Support varieties for Lie superalgebras

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Recollections

Support varieties

Suppose $\mathrm{H}^\bullet(A, k)$ is "commutative" and finitely generated. Then the maximal ideal spectrum

$$|A| = \mathrm{Max}\left(\mathrm{H}^\bullet(A, k)\right)$$

is an affine algebraic variety. Given an A-module M, have a map

$$\mathrm{H}^\bullet(A, k) \to \mathrm{Ext}_A^\bullet(M, M)$$

with annihilator ideal $I_A(M)$.

Support varieties

The **cohomological support variety** associated to M is

$$|A|_M = \mathrm{Max}\left(\mathrm{H}^\bullet(A, k)/I_A(M)\right),$$

a closed subvariety of the **cohomological spectrum** $|A|$.

Support varieties thus attach geometric invariants to A-modules.

Classical Results

k a field of positive characteristic

Carlson's Conjecture (proved by Avrunin–Scott ca. 1982)

Let G be a finite group, and let M be a f.g. kG-module. Then

$$|kG|_M = \bigcup_{\substack{E \leq G \\ \text{elem abel}}} \operatorname{res}_{G,E} |kE|_M$$

Friedlander–Parshall (ca. 1986)

Let $\mathfrak{g}$ be a f.d. restricted Lie algebra (and some other hypotheses). Then for each f.d. $V(\mathfrak{g})$-module M,

$$|V(\mathfrak{g})|_M = \left\{ X \in \mathfrak{g} : X^{[p]} = 0 \text{ and } M|_{\langle X \rangle} \text{ is not free} \right\} \cup \{0\}\,.$$

Later generalizations

Suslin–Friedlander–Bendel (ca. 1997)

Always have a homeomorphism

$$|V(\mathfrak{g})|_M \simeq \left\{X \in \mathfrak{g} : X^{[p]} = 0 \text{ and } M|_{\langle X\rangle} \text{ is not free}\right\} \cup \{0\}\,.$$

More generally, they describe $|G|_M$ for infinitesimal group scheme G in terms of the variety of 1-parameter subgroups $\nu : \mathbb{G}_{a(r)} \to G$.

Friedlander–Pevtsova (2000s)

Describe $|G|_M$ for G a finite group scheme in terms of Π-points.

Question

How much generalizes to $\mathbb{Z}$- or $\mathbb{Z}/2\mathbb{Z}$-graded (super) settings?

Super linear algebra

What does it mean to be "super"?

Something is "super" if it has a compatible $\mathbb{Z}/2\mathbb{Z}$-grading.

- Superspaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$, $W = W_{\overline{0}} \oplus W_{\overline{1}}$
- Induced gradings on tensor products, linear maps, etc.

$$(V \otimes W)_\ell = \bigoplus_{i+j=\ell} V_i \otimes W_j$$

$$\mathrm{Hom}_k(V,W)_\ell = \{f \in \mathrm{Hom}_k(V,W) : f(V_i) \subseteq W_{i+\ell}\}$$

- $V \otimes W \cong W \otimes V$ via the **supertwist** $v \otimes w \mapsto (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$

Define (Hopf) superalgebras and 'super' (co)commutativity in terms of the "usual diagrams," but use the supertwist when objects pass.

Simplest possible example

Exterior algebra of a finite-dimensional vector space V

The exterior algebra $\Lambda(V)$ is a (super)commutative superalgebra:

$$ab = (-1)^{\overline{a}\cdot\overline{b}} ba$$

It is also a (super)cocommutative Hopf superalgebra:

$$\begin{aligned}
&\Delta(uv) \\
&= \Delta(u)\Delta(v) \\
&= (u \otimes 1 + 1 \otimes u)(v \otimes 1 + 1 \otimes v) \\
&= (u \otimes 1)(v \otimes 1) + (u \otimes 1)(1 \otimes v) + (1 \otimes u)(v \otimes 1) + (1 \otimes u)(1 \otimes v) \\
&= (uv \otimes 1) + (u \otimes v) - (v \otimes u) + (1 \otimes uv) \\
&= (uv \otimes 1) + (1 \otimes uv)
\end{aligned}$$

Hopf superalgebras

Examples of Hopf superalgebras

– Ordinary Hopf algebras (as purely even superalgebras).

