Superized Troesch complexes and cohomology for strict polynomial superfunctors

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Strict polynomial functors

Category of vector spaces

Let k be a field (later, characteristic $p > 0$, then $p \geq 3$).

Let $\mathcal{V}$ be the category of finite-dimensional k-vector spaces.

The symmetric group Σ_d acts on $V^{\otimes d}$ by place permutation. Set

$$\Gamma^d(V) = (V^{\otimes d})^{\Sigma_d}.$$

The category $\Gamma^d\mathcal{V}$

Let $\Gamma^d\mathcal{V}$ be the category whose objects are the same as those in $\mathcal{V}$, but in which spaces of morphisms are defined by

$$\mathrm{Hom}_{\Gamma^d\mathcal{V}}(V, W) = \Gamma^d \, \mathrm{Hom}_k(V, W) \cong \mathrm{Hom}_{k\Sigma_d}(V^{\otimes d}, W^{\otimes d}),$$

and composition is that of $k\Sigma_d$-module homomorphisms.

Strict polynomial functors, after Pirashvili

The category $\mathcal{P}_d$

The category $\mathcal{P}_d$ of homogeneous degree-d strict polynomial functors is the category of linear functors

$$F : \Gamma^d \mathcal{V} \to \mathcal{V},$$

i.e., functors such that for all $V, W \in \mathcal{V}$, the function

$$F_{V,W} : \mathrm{Hom}_{k\boldsymbol{\Sigma}_d}(V^{\otimes d}, W^{\otimes d}) \to \mathrm{Hom}_k(F(V), F(W))$$

is a linear map.

Examples of homogeneous strict polynomial functors

- $\bigotimes^d : V \mapsto V^{\otimes d}$ — d-th tensor power
- $\Gamma^d : V \mapsto \Gamma^d(V) = (V^{\otimes d})^{\Sigma_d}$ — d-th divided power
- $\Lambda^d : V \mapsto \Lambda^d(V)$ — d-th exterior power
- $S^d : V \mapsto S^d(V) = (V^{\otimes d})_{\Sigma_d}$ — d-th symmetric power
- $\Gamma^{d,W} : V \mapsto \Gamma^d(\mathrm{Hom}_k(W, V))$ — projective object in $\mathcal{P}_d$
- $S^d_W : W \mapsto S^d(W \otimes V)$ — injective object in $\mathcal{P}_d$

Suppose k is a field of characteristic $p > 0$. Let $r \geq 1$.

- $I^{(r)} : V \mapsto V^{(r)}$ — r-th Frobenius twist, $I^{(r)} \in \mathcal{P}_{p^r}$

The p^r-power map induces an embedding $I^{(r)} \hookrightarrow S^{p^r}$.

Why (else) do we care about strict polynomial functors?

Theorem (Friedlander–Suslin)

Let $V \in \mathcal{V}$. If $\dim_k(V) = n \geq d$, then evaluation on V

$$F \mapsto F(V)$$

defines an equivalence of categories $\mathcal{P}_d \simeq S(n,d)$-mod.

Theorem (Friedlander–Suslin)

Extension classes in $\mathrm{Ext}^{\bullet}_{\mathcal{P}}(I^{(r)}, I^{(r)})$ restrict nontrivially to GL_n and its Frobenius kernel $GL_{n(r)}$, and provide generators for the cohomology of finite subgroup schemes of $GL_{n(r)}$.

Other extension classes in for strict polynomial functors play a role in more general cohomological finite-generation results by Touzé and van ker Kallen.

Troesch complexes, after Touzé

Goal

Given $m, r \geq 1$, describe an injective resolution in $\mathcal{P}_{p^r m}$ of $S^{m(r)}$.

It's only a 20 minute talk, so let's stick to the case $r = 1$.

Let Ш be the graded k-space with basis $\text{ш}_0, \ldots, \text{ш}_{p-1}$, $\deg(\text{ш}_i) = i$.

