Cohomology and support varieties for (unipotent) finite supergroup schemes

Christopher M. Drupieski, DePaul University
Joint work with Jonathan Kujawa, University of Oklahoma

2018 AMS Spring Eastern Sectional Meeting, Northeastern University
Special Session on Hopf Algebras, Tensor Categories, and Homological Algebra

Let A be a Hopf algebra over k, a field of characteristic $p \geq 3$.

Suppose $\mathrm{H}^\bullet(A,k) = \mathrm{Ext}_A^\bullet(k,k)$ is finitely generated as a k-algebra.

Cohomological spectrum and support varieties

The **cohomological spectrum** of A is the affine algebraic variety

$$|A| = \mathrm{MaxSpec}\left(\mathrm{H}^\bullet(A,k)\right).$$

Given an A-module M, let $I_A(M)$ be the kernel of the map

$$\mathrm{H}^\bullet(A,k) = \mathrm{Ext}_A^\bullet(k,k) \xrightarrow{-\otimes M} \mathrm{Ext}_A^\bullet(M,M).$$

The **cohomological support variety** associated to M is

$$|A|_M = \mathrm{MaxSpec}\left(\mathrm{H}^\bullet(A,k)/I_A(M)\right).$$

a closed subvariety of the cohomological spectrum.

It is an open question whether $\mathrm{H}^\bullet(A, k)$ is finitely-generated for all finite-dimensional Hopf algebras, but finite generation has been verified in a number of cases, including:

- group algebras of finite groups (Golod, Venkov, Evens 1961)
- f.d. graded connected cocom. Hopf algebras (Wilkerson 1981)
- **f.d. cocommutative Hopf algebras** (Friedlander–Suslin 1997)
- **f.d. cocommutative Hopf superalgebras** (Drupieski 2016)

In these contexts, cohomological support varieties will have sensible properties.

Equivalences

- finite group scheme $G \leftrightarrow$ f.d. cocommutative Hopf algebra kG
- infinitesimal group scheme $G \leftrightarrow$ f.d. cocom. Hopf algebra kG such that the dual Hopf algebra $(kG)^* = k[G]$ is local

Suslin–Friedlander–Bendel (1997)

Let G be an infinitesimal group scheme of height $\leq r$. Then there exists a homeomorphism

$$|kG| \cong V_r(G) := \mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, G).$$

For $G = GL_{n(r)}$, the r-th Frobenius kernel of GL_n, one has

$$V_r(GL_{n(r)}) \cong \left\{(\alpha_0, \ldots, \alpha_{r-1}) \in \mathfrak{gl}_n^{\times r} : \alpha_i^p = 0, [\alpha_i, \alpha_j] = 0, \forall i, j\right\}.$$

If $\nu : \mathbb{G}_{a(r)} \to G$ is a one-parameter subgroup, and if M is a rational G-module, then M pulls back to a rational $\mathbb{G}_{a(r)}$-module, $\nu^*(M)$.

Equivalently, $\nu^*(M)$ is a module over the group algebra

$$k\mathbb{G}_{a(r)} = k[\mathbb{G}_{a(r)}]^\# = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p).$$

Suslin–Friedlander–Bendel (1997)

Let G be infinitesimal of height $\leq r$. If M is a finite-dimensional rational G-module, then

$$|kG|_M \cong \left\{\nu \in V_r(G) : \nu^*(M) \text{ is not free over } k[u_{r-1}]/(u_{r-1}^p)\right\}.$$

So, for example, the projectivity of M can be detected by restrictions along various ν to algebras of the form $k[u]/(u^p)$.

Our motivating question

(How) can this be generalized to supergroups?

Wikipedia definition of a supergroup

A **supergroup** is a music group whose members are already successful as solo artists or as part of other groups or well known in other musical professions.

What do we mean by "super"?

On object is "super" if it is appropriately graded by $\mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$.

- Super vector spaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$
- $V \otimes W \cong W \otimes V$ via the **supertwist** $v \otimes w \mapsto (-1)^{\overline{v} \cdot \overline{w}} w \otimes v$

Define Hopf superalgebras to be Hopf algebra objects in the (tensor) category of vector superspaces.

Examples of Hopf superalgebras

- Ordinary Hopf algebras (as purely even superalgebras)
- $\mathbb{Z}$-graded Hopf algebras in the sense of Milnor and Moore
- Enveloping superalgebras of (restricted) Lie superalgebras
- Exterior algebra $\Lambda(V)$ over a (purely odd) vector space V (both commutative and cocommutative in the super sense)

Cautionary example

Let $A = k[u,v]/\langle u^p, v^2\rangle$ with $\overline{u} = \overline{0}$ and $\overline{v} = \overline{1}$.

Then A is a Hopf superalgebra with u and v both primitive.

