Support schemes for infinitesimal unipotent supergroups

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Support schemes

Let A be a Hopf algebra over a field k, $\mathrm{char}(k) = p > 0$.

Then $\mathrm{H}^\bullet(A, k)$ is a graded-commutative algebra.

Cohomological spectrum and support varieties

The **cohomological spectrum** of A is the affine scheme

$$|A| = \mathrm{Spec}\left(\mathrm{H}^\bullet(A, k)\right).$$

Given an A-module M, let $I_A(M)$ be the kernel of the (k-algebra) map

$$\mathrm{H}^\bullet(A, k) = \mathrm{Ext}_A^\bullet(k, k) \xrightarrow{-\otimes M} \mathrm{Ext}_A^\bullet(M, M).$$

The **cohomological support scheme** associated to M is

$$|A|_M = \mathrm{Spec}\left(\mathrm{H}^\bullet(A, k)/I_A(M)\right).$$

Properties of support schemes

Let A be a finite-dimensional Hopf algebra such that $\mathrm{H}^\bullet(A,k)$ is a finitely-generated k-algebra, and such that $\mathrm{H}^\bullet(A,M)$ is finite over $\mathrm{H}^\bullet(A,k)$ for each finite-dimensional A-module M.

Then one has the following good properties for cohomological support schemes of finite-dimensional modules:

Some good properties of support schemes

- $\dim |A|_M = \mathrm{cx}_A(M)$, the complexity of M as an A-module.
- $|A|_M = \{0\} \iff M$ is projective $\iff M$ is injective.
- $|A|_{M \oplus N} = |A|_M \cup |A|_N$.
- If A is cocommutative, then $|A|_{M \otimes N} \subseteq |A|_M \cap |A|_N$.

Finite groups

Case $A = kG$, group algebra of a finite group G, $\mathrm{char}(k) = p > 0$.

- If $E = (\mathbb{Z}/p\mathbb{Z})^n$ is elementary abelian, then $|kE|$ is affine n-space.
- In general, Quillen (1971) showed that $|kG|$ is stratified by pieces coming from the elementary abelian p-subgroups of G.

$$|kG| = \bigcup_{\substack{E \leq G \\ E \text{ elem. ab.}}} \mathrm{res}^*_{G,E}(|kE|)$$

- In the 1980s, Carlson conjectured, and Avrunin and Scott proved, that $|kG|_M$ can also be computed in terms of data coming from the elementary abelian p-subgroups of G.

$$|kG|_M = \bigcup_{\substack{E \leq G \\ E \text{ elem. ab.}}} \mathrm{res}^*_{G,E}(|kE|_M)$$

Restricted Lie algebras

Suppose k is a field of characteristic $p > 0$.

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie superalgebra over k.

Let $V(\mathfrak{g})$ be the restricted enveloping algebra of $\mathfrak{g}$.

Friedlander–Parshall (1986); Suslin–Friedlander–Bendel (1997)

There is a homeomorphism

$$|V(\mathfrak{g})| \simeq \mathcal{N}_p(\mathfrak{g}) = \left\{ x \in \mathfrak{g} : x^{[p]} = 0 \right\}.$$

For each finite-dimensional $V(\mathfrak{g})$-module M, one has

$$|V(\mathfrak{g})|_M \simeq \left\{ x \in \mathcal{N}_p(\mathfrak{g}) : M|_{\langle x \rangle} \right\}.$$

In particular, projectivity of modules is detected by restriction to subalgebras of $V(\mathfrak{g})$ of the form $k[x]/(x^p)$.

$$|V(\mathfrak{g})| \simeq \mathcal{N}_p(\mathfrak{g}) = \left\{x \in \mathfrak{g} : x^{[p]} = 0\right\}$$

$$|V(\mathfrak{g})|_M \simeq \left\{x \in \mathcal{N}_p(\mathfrak{g}) : M|_{\langle x \rangle}\right\}$$

From these explicit descriptions for $|V(\mathfrak{g})|$ and $|V(\mathfrak{g})|_M$, we get:

Naturality

If $\mathfrak{h} \subseteq \mathfrak{g}$ is a restricted Lie subalgebra, then

$$|V(\mathfrak{h})| \simeq \mathcal{N}_p(\mathfrak{h}) \subseteq \mathcal{N}_p(\mathfrak{g}) \simeq |V(\mathfrak{g})|$$

and under this identification,

$$|V(\mathfrak{h})|_M = |V(\mathfrak{g})|_M \cap \mathfrak{h}.$$

Infinitesimal group schemes

Equivalences

- finite group scheme $G \leftrightarrow$ f.d. cocommutative Hopf algebra kG
- infinitesimal group scheme $G \leftrightarrow$ f.d. cocom. Hopf algebra kG such that the dual Hopf algebra $(kG)^* = k[G]$ is local

Examples

- If G is an ordinary finite group, then its usual group algebra is the group algebra of a finite group scheme.
- If $\mathfrak{g}$ is a finite-dimensional restricted Lie algebra, then there exists an infinitesimal group scheme such that $kG = V(\mathfrak{g})$.