Consider the functor $S(\text{Ш} \otimes -) : U \mapsto S(\text{Ш} \otimes U)$.

$$S(\text{Ш} \otimes U) \cong S(\text{ш}_0 \otimes U) \otimes S(\text{ш}_1 \otimes U) \otimes \cdots \otimes S(\text{ш}_{p-1} \otimes U)$$

$S(\text{Ш} \otimes U)$ inherits an $\mathbb{N}$-grading from that on Ш:

$$S^n(\text{Ш} \otimes U)^\ell \cong \bigoplus_{\substack{i_0+i_1+\cdots+i_p=n \\ i_0\cdot 0+i_1\cdot 1+\cdots+i_{p-1}\cdot(p-1)=\ell}} S^{i_0}(U) \otimes S^{i_1}(U) \otimes \cdots \otimes S^{i_{p-1}}(U).$$

Troesch complexes, after Touzé

Define $\rho : \text{Ш} \to \text{Ш}$ by $\quad \rho(\text{ш}_i) = \begin{cases} \text{ш}_{i+1} & \text{if } 0 \le i \le p-2, \\ 0 & \text{if } i = p-1. \end{cases}$

Define $d : S^n(\text{Ш} \otimes U)^\ell \to S^n(\text{Ш} \otimes U)^{\ell+1}$ to be the composite

$$S^n(\text{Ш} \otimes U) \xrightarrow{\Delta} S^{n-1}(\text{Ш} \otimes U) \otimes S^1(\text{Ш} \otimes U)$$
$$\xrightarrow{\text{id} \otimes S(\rho \otimes \text{id}_U)} S^{n-1}(\text{Ш} \otimes U) \otimes S^1(\text{Ш} \otimes U) \xrightarrow{m} S^n(\text{Ш} \otimes U).$$

Remark

For $r = 1$, the map d is simply the algebra derivation on $S(\text{Ш} \otimes U)$ induced by the vector space map $\rho \otimes \text{id}_U : \text{Ш} \otimes U \to \text{Ш} \otimes U$.

Troesch complexes, after Touzé

Now $d : S^n(Ш \otimes -)^\ell \to S^n(Ш \otimes -)^{\ell+1}$ is a p-differential, i.e., $d^p = 0$.

Then the contraction

$$B_n^\bullet : S^n(Ш \otimes -)^0 \xrightarrow{d} S^n(Ш \otimes -)^1 \xrightarrow{d^{p-1}} S^n(Ш \otimes -)^p \xrightarrow{d} S^n(Ш \otimes -)^{p+1} \xrightarrow{d^{p-1}} S^n(Ш \otimes -)^{2p} \xrightarrow{d} \cdots$$

is an ordinary cochain complex with

$$B_n^{2i} = S^n(Ш \otimes -)^{pi} \quad \text{and} \quad B_n^{2i+1} = S^n(Ш \otimes -)^{pi+1}.$$

Theorem (Troesch)

$B_n^\bullet$ is acyclic if $p \nmid n$, and is an injective resolution of $S^{m(1)}$ if $n = pm$.

More generally, he constructs an injective resolution of $S^{m(r)}$, $r \geq 1$.

Note: For fixed n, one has $B_n^i = 0$ for $i \gg 0$.

Why are Troesch complexes the bee's knees?

Yoneda isomorphism, compatible with $\mathbb{Z}$-gradings

Let $F \in \mathcal{P}_m$. Let $F^{(1)} = F \circ I^{(1)}$. Then

$$\mathrm{Hom}_{\mathcal{P}}(F^{(1)}, S^{pm}(\text{Ш} \otimes -)) \cong F^{\#}(\text{Ш}^{(1)})$$

is concentrated in $\mathbb{Z}$-degrees divisible by p.

Then $\mathrm{Hom}_{\mathcal{P}}(F^{(1)}, B^{\bullet}_{pn})$ is concentrated in even degrees.

