Support varieties for Lie superalgebras and finite graded group schemes

Brian Parshall Day

Christopher Drupieski
DePaul University

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Unless stated otherwise, work over an algebraically closed field k of characteristic $p > 0$.

Table of contents

Recollections

Support varieties

Suppose $\mathrm{H}^\bullet(A,k)$ is "commutative" and finitely generated. Then the maximal ideal spectrum

$$|A| = \mathrm{Max}\left(\mathrm{H}^\bullet(A,k)\right)$$

is an affine algebraic variety. Given an A-module M, have a map

$$\mathrm{H}^\bullet(A,k) \to \mathrm{Ext}_A^\bullet(M,M)$$

with annihilator ideal $I_A(M)$.

Support varieties

The **cohomological support variety** associated to M is

$$|A|_M = \mathrm{Max}\left(\mathrm{H}^\bullet(A,k)/I_A(M)\right),$$

a closed subvariety of the **cohomological spectrum** $|A|$.

Friedlander & Parshall (1986)

Given a finite-dimensional restricted Lie algebra $\mathfrak{g}$, let $V(\mathfrak{g})$ be its restricted enveloping algebra. There exists a map

$$\Phi^* : S(\mathfrak{g}^*)^{(1)} \to \mathrm{H}^\bullet(V(\mathfrak{g}), k).$$

and from this one gets a finite morphism $\Phi : |V(\mathfrak{g})| \to \mathfrak{g}$.

Friedlander–Parshall (ca. 1986)

Given a finite-dimensional $V(\mathfrak{g})$-module M,

$$\Phi\left(|V(\mathfrak{g})|_M \right) = \left\{ X \in \mathfrak{g} : X^{[p]} = 0 \text{ and } M|_{\langle X \rangle} \text{ is not free} \right\} \cup \{0\}.$$

Jantzen: First did cases $p = 2$ with M arbitrary, and $p \geq 3$ with $M = k$.

Later generalizations

Suslin–Friedlander–Bendel (ca. 1997)

Always have a homeomorphism

$$|V(\mathfrak{g})|_M \simeq \left\{X \in \mathfrak{g} : X^{[p]} = 0 \text{ and } M|_{\langle X \rangle} \text{ is not free}\right\} \cup \{0\}.$$

More generally, they describe $|G|_M$ for infinitesimal group scheme G in terms of the variety of 1-parameter subgroups $\nu : \mathbb{G}_{a(r)} \to G$ (first investigated by **Brian's PhD student David Gross!**).

Friedlander–Pevtsova (2000s)

Describe $|G|_M$ for G a finite group scheme in terms of Π-points.

Question

How much of this generalizes to $\mathbb{Z}$- or $\mathbb{Z}/2\mathbb{Z}$-graded settings?

Super linear algebra

What does it mean to be "super"?

Something is "super" if it has a compatible $\mathbb{Z}/2\mathbb{Z}$-grading.

- Superspaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$, $W = W_{\overline{0}} \oplus W_{\overline{1}}$
- Induced gradings on tensor products, linear maps, etc.

$$(V \otimes W)_\ell = \bigoplus_{i+j=\ell} V_i \otimes W_j$$

$$\mathrm{Hom}_k(V, W)_\ell = \{f \in \mathrm{Hom}_k(V, W) : f(V_i) \subseteq W_{i+\ell}\}$$

- $V \otimes W \cong W \otimes V$ via the **supertwist** $v \otimes w \mapsto (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$

Define (Hopf) superalgebras and 'super' (co)commutativity in terms of the "usual diagrams," but use the supertwist when objects pass.

Simplest possible example

Exterior algebra of a finite-dimensional vector space V

The exterior algebra $\Lambda(V)$ is a (super)commutative superalgebra:

$$ab = (-1)^{\overline{a}\cdot\overline{b}}ba$$

It is also a (super)cocommutative Hopf superalgebra:

$$\begin{aligned}
&\Delta(uv) \\
&= \Delta(u)\Delta(v) \\
&= (u \otimes 1 + 1 \otimes u)(v \otimes 1 + 1 \otimes v) \\
&= (u \otimes 1)(v \otimes 1) + (u \otimes 1)(1 \otimes v) + (1 \otimes u)(v \otimes 1) + (1 \otimes u)(1 \otimes v) \\
&= (uv \otimes 1) + (u \otimes v) - (v \otimes u) + (1 \otimes uv) \\
&= (uv \otimes 1) + (1 \otimes uv)
\end{aligned}$$