Suslin–Friedlander–Bendel (1997)

Let G be an infinitesimal group scheme of height $\leq r$. Then there exists a homeomorphism

$$|kG| \cong V_r(G) := \mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, G).$$

Call $V_r(G)$ the scheme of one-parameter subgroups in G.

Example

For $G = GL_{n(r)}$, the r-th Frobenius kernel of GL_n, one has

$$V_r(GL_{n(r)}) \cong \left\{(\alpha_0, \ldots, \alpha_{r-1}) \in \mathfrak{gl}_n^{\times r} : \alpha_i^p = 0, [\alpha_i, \alpha_j] = 0, \forall i, j\right\}.$$

If $\nu : \mathbb{G}_{a(r)} \to G$ is a one-parameter subgroup, and if M is a rational G-module, then M pulls back to a rational $\mathbb{G}_{a(r)}$-module, $\nu^*(M)$.

Equivalently, $\nu^*(M)$ is a module over the group algebra

$$k\mathbb{G}_{a(r)} = \left(k[T]/(T^{p^r})\right)^{\#} = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p).$$

Suslin–Friedlander–Bendel (1997)

Let G be infinitesimal of height $\leq r$. If M is a finite-dimensional rational G-module, then

$$|kG|_M \cong \left\{\nu \in V_r(G) : \nu^*(M) \text{ is not free over } k[u_{r-1}]/(u_{r-1}^p)\right\}.$$

Consequences: Naturality, $\otimes$-property $|kG|_{M\otimes N} = |kG|_M \cap |kG|_N$, ...

π-points

Let G be a finite group scheme over the field k. A **π-point** is a flat map of K-algebras $\alpha_K : K[t]/(t^p) \to KG$ for some field extension K/k such that α factors through the group algebra of some unipotent abelian subgroup scheme $U_K \subseteq G_K$.

Say that α_K is equivalent to β_L if for all finite-dimensional M one has

$$\alpha^*(M_K) \text{ is projective} \iff \beta^*(M_L) \text{is projective}$$

Let $\Pi(G)$ be the set of equivalence classes of π-points.

Friedlander–Pevtsova (1997)

There is a natural homeomorphism

$$\Psi_G : \Pi(G) \xrightarrow{\sim} \mathrm{Proj}\,(\mathrm{H}^\bullet(G, k))$$

which restricts to $|G|_M \simeq \{[\alpha] : \alpha^*(M_K) \text{ is not projective}\}$.

How can we begin to generalize this to supergroups?

Supergroups

A group in stratigraphy is a lithostratigraphic unit, a part of the geologic record or rock column that consists of defined rock strata. Groups are generally divided into individual formations. Groups may sometimes be divided into “subgroups” and are themselves sometimes grouped into “**supergroups**.”

Superalgebra

Super $\equiv$ graded by $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$

- Super vector spaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$
- $V \otimes W \cong W \otimes V$ via the supertwist $v \otimes w \mapsto (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$

Define Hopf superalgebras to be Hopf algebra objects in the (tensor) category of vector superspaces.

Examples of Hopf superalgebras

- Ordinary Hopf algebras (as purely even superalgebras)
- $\mathbb{Z}$-graded Hopf algebras in the sense of Milnor and Moore
- Enveloping superalgebras of (restricted) Lie superalgebras
- Exterior algebra $\Lambda(V)$ over a (purely odd) vector space V (both commutative and cocommutative in the super sense)

Infinitesimal unipotent supergroups

kG is the group algebra of a **finite supergroup (scheme)** G if:

- kG is a finite-dimensional cocommutative Hopf superalgebra.

Then G is **infinitesimal** if

- the dual Hopf superalgebra $k[G] = kG^{\#}$ is local,

and G is **unipotent** if

- kG is a local k-algebra.