Corollary

$$\mathrm{Ext}^{\bullet}_{\mathcal{P}}(I^{(1)}, I^{(1)}) \cong \mathrm{Hom}_{\mathcal{P}}(I^{(1)}, B^{\bullet}_{p}) \cong E_1,$$

where E_1 the space Ш regraded so that $\mathrm{deg}(\text{ш}_i) = 2i$ $(0 \leq i < p)$.

More generally, Touzé applies Troesch's complexes to give short proofs of Ext-calculations between many twisted functors.

Strict polynomial superfunctors

Category of vector superspaces

Let $\mathcal{V}$ be the category of finite-dimensional k-vector superspaces.

$$V = V_{\overline{0}} \oplus V_{\overline{1}}$$

The symmetric group Σ_d acts on $V^{\otimes d}$ by signed place permutations.

$$V \otimes W \cong W \otimes V, \quad v \otimes w \mapsto (-1)^{\overline{v} \cdot \overline{w}} w \otimes v$$

The category $\Gamma^d \mathcal{V}$

Let $\Gamma^d \mathcal{V}$ be the category whose objects are the same as those in $\mathcal{V}$, but in which spaces of morphisms are defined by

$$\mathrm{Hom}_{\Gamma^d \mathcal{V}}(V, W) = \Gamma^d \, \mathrm{Hom}_k(V, W) \cong \mathrm{Hom}_{k\Sigma_d}(V^{\otimes d}, W^{\otimes d}),$$

and composition is that of $k\Sigma_d$-module homomorphisms.

Strict polynomial superfunctors, after Axtell

The category $\mathcal{P}_d$

The category $\boldsymbol{\mathcal{P}}_d$ of homogeneous degree-d strict polynomial superfunctors is the category of even linear functors

$$F : \boldsymbol{\Gamma}^d \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}},$$

i.e., functors such that for all $V, W \in \boldsymbol{\mathcal{V}}$, the function

$$F_{V,W} : \mathrm{Hom}_{k\boldsymbol{\Sigma}_d}(V^{\otimes d}, W^{\otimes d}) \to \mathrm{Hom}_k(F(V), F(W))$$

is an even (i.e., $\mathbb{Z}_2$-degree preserving) linear map.

Examples of homogeneous strict polynomial superfunctors

- $\mathbf{\Pi} \in \mathcal{P}_1$ — parity change functor
- $\mathbf{\Gamma}^d : V \mapsto \mathbf{\Gamma}^d(V) = (V^{\otimes d})^{\Sigma_d}$ — $\mathbf{\Gamma}(V) \cong \Gamma(V_{\bar{0}}) \otimes \Lambda(V_{\bar{1}})$
- $\mathbf{A}^d : V \mapsto [\mathrm{sgn} \otimes (V^{\otimes d})]^{\Sigma_d}$ — $\mathbf{A}(V) \cong \Lambda(V_{\bar{0}}) \otimes \Gamma(V_{\bar{1}})$
- $\mathbf{\Lambda}^d : V \mapsto \mathbf{\Lambda}^d(V)$ — $\mathbf{\Lambda}(V) \cong \Lambda(V_{\bar{0}}) \otimes S(V_{\bar{1}})$
- $\mathbf{S}^d : V \mapsto \mathbf{S}^d(V) = (V^{\otimes d})_{\Sigma_d}$ — $\mathbf{S}(V) \cong S(V_{\bar{0}}) \otimes \Lambda(V_{\bar{1}})$
- $\mathbf{\Gamma}^d(\mathrm{Hom}_k(W, -))$, $\mathbf{A}^d(\mathrm{Hom}_k(W, -))$ — projective objects
- $\mathbf{S}^d(W \otimes -)$, $\mathbf{\Lambda}^d(W \otimes -)$ — injective objects

For $r \geq 1$,

- $I^{(r)} = I_0^{(r)} \oplus I_1^{(r)}$ where $I_0^{(r)}(V) = V_{\bar{0}}^{(r)}$ and $I_1^{(r)}(V) = V_{\bar{1}}^{(r)}$

Power maps induce embeddings $I_0^{(r)} \hookrightarrow \mathbf{S}^{p^r}$ and $I_1^{(r)} \hookrightarrow \mathbf{\Lambda}^{p^r}$.