Hopf superalgebras

Examples of Hopf superalgebras

- Ordinary Hopf algebras (as purely even superalgebras).
- **$\mathbb{Z}$-graded Hopf algebras in the sense of Milnor and Moore**
- Enveloping superalgebras of (restricted) Lie superalgebras

Recall that a **Lie superalgebra** is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ equipped with an even map $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that for homogeneous x, y, z,

- $[x, y] = -(-1)^{\bar{x} \cdot \bar{y}}[y, x]$
- $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x} \cdot \bar{y}}[y, [x, z]]$
- $[x, x] = 0$ if $x \in \mathfrak{g}_{\bar{0}}$ and $p = 2$
- $[x, [x, x]] = 0$ if $x \in \mathfrak{g}_{\bar{1}}$ and $p = 3$

Say that $\mathfrak{g}$ is **restricted** if $\mathfrak{g}_{\bar{0}}$ is an ordinary restricted Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a restricted $\mathfrak{g}_{\bar{0}}$-module under the adjoint action.

Supergroup schemes

Classical correspondences

affine group schemes $\leftrightarrow$ cocommutative Hopf algebras

finite group schemes $\leftrightarrow$ f.d. cocommutative Hopf algebras

height-one group schemes $\leftrightarrow$ f.d. restricted Lie algebras

Super correspondences

affine supergroup schemes $\leftrightarrow$ cocommutative Hopf superalgebras

finite supergroup schemes $\leftrightarrow$ f.d. cocommut. Hopf superalgebras

height-one supergroup schemes $\leftrightarrow$ f.d. res. Lie superalgebras

Finite supergroup schemes, $p = 0$

Simplest example

Theorem

Let V be a finite-dimensional space. Then $\mathrm{H}^\bullet(\Lambda(V), k) \cong S^\bullet(V^*)$.

The cohomology ring is **graded-(super)commutative** in the sense

$$ab = (-1)^{\deg(a)\cdot\deg(b)+\overline{a}\cdot\overline{b}}ba.$$

Aramova–Avramov–Herzog (2000)

Let M be a finite-dimensional $\Lambda(V)$-supermodule. Then

$$|\Lambda(V)|_M \cong \left\{v \in V : M|_{\langle v\rangle} \text{ is not free}\right\}.$$

In the theorem, $\langle v\rangle$ refers to an algebra isomorphic to $\Lambda(v) \cong k[v]/\langle v^2\rangle$.

In characteristic 0, this is most of the complete picture!

Classification in characteristic zero

Suppose k is an algebraically closed field of characteristic 0.

Kostant

Let A be a cocommutative Hopf superalgebra over k. Then

$$A \cong U(\mathfrak{g})\#kG$$

for some Lie superalgebra $\mathfrak{g}$ over k and some subgroup $G \leq \mathrm{Aut}(\mathfrak{g})$.

Corollary

Let A be a finite-dimensional cocommutative Hopf superalgebra over k. Then $A \cong \Lambda(V)\#kG$ for some finite group G and some f.d. purely odd kG-module V.

Given $\Lambda(V)\#kG$ as in the Corollary, denote the corresponding finite supergroup scheme by $V \rtimes G$.

Theorem

Let $V \rtimes G$ be a finite k-supergroup scheme. Let M and N be $V \rtimes G$-supermodules. Then $\mathrm{Ext}^\bullet_{V\rtimes G}(M, N) \cong \mathrm{Ext}^\bullet_{\Lambda(V)}(M, N)^G$. In particular,

$$\mathrm{H}^\bullet(V \rtimes G, k) \cong \mathrm{H}^\bullet(\Lambda(V), k)^G \cong S^\bullet(V^*)^G.$$

Corollary

Let $V \rtimes G$ be a finite k-supergroup scheme, and let M be a finite-dimensional $V \rtimes G$-supermodule. Then

$$|V \rtimes G| \cong V/G, \quad \text{the quotient of } V \text{ by } G \text{, and}$$
$$|V \rtimes G|_M \cong \left\{[v] \in V/G : M|_{\langle v\rangle} \text{ is not free}\right\} \cup \{0\}.$$