Define M to be the A-supermodule with homogeneous basis

$$\{x_0, \ldots, x_{p-1}, y_0, \ldots, y_{p-1}\}, \quad x_i \text{ even}, \quad y_i \text{ odd},$$

such that $u.x_i = x_{i+1}$, $u.y_i = y_{i+1}$, $v.x_i = y_{i+1}$, and $v.y_i = x_{i+p-1}$.

Claim: M is projective over all proper cyclic subalgebras of A, but is not projective over A itself. So in contrast to the classical theory, need more than just cyclic subalgebras to detect projectivity.

What supergroups play the role of $\mathbb{G}_{a(r)}$ in the super theory?

Now discuss some results that hint at the possible answer...

Multiparameter supergroups

Let $f = T^{p^t} + \sum_{i=1}^{t-1} a_i T^{p^i} \in k[T]$ be a p-polynomial (no linear term).

Let $\eta \in k$ be a scalar.

The infinitesimal multiparameter supergroup $\mathbb{M}_{r;f,\eta}$

$$k\mathbb{M}_{r;f,\eta} = k[u_0, \ldots, u_{r-1}, v]/\langle u_0^p, \ldots, u_{r-2}^p, u_{r-1}^p + v^2, f(u_{r-1}) + \eta u_0 \rangle$$

– $u_0, \ldots, u_{r-1}$ are even; coproducts look like they do in $k\mathbb{G}_{a(r)}$

– u_{r-1}^p is primitive, v is an odd primitive generator

For $r = 1$, this reduces to

$$k[u, v]/\langle u^p + v^2, f(u) + \eta u \rangle.$$

Cohomological spectrum of $GL_{m|n(r)}$

Super analogue of the commuting variety for GL_n:

$$V_r(GL_{m|n})(A) = \Big\{(\alpha_0, \dots, \alpha_{r-1}, \beta) \in (\mathrm{Mat}_{m|n}(A)_{\overline{0}})^{\times r} \times \mathrm{Mat}_{m|n}(A)_{\overline{1}} : [\alpha_i, \alpha_j] = [\alpha_i, \beta] = 0 \text{ for all } 0 \le i, j \le r-1, \alpha_i^p = 0 \text{ for all } 0 \le i \le r-2, \text{ and } \alpha_{r-1}^p + \beta^2 = 0\Big\}.$$

Drupieski–Kujawa (arXiv Jan 2017)

There is a finite surjective morphism of varieties

$$\left|GL_{m|n(r)}\right| \to V_r(GL_{m|n})(k).$$

Get this by pasting together contributions coming from different multiparameter supergroups.

Better results for unipotent supergroup schemes

Benson–Iyengar–Krause–Pevtsova (announced July 2017)

For **unipotent** finite supergroup schemes, projectivity of modules and nilpotents in cohomology are detected (after field extension) by restriction to **'elementary'** subsupergroup schemes.

The ***infinitesimal* elementary** supergroups are precisely the **unipotent multiparameter supergroups**, together with $\mathbb{G}_{a(r)}$ and $\mathbb{G}_a^-$.

The height-one infinitesimal elementary supergroups have group algebras of the form

- $k\mathbb{G}_{a(1)} = k[u]/(u^p)$ with u even
- $k\mathbb{G}_a^- = k[v]/(v^2)$ with v odd
- $k\mathbb{M}_{1;s} = k[u, v]/(u^p + v^2, u^{p^s})$ for $s \geq 1$

Applying the BIKP detection theorem

There is an affine (non-algebraic!) k-supergroup scheme $\mathbb{M}_r$ that surjects onto all infinitesimal unipotent k-supergroups of height $\leq r$.

$$k\mathbb{M}_r = k[[u_0, \dots, u_{r-1}, v]]/\langle u_0^p, \dots, u_{r-2}^p, u_{r-1}^p + v^2 \rangle$$

Note that $k\mathbb{M}_1 = k[[u, v]]/\langle u^p + v^2 \rangle \subset k\mathbb{M}_r$.

As an ungraded algebra, $k\mathbb{M}_1$ is a hypersurface ring.

Drupieski–Kujawa (arXiv Dec 2017 and forthcoming...)

Suppose $k = \overline{k}$. Let G be an infinitesimal unipotent k-supergroup scheme of height $\leq r$. Then there is a homeomorphism of varieties

$$|kG| \cong \mathcal{N}_r(G) := \mathrm{Hom}_{Grp}(\mathbb{M}_r, G).$$

If M is a finite-dimensional rational G-supermodule, then

$$|kG|_M \cong \left\{ \phi \in \mathcal{N}_r(G) : \mathrm{injdim}_{\mathbb{M}_1}(\phi^* M) = \infty \right\}.$$