Why (else) do we care about strict polynomial superfunctors?

Theorem (Axtell)

Let $V \in \boldsymbol{\mathcal{V}}$. If $V \cong k^{m|n}$ and $m, n \geq d$, then evaluation on V

$$F \mapsto F(V)$$

defines an equivalence of categories $\boldsymbol{\mathcal{P}}_d \simeq S(m|n, d)$-smod.

Theorem (Drupieski)

Extension classes in $\mathrm{Ext}^{\bullet}_{\boldsymbol{\mathcal{P}}}(I^{(r)}, I^{(r)})$ restrict nontrivially to the affine supergroup scheme $GL_{m|n}$ and its Frobenius kernel $GL_{m|n(r)}$, and together with the generators exhibited by Friedlander and Suslin, give generators for the cohomology of finite supergroup schemes.

Would like to better understand other extension groups in $\boldsymbol{\mathcal{P}}$, e.g., between Frobenius twists of classical exponential functors.

Naive generalization of Troesch's construction

Consider Ш as a $\mathbb{Z}$-graded superspace of purely even superdegree.

For $U = U_{\bar{0}} \oplus U_{\bar{1}}$, consider

$$S(\text{Ш} \otimes U) \cong S(\text{ш}_0 \otimes U) \otimes S(\text{ш}_1 \otimes U) \otimes \cdots \otimes S(\text{ш}_{p-1} \otimes U).$$

Define $d : S(\text{Ш} \otimes U)^{\ell} \to S(\text{Ш} \otimes U)^{\ell+1}$ exactly as before.

Cocycles (by virtue of d being a derivation when $r = 1$)

For $u \in U_{\bar{0}}$, get

$$(\text{ш}_0 \otimes u)^p \in S^p(\text{Ш} \otimes U)^0.$$

New for super: If $u \in U_{\bar{1}}$, get

$$u^{(1)} := (\text{ш}_0 \otimes u) \otimes (\text{ш}_1 \otimes u) \otimes \cdots \otimes (\text{ш}_{p-1} \otimes u) \in S^p(\text{Ш} \otimes U)^{p(p-1)/2}$$

in the exterior algebra part of $S(\text{Ш} \otimes U) \cong S(\text{Ш} \otimes U_{\bar{0}}) \otimes \Lambda(\text{Ш} \otimes U_{\bar{1}})$

Naive generalization of Troesch's construction

Let $B_n^\bullet$ be the contracted complex of strict polynomial superfunctors

$$B_n^\bullet : S^n(Ш \otimes -)^0 \xrightarrow{d} S^n(Ш \otimes -)^1 \xrightarrow{d^{p-1}} S^n(Ш \otimes -)^p \xrightarrow{d} S^n(Ш \otimes -)^{p+1} \xrightarrow{d^{p-1}} S^n(Ш \otimes -)^{2p} \xrightarrow{d} \cdots$$

Theorem (Drupieski–Kujawa)

$$\mathrm{H}^\bullet(B_n) \cong \begin{cases} 0 & \text{if } p \nmid n, \\ S^{m(1)} & \text{if } n = pm. \end{cases}$$

In the latter case, for $0 \leq \ell \leq m$, the summand

$$(S^{m-\ell} \circ I_0^{(1)}) \otimes (\Lambda^\ell \circ I_1^{(1)})$$

of $S^{m(1)}$ is in cohomological degree $\ell \cdot (p-1)$.