Some important (for us) Hopf superalgebras

- $\mathbb{P}_r = k[u_0, \ldots, u_{r-1}, v]/(u_0^p, \ldots, u_{r-2}^p, u_{r-1}^p + v^2)$, where
    - $u_0, \ldots, u_{r-1}$ are of even superdegree, v is of odd superdegree,
    - coproducts for $u_0, \ldots, u_{r-1}$ look like they do in $k\mathbb{G}_{a(r)}$,
    - u_{r-1}^p and v are primitive.
- $\mathbb{P}_r$ is a commutative complete intersection.
- In particular, $\mathbb{P}_1 = k[u, v]/(u^p + v^2)$ is a hypersurface ring.

Support space

$$\mathbb{P}_r = k[u_0, \ldots, u_{r-1}, v]/(u_0^p, \ldots, u_{r-2}^p, u_{r-1}^p + v^2)$$

Lemma

Let G be a finite k-supergroup scheme. Then the functor from commutative k-algebras to sets,

$$V_r(G) : A \mapsto V_r(G)(A) = \mathrm{Hom}_{sHopf/A}(\mathbb{P}_r \otimes_k A, kG \otimes_k A),$$

admits the structure of an affine scheme of finite type over k.

Support schemes for modules

$\mathbb{P}_1 = k[u, v]/(u^p + v^2)$

Superalgebra map $\iota : \mathbb{P}_1 \hookrightarrow \mathbb{P}_r$ defined by $\iota(u) = u_{r-1}$ and $\iota(v) = v$.

The support scheme $V_r(G)_M$

Let G be a finite k-supergroup scheme and M a finite-dimensional kG-supermodule. Set

$$V_r(G)_M = \left\{ \nu \in \mathrm{Hom}_{sHopf}(\mathbb{P}_r, kG) : \mathrm{projdim}_{\mathbb{P}_1}(\iota^* \nu^* M) = \infty \right\}.$$

Proposition

Then $V_r(G)_M$ is a Zariski closed conical subset of $V_r(G)$.

Key ingredient of the proof: Explicit $\mathbb{P}_1$-projective resolution of k constructed via Eisenbud's theory of matrix factorizations.

$$V_r(G) = \mathrm{Hom}_{sHopf}(\mathbb{P}_r, kG)$$
$$V_r(G)_M = \left\{\nu \in V_r(G) : \mathrm{projdim}_{\mathbb{P}_1}(\iota^*\nu^*M) = \infty\right\}$$

Remarks

- If G is an ordinary finite group scheme, i.e., kG is purely even, Then $V_r(G) = \mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, G)$, as defined by SFB.
- If $kG = V(\mathfrak{g})$ for a f.d. restricted Lie superalgebra $\mathfrak{g}$, and if $p \geq 5$, then points in $V_1(G)$ identify with subalgebras of $\mathfrak{g}$ generated by $u \in \mathfrak{g}_{\bar{0}}$ and $v \in \mathfrak{g}_{\bar{1}}$ such that $[u, v] = 0$ and $u^{[p]} + \frac{1}{2}[v, v] = 0$.
- **Naturality**: If H is a closed subsupergroup of G, then $V_r(H)$ is closed in $V_r(G)$, and $V_r(H)_M = V_r(G)_M \cap V_r(H)$.
- **Stratification**: $V_r(G)_M = \bigcup_{kE \leq kG} V_r(E)_M$
  kE ranges over fin. dim. Hopf quotients of $\mathbb{P}_r$ (described later).

Main Theorem

Drupieski–Kujawa (arXiv 2018)

Let G be an **infinitesimal unipotent** supergroup scheme of height $\leq r$. Then there is a natural k-algebra map $\psi : H(G, k) \to k[V_r(G)]$, which defines a universal homeomorphism of schemes

$$|G| \simeq V_r(G) = \mathrm{Hom}_{sHopf}(\mathbb{P}_r, kG).$$

This restricts for each finite-dimensional kG-supermodule M to a homeomorphism

$$|G|_M \simeq V_r(G)_M = \left\{\nu \in V_r(G) : \mathrm{projdim}_{\mathbb{P}_1}(\iota^* \nu^* M) = \infty\right\}.$$

If G is a purely even, theorem reduces to unipotent case of SFB.

Proof ingredients: ‘superization’ of SFB + detection theorem of BIKP.

Example

Let $G = \mathbb{G}_{a(1)} \times \mathbb{G}_a^-$.

