}(\mathrm{H}^\bullet(V(\mathfrak{g}), k))$ look like?

For $p = 2$, we can appeal to the underlying ordinary Lie algebra $\overline{\mathfrak{g}}$, and the identification $V(\mathfrak{g}) \cong V(\overline{\mathfrak{g}})$.

Spectrum in characteristic 2

$$\begin{aligned} |V(\mathfrak{g})| &= |V(\overline{\mathfrak{g}})| \\ &\simeq \mathcal{N}_p(\overline{\mathfrak{g}}) \\ &= \{z \in \overline{\mathfrak{g}} : z^{\{2\}} = 0\} \\ &= \{z = z_{\overline{0}} + z_{\overline{1}} \in \mathfrak{g} : (z_{\overline{0}})^{[2]} + q(z_{\overline{1}}) = 0 \text{ and } [z_{\overline{1}}, z_{\overline{0}}] = 0\}. \end{aligned}$$

In characteristic $p \geq 3$, we can currently only show that an identification like this holds up to a finite morphism of varieties.

Support of a module in characteristic 2

Let M be a finite-dimensional $\mathfrak{g}$-supermodule. Then

$$|V(\mathfrak{g})|_M = |V(\overline{\mathfrak{g}})|_M \simeq \{z \in \mathcal{N}_p(\overline{\mathfrak{g}}) : M|_{\langle z\rangle} \text{ is not free}\}.$$

Let $P = k[u, v]/(u^p + v^2)$, with u even and v odd.

Given $z = z_{\overline{0}} + z_{\overline{1}} \in \mathfrak{g}$ with $z^{\{2\}} = 0$, let $\sigma_z : P \to V(\mathfrak{g})$ be the algebra map defined by $\sigma_z(u) = z_{\overline{0}}$ and $\sigma_z(v) = z_{\overline{1}}$.

Reinterpreting support of a module in characteristic 2

Let M be a finite-dimensional $\mathfrak{g}$-supermodule. Then

$$|V(\mathfrak{g})|_M \simeq \{z \in \mathfrak{g} : z^{\{2\}} = 0 \text{ and } \operatorname{projdim}_P(M\downarrow_{\sigma_z}) = \infty\}.$$

In characteristic $p \geq 3$, we can show that a description like this for $|V(\mathfrak{g})|_M$ holds when $\mathfrak{g}$ is p-nilpotent (conjecturally for arbitrary $\mathfrak{g}$).

Tensor triangular geometry in characteristic 2

Let $A = \mathrm{H}^\bullet(V(\mathfrak{g}), k) = \mathrm{H}^\bullet(V(\overline{\mathfrak{g}}), k)$. Then A is graded both by the cohomological degree and by superdegree.

Let $\mathrm{Proj}(A)$ be the set of all $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\mathfrak{p}$ is homogeneous with respect to the cohomological grading.

Theorem (Benson–Iyengar–Krause–Pevtsova)

There is a canonical homeomorphism

$$\mathrm{Proj}\left(\mathrm{H}^\bullet(V(\overline{\mathfrak{g}}), k)\right) \simeq \mathrm{Spc}\left(\mathrm{stmod}_{V(\overline{\mathfrak{g}})}\right),$$

and there are inverse bijections

$$\{\text{specialization closed subsets } \mathcal{V} \text{ of } \mathrm{Proj}(\mathrm{H}^\bullet(V(\overline{\mathfrak{g}}), k))\} \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \left\{\text{thick } \otimes\text{-ideal subcategories } \mathcal{J} \text{ of } \mathrm{stmod}_{V(\overline{\mathfrak{g}})}\right\}.$$

Tensor triangular geometry in characteristic 2

Let $A = \mathsf{H}^\bullet(V(\mathfrak{g}), k) = \mathsf{H}^\bullet(V(\overline{\mathfrak{g}}), k)$. Then A is graded both by the cohomological degree and by superdegree.

$\mathsf{Proj}_s(A)$

Say that a bi-homogeneous ideal $P \subset A$ is s-prime if it is prime among the set of bi-homogeneous ideals in P.

Let $\mathsf{Proj}_s(A)$ be the set of all (bi-homogeneous) s-prime ideals that are properly contained in $\mathsf{H}^{>0}(V(\mathfrak{g}), k)$.

The Zariski topology on $\mathsf{Proj}_s(A)$ is defined via closures of bi-homogeneous ideals $I \subset A$. There is a continuous surjection

$$\varphi : \mathsf{Proj}(A) \to \mathsf{Proj}_s(A)$$

where $\varphi(\mathfrak{p}) = \mathfrak{p}_s$ is the largest bi-homogenous subideal of $\mathfrak{p}$.

Tensor triangular geometry in characteristic 2

Let st-smod$_{V(\mathfrak{g})}$ be the stable module category of finite-dimensional $V(\mathfrak{g})$-supermodules.

Theorem

There is a canonical homeomorphism

$$\mathrm{Proj}_s\left(\mathrm{H}^\bullet(V(\mathfrak{g}), k)\right) \simeq \mathrm{Spc}\left(\text{st-smod}_{V(\mathfrak{g})}\right),$$

and there are inverse bijections

$$\{\text{specialization closed subsets } \mathcal{V} \text{ of } \mathrm{Proj}_s(\mathrm{H}^\bullet(V(\overline{\mathfrak{g}}), k))\} \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \left\{\text{thick } \otimes\text{-ideal subcategories } \mathcal{J} \text{ of st-smod}_{V(\mathfrak{g})}\right\}.$$