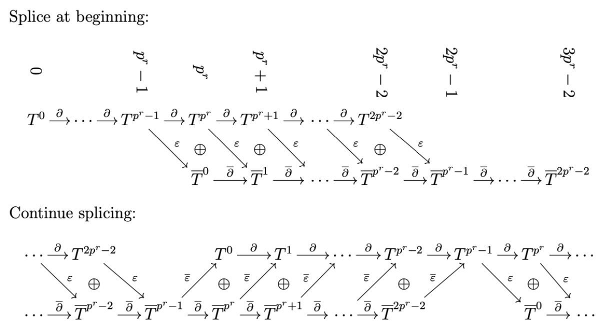
Resolutions of injectives

In the case $n = p$, get a complex of injective objects

$$B_p^0 \to B_p^1 \to \cdots \to B_p^{p-1} \to \cdots \to B_p^{2(p-1)}$$

with $\quad \mathrm{H}^0(B_p) \cong I_0^{(1)}, \quad \mathrm{H}^{p-1}(B_p) \cong I_1^{(1)}, \quad$ and $\quad \mathrm{H}^i(B_p) = 0$ otherwise.

These complexes can be spliced together:



Calculations

End result of splicing

For all $r \geq 1$, construct periodic injective resolutions

$$I_0^{(r)} \to J(r) \quad \text{and} \quad I_1^{(r)} \to \bar{J}(r).$$

Corollary ("quick" recalculation)

$$\mathrm{Ext}^{\bullet}_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) \cong \mathrm{Hom}_{\mathcal{P}}(I_0^{(r)}, J(r)) \cong \bigoplus_{n \geq 0} E_r\langle 2np^r \rangle$$

$$\mathrm{Ext}^{\bullet}_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}) \cong \mathrm{Hom}_{\mathcal{P}}(I_1^{(r)}, J(r)) \cong \bigoplus_{n \geq 0} E_r\langle (2n+1)p^r \rangle$$

where $E_r = \bigoplus_{0 \leq i < p^r} k\langle 2i \rangle$.

More calculations (after Franjou, Friedlander, Scorichenko, and Suslin)

For $1 \leq j \leq r$ and $\ell \in \{0,1\}$, set

$$V_{j,\ell} = \mathrm{Ext}^\bullet_{\mathcal{P}}(I_\ell^{(r)}, S_0^{p^{r-j}(j)}), \qquad W_{j,\ell} = \mathrm{Ext}^\bullet_{\mathcal{P}}(I_\ell^{(r)}, \Lambda_0^{p^{r-j}(j)}),$$
$$\overline{V}_{j,\ell} = \mathrm{Ext}^\bullet_{\mathcal{P}}(I_\ell^{(r)}, S_1^{p^{r-j}(j)}), \qquad \overline{W}_{j,\ell} = \mathrm{Ext}^\bullet_{\mathcal{P}}(I_\ell^{(r)}, \Lambda_1^{p^{r-j}(j)}).$$

Using the superized Troesch complexes in lieu of the de Rham and Koszul complexes:

Theorem

Let $\ell \in \{0,1\}$. For all $d \geq 1$ and all $1 \leq j \leq r$, the cup product maps

$$(V_{j,\ell})^{\otimes d} \to \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, S_0^{dp^{r-j}(j)}), \qquad (W_{j,\ell})^{\otimes d} \to \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, \Lambda_0^{dp^{r-j}(j)}),$$
$$(\overline{V}_{j,\ell})^{\otimes d} \to \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, S_1^{dp^{r-j}(j)}), \qquad (\overline{W}_{j,\ell})^{\otimes d} \to \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, \Lambda_1^{dp^{r-j}(j)})$$

factor to induce isomorphisms of graded vector spaces

$$S^d(V_{j,\ell}) \cong \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, S_0^{dp^{r-j}(j)}), \qquad \Lambda^d(W_{j,\ell}) \cong \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, \Lambda_0^{dp^{r-j}(j)}),$$
$$S^d(\overline{V}_{j,\ell}) \cong \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, S_1^{dp^{r-j}(j)}), \qquad \Lambda^d(\overline{W}_{j,\ell}) \cong \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, \Lambda_1^{dp^{r-j}(j)}).$$