$kG = k[s]/(s^p) \otimes k[t]/(t^2)$

$\mathrm{H}^\bullet(G, k) \cong k[x, y] \otimes \Lambda(\lambda)$

Then $|kG| \cong k^2 \cong V_1(G) = \mathrm{Hom}_{sHopf}(\mathbb{P}_1, kG)$.

$(c, d) \in k^2$ defines $\nu_{(c,d)} : \mathbb{P}_1 \to kG, \quad \nu_{(c,d)}(u) = d \cdot s, \quad \nu_{(c,d)}(v) = c \cdot t.$

Let $L = L_{(\mu,a)}$ be the $2p$-dimensional kG-supermodule ...

draw on board

Proposition

$V_1(G)_L$ is the affine line in $V_1(G)$ through (μ, a).

Toward the non-unipotent (infinitesimal) case

For arbitrary infinitesimal G, we don't have a description for $|G|$. However, previous calculations show that, up to a finite morphism, $\left|GL_{m|n(r)}\right|$ seems to identify with $V_r(GL_{m|n(r)})(k)$.

$$V_r(GL_{m|n(r)}) \cong \Big\{(\alpha_0, \ldots, \alpha_{r-1}, \beta) \in \mathfrak{gl}(m|n)_{\bar{0}}^{\times r} \times \mathfrak{gl}(m|n)_{\bar{1}} : \\ [\alpha_i, \alpha_j] = [\alpha_i, \beta] = 0 \text{ for all } 0 \le i, j \le r-1, \\ \alpha_i^p = 0 \text{ for all } 0 \le i \le r-2, \text{ and } \alpha_{r-1}^p + \beta^2 = 0\Big\}.$$

So, betting on the inertia of truth, perhaps $V_r(G)$ is the correct ambient scheme to consider even for non-unipotent G?

$V_r(G)_M$ also makes sense, and is closed in $V_r(G)$, for non-unipotent G.

Let $f = T^{p^t} + \sum_{i=1}^{t-1} a_i T^{p^i} \in k[T]$ be a p-polynomial (no linear term).

Let $\eta \in k$ be a scalar.

The infinitesimal multiparameter supergroup $\mathbb{M}_{r;f,\eta}$

$$k\mathbb{M}_{r;f,\eta} = \mathbb{P}_r / \langle f(u_{r-1}) + \eta u_0 \rangle$$

Proposition

Every finite-dimensional Hopf quotient of $\mathbb{P}_r$ is of the form

- $k\mathbb{G}_{a(s)}$ for some $0 \leq s \leq r$,
- $k\mathbb{G}_a^- = k[v]/\langle v^2 \rangle$, or $\qquad \mathbb{G}_a^-(A) = (A_{\bar{1}}, +)$
- $k\mathbb{M}_{s;f,\eta}$ for some $1 \leq s \leq r$ and some f, η as above.

Benson–Iyengar–Krause–Pevtsova

For **unipotent** finite supergroup schemes, projectivity of modules and nilpotence in cohomology are detected (after field extension) by restriction to **'elementary'** subsupergroup schemes.

The *infinitesimal* elementary k-supergroup schemes are

- $\mathbb{G}_{a(r)}$ for $r \geq 0$,
- $\mathbb{G}_{a(r)} \times \mathbb{G}_a^-$ for $r \geq 0$,
- $\mathbb{M}_{r;T^{p^s},0}$ for $r, s \geq 1$,
- $\mathbb{M}_{r,T^{p^s},\eta}$ for $r \geq 2$, $s \geq 1$, and $0 \neq \eta \in k$.

The group algebras of these each occur as Hopf quotients of $\mathbb{P}_r$.

Roughly: $\mathbb{M}_{r;f,\eta}$ is unipotent if the polynomial f is a single monomial.

Question

For arbitrary infinitesimal supergroups, is projectivity of modules and nilpotence in cohomology detected (after field extension) by restriction to finite-dimensional Hopf superalgebra quotients of $\mathbb{P}_r$?

Seems likely that the hardest part of extending $|G|_M \simeq V_r(G)_M$ to the non-unipotent case will be answering this question.

For non-unipotent G, you need more than just the unipotent elementary subsupergroup schemes to detect nilpotents in cohomology. For example, the cohomology of

$$k\mathbb{M}_{1;T^p,-1} = k[u,v]/(u^p + v^2, u^p - u)$$

is not detected by restriction to unipotent subgroup schemes.

The end?