Finite-dim’l Lie superalgebras

Background

Cohomology of Lie superalgebras

$\mathrm{H}^\bullet(\mathfrak{g}, k)$ is the cohomology ring of enveloping superalgebra $U(\mathfrak{g})$

$\mathrm{H}^\bullet(\mathfrak{g}, k)$ can be computed via the super Koszul resolution $(\boldsymbol{\Lambda}(\mathfrak{g}^*), \partial)$

As a superalgebra, $\boldsymbol{\Lambda}(\mathfrak{g}^*) \cong \Lambda(\mathfrak{g}^*_{\bar{0}})\ {}^{g}\!\!\otimes S(\mathfrak{g}^*_{\bar{1}})$.

Results in characteristic zero

$\mathrm{H}^\bullet(\mathfrak{g}, k)$ can be either finite-dimensional or infinite-dimensional

If $\mathfrak{g} = \mathfrak{g}_{\bar{1}}$, then $U(\mathfrak{g}) = \Lambda(\mathfrak{g})$ and $\mathrm{H}^\bullet(\mathfrak{g}, k) \cong S(\mathfrak{g}^*)$.

If $\mathfrak{g} = \mathfrak{gl}(m|n)$, then $\mathrm{H}^\bullet(\mathfrak{g}, k)$ is a f.d. exterior algebra [Fuks–Leites]

Competing variety theories in characteristic zero

Duflo–Serganova (arXiv 2005)

Given a $\mathfrak{g}$-supermodule M, defined the *associated variety*

$$X_M = \left\{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0 \text{ and } M|_{\langle x \rangle} \text{ is not free}\right\}$$

Relatively simple $GL_m \times GL_n$ orbit structure when $\mathfrak{g} = \mathfrak{gl}(m|n)$.

Detect projectivity in category $\mathcal{F}$ of f.d. $\mathfrak{g}$-supermodules s.s. over $\mathfrak{g}_{\bar{0}}$.

Not defined via cohomology.

Boe–Kujawa–Nakano (2009, 2010, 2011, 2012)

Support varieties in terms of relative cohomology $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; k)$.

Work in category $\mathcal{F}$ of f.d. $\mathfrak{g}$-supermodules that are s.s. over $\mathfrak{g}_{\bar{0}}$.

Variety theory can measure *defect* of $\mathfrak{g}$ and *atypicality* of modules.

For the rest of this talk, assume that k is of characteristic $p \geq 3$.

Positive characteristic (k algebraically closed, $p \geq 3$)

Super Koszul complex $\mathbf{\Lambda}(\mathfrak{g}^*) \cong \Lambda(\mathfrak{g}_{\bar{0}}^*) \, {}^g\!\otimes \, S(\mathfrak{g}_{\bar{1}}^*)$

p-th powers in $S(\mathfrak{g}_{\bar{1}}^*) \subset \mathbf{\Lambda}(\mathfrak{g}^*)$ consist of cocycles, so get a map

$$\varphi : S(\mathfrak{g}_{\bar{1}}^*[p])^{(1)} \to \mathrm{H}^\bullet(\mathfrak{g}, k).$$

Study $|\mathfrak{g}| := \mathrm{Max}\,(\mathrm{H}^\bullet(\mathfrak{g}, k))$ via this map.

Theorem

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra. Let M be a finite-dimensional $\mathfrak{g}$-supermodule. Then there are homeomorphisms

$$|\mathfrak{g}| \cong \{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0\}$$
$$|\mathfrak{g}|_M \cong \left\{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0 \text{ and } M|_{\langle x \rangle} \text{ is not free}\right\} \cup \{0\}\,.$$

Since $[x, x] = 0$, have isomorphism of algebras $\langle x \rangle \cong k[x]/\langle x^2 \rangle$.

Identical in definition to the Duflo–Serganova varieties!

Cohomology of finite supergroup schemes

CFG for finite supergroup schemes

First step toward support varieties: cohomological finite generation

Drupieski (Adv. Math. 2016)

Let G be a finite supergroup scheme over k and let M be a finite-dimensional G-supermodule. Then $\mathrm{H}^\bullet(G,k)$ is a finitely-generated k-superalgebra and $\mathrm{H}^\bullet(G,M)$ is finite over $\mathrm{H}^\bullet(G,k)$.