– $\mathbb{Z}$-graded Hopf algebras in the sense of Milnor and Moore

– Enveloping superalgebras of (restricted) Lie superalgebras

Recall that a **Lie superalgebra** is a superspace $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ equipped with an even map $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that for homogeneous x, y, z,

- $[x, y] = -(-1)^{\overline{x} \cdot \overline{y}}[y, x]$
- $[x, [y, z]] = [[x, y], z] + (-1)^{\overline{x} \cdot \overline{y}}[y, [x, z]]$
- $[x, x] = 0$ if $x \in \mathfrak{g}_{\overline{0}}$ and $char(k) = p = 2$
- $[x, [x, x]] = 0$ if $x \in \mathfrak{g}_{\overline{1}}$ and $char(k) = p = 3$

Say that $\mathfrak{g}$ is **restricted** if $\mathfrak{g}_{\overline{0}}$ is an ordinary restricted Lie algebra and $\mathfrak{g}_{\overline{1}}$ is a restricted $\mathfrak{g}_{\overline{0}}$-module under the adjoint action.

Supergroup schemes

Classical correspondences

affine group schemes $\leftrightarrow$ cocommutative Hopf algebras

finite group schemes $\leftrightarrow$ f.d. cocommutative Hopf algebras

height-one group schemes $\leftrightarrow$ f.d. restricted Lie algebras

Super correspondences

affine supergroup schemes $\leftrightarrow$ cocommutative Hopf superalgebras

finite supergroup schemes $\leftrightarrow$ f.d. cocommut. Hopf superalgebras

height-one supergroup schemes $\leftrightarrow$ f.d. res. Lie superalgebras

Some characteristic zero theory

Simplest example

Theorem

Let V be a finite-dimensional space. Then $\mathrm{H}^\bullet(\Lambda(V), k) \cong S^\bullet(V^*)$.

The cohomology ring is **graded-(super)commutative** in the sense

$$ab = (-1)^{\deg(a)\cdot\deg(b)+\overline{a}\cdot\overline{b}} ba.$$

Aramova–Avramov–Herzog (2000)

Let M be a finite-dimensional $\Lambda(V)$-supermodule. Then

$$|\Lambda(V)|_M \cong \left\{ v \in V : M|_{\langle v \rangle} \text{ is not free} \right\}.$$

In the theorem, $\langle v \rangle$ refers to an algebra isomorphic to $\Lambda(v) \cong k[v]/\langle v^2 \rangle$.

In characteristic 0, this is most of the complete picture!

Classification in characteristic zero

Suppose k is an algebraically closed field of characteristic 0.

Kostant

Let A be a cocommutative Hopf superalgebra over k. Then

$$A \cong U(\mathfrak{g}) \# kG$$

for some Lie superalgebra $\mathfrak{g}$ over k and some subgroup $G \leq \mathrm{Aut}(\mathfrak{g})$.

Corollary

Let A be a finite-dimensional cocommutative Hopf superalgebra over k. Then $A \cong \Lambda(V) \# kG$ for some finite group G and some f.d. purely odd kG-module V.

Given $\Lambda(V) \# kG$ as in the Corollary, denote the corresponding finite supergroup scheme by $V \rtimes G$.

Cohomology and support varieties

Theorem

Let $V \rtimes G$ be a finite k-supergroup scheme. Let M and N be $V \rtimes G$-supermodules. Then $\operatorname{Ext}^{\bullet}_{V \rtimes G}(M, N) \cong \operatorname{Ext}^{\bullet}_{\Lambda(V)}(M, N)^G$. In particular,

$$\mathrm{H}^{\bullet}(V \rtimes G, k) \cong \mathrm{H}^{\bullet}(\Lambda(V), k)^G \cong S^{\bullet}(V^*)^G.$$

Corollary

Let $V \rtimes G$ be a finite k-supergroup scheme, and let M be a finite-dimensional $V \rtimes G$-supermodule. Then

$$|V \rtimes G| \cong V/G, \quad \text{the quotient of } V \text{ by } G \text{, and}$$
$$|V \rtimes G|_M \cong \left\{ [v] \in V/G : M|_{\langle v \rangle} \text{ is not free} \right\} \cup \{0\}.$$

Lie superalgebras

Background

‘Ordinary’ cohomology of Lie superalgebra

$\mathrm{H}^\bullet(\mathfrak{g}, k)$ is the cohomology ring of enveloping superalgebra $U(\mathfrak{g})$

$\mathrm{H}^\bullet(\mathfrak{g}, k)$ can be computed via the super Koszul resolution $(\boldsymbol{\Lambda}(\mathfrak{g}^*), \partial)$

As a superalgebra, $\boldsymbol{\