Proved by way of cohomology calculations in the category of **strict polynomial superfunctors**, analogous to the argument for ordinary finite group schemes by Friedlander and Suslin.

Remark

If A is a Hopf superalgebra, then the smash product $A\#(\mathbb{Z}/2\mathbb{Z})$ is an ordinary Hopf algebra, and $\mathrm{H}^\bullet(A\#(\mathbb{Z}/2\mathbb{Z}),k) \cong \mathrm{H}^\bullet(A,k)_{\overline{0}}$.

Example of an ordinary strict polynomial functor

Suppose V has basis $\{u, v\}$ and W has basis $\{x, y\}$.

Then $S^2(V)$ has basis $\left\{u^2, uv, v^2\right\}$ and $S^2(W)$ has basis $\left\{x^2, xy, y^2\right\}$.

Let $\phi : V \to W$ be the linear map with associated matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

The linear map $S^2(\phi) : S^2(V) \to S^2(W)$ is defined for $f \in S^2(V)$ by

$$S^2(\phi)(f(u, v)) = f(\phi(u), \phi(v)).$$

The associated matrix for $S^2(\phi)$ is then

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & (ad + cb) & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

Strict polynomial superfunctors

Examples of strict polynomial superfunctors

Π parity flip functor	$(\Pi V)_{\bar{0}} = V_{\bar{1}}$, $(\Pi V)_{\bar{1}} = V_{\bar{0}}$
$\boldsymbol{\Gamma}^d(V) = (V^{\otimes d})^{\Sigma_d}$ super-symmetric tensors	$\boldsymbol{\Gamma}(V) = \Gamma(V_{\bar{0}}) \otimes \Lambda(V_{\bar{1}})$
$\boldsymbol{S}^d(V) = (V^{\otimes d})_{\Sigma_d}$ super-symmetric power	$\boldsymbol{S}(V) = S(V_{\bar{0}}) \otimes \Lambda(V_{\bar{1}})$
$\boldsymbol{\Lambda}^d(V)$ super-exterior power	$\boldsymbol{\Lambda}(V) = \Lambda(V_{\bar{0}}) \; {}^{g}\!\!\otimes S(V_{\bar{1}})$
$\boldsymbol{A}^d(V)$ super-alternating tensors	$\boldsymbol{A}(V) = \Lambda(V_{\bar{0}}) \; {}^{g}\!\!\otimes \Gamma(V_{\bar{1}})$
$\boldsymbol{I}^{(r)}(V) = V^{(r)}$ r-th Frobenius twist ($r \geq 1$)	$\boldsymbol{I}^{(r)} = \boldsymbol{I}_0{}^{(r)} \oplus \boldsymbol{I}_1{}^{(r)}$

Non-example: $V \mapsto V_{\bar{0}}$ (incompatible with composing odd maps)

- SPSFs can restrict to ordinary SPFs in two different ways
- Ordinary SPFs in general don't seem lift to SPSFs
- Frobenius twists of SPFs lift to SPSFs in several different ways

Main calculation: structure of the extension algebra

$$\mathrm{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)}) = \begin{pmatrix} \mathrm{Ext}^\bullet_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) & \mathrm{Ext}^\bullet_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}) \\ \mathrm{Ext}^\bullet_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}) & \mathrm{Ext}^\bullet_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)}) \end{pmatrix}$$

Cohomology of strict polynomial superfunctors

Drupieski (2016)

$\text{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)})$ is generated as an algebra by extension classes

- $\boldsymbol{e}_i' \in \text{Ext}^{2p^{i-1}}_{\mathcal{P}}({I_0}^{(r)}, {I_0}^{(r)})$ and $\boldsymbol{e}_i'' \in \text{Ext}^{2p^{i-1}}_{\mathcal{P}}({I_1}^{(r)}, {I_1}^{(r)})$ $\qquad (1 \leq i \leq r)$
- $\boldsymbol{c}_r \in \text{Ext}^{p^r}_{\mathcal{P}}({I_1}^{(r)}, {I_0}^{(r)})$ and $\boldsymbol{c}_r^\Pi \in \text{Ext}^{p^r}_{\mathcal{P}}({I_0}^{(r)}, {I_1}^{(r)})$

These generators satisfy:

- $(\boldsymbol{e}_i')^p = 0 = (\boldsymbol{e}_i'')^p$ if $1 \leq i \leq r-1$.
- $(\boldsymbol{e}_r')^p = \boldsymbol{c}_r \circ \boldsymbol{c}_r^\Pi$ and $(\boldsymbol{e}_r'')^p = \boldsymbol{c}_r^\Pi \circ \boldsymbol{c}_r$.
- The $\boldsymbol{e}_i', \boldsymbol{e}_i''$ generate a commutative subalgebra.
- The $\boldsymbol{e}_i'$ restrict to Friedlander and Suslin's extension classes
- Have $\boldsymbol{e}_i' \circ \boldsymbol{c}_r = \pm \boldsymbol{c}_r \circ \boldsymbol{e}_i''$. But is it $+$ or $-$? (It is $+$ for $i = r$.)

Support varieties for infinitesimal supergroup schemes

Extension classes give a finite morphism

Let $G \subset GL(m|n)$ be infinitesimal of height $\leq r$.

Evaluation and restriction maps

$$\begin{aligned} \mathrm{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)}) &\to \mathrm{Ext}^\bullet_{GL(m|n)}((k^{m|n})^{(r)}, (k^{m|n})^{(r)}) \\ &\cong \mathrm{Ext}^\bullet_{GL(m|n)}(k, \mathfrak{gl}(m|n)^{(r)}) \\ &\to \mathrm{Ext}^\bullet_{G}(k, \mathfrak{gl}(m|n)^{(r)}) \\ &\cong \mathrm{Hom}_k(\mathfrak{gl}(m|n)^{*(r)}, \mathrm{H}^\bullet(G, k)) \end{aligned}$$

For $r = 1$, the strict polynomial superfunctor extension classes give rise to a superalgebra homomorphism over which $\mathrm{H}^\bullet(G, k)$ is finite:

$$\varphi : S(\mathfrak{gl}(m|n)^*_{\bar{0}}[2])^{(1)} \otimes S(\mathfrak{gl}(m|n)^*_{\bar{1}}[p])^{(1)} \to \mathrm{H}^\bullet(G, k).$$

Induced finite map of varieties $|G| \to \mathfrak{gl}(m|n)$ with image $V_G(k)$.

Restricted Lie superalgebras

Theorem

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie superalgebra. Then

$$V_{\mathfrak{g}}(k) \cong \{x + y \mid x \in \mathfrak{g}_{\bar{0}}, y \in \mathfrak{g}_{\bar{1}}, [x, y] = 0, x^{[p]} = y^2\}$$

where $y^2 := \frac{1}{2}[y, y]$.

- Relations come from the functor cohomology calculations
- Sufficiency comes from explicit calculations for the restricted subalgebra generated by x and y, using an “explicit” projective resolution constructed by Iwai–Shimada and May.
- Agrees with results of Nakano & Palmieri (1998) for finite-dimensional subalgebras of the Steenrod algebra
- Support variety $V_{\mathfrak{g}}(M)$ of a nontrivial supermodule M?

Arbitrary infinitesimal supergroup schemes

Now let $G \subset GL(m|n)$ be a height-r infinitesimal supergroup scheme.

The polynomial superfunctor classes give rise to a homomorphism

$$\left[\bigotimes_{i=1}^{r} S(\mathfrak{gl}(m|n)^*_{\bar{0}}[2p^{i-1}])^{(r)}\right] \otimes S(\mathfrak{gl}(m|n)^*_{\bar{1}}[p^r])^{(r)} \to \mathrm{H}^\bullet(G,k)$$

over whose image $\mathrm{H}^\bullet(G,k)$ is finite.

Possible description for $|G|$ à la Suslin–Friedlander–Bendel?

Set of all r-tuples $(x_0, \ldots, x_{r-1}, y)$ such that

- $x_i \in \mathfrak{g}_{\bar{0}}$ for $0 \leq i \leq r-1$, and $y \in \mathfrak{g}_{\bar{1}}$
- Entries pairwise commute
- $x_i^{[p]} = 0$ for $0 \leq i \leq r-2$
- $x_{r-1}^{[p]} = y^2$

Open and ongoing topics

- Completely identify the spectrum of $\mathrm{H}^\bullet(G, k)$ or $\mathrm{H}^\bullet(V(\mathfrak{g}), k)$
- Rank variety description for support varieties?
- Super one-parameter subgroups?
- Super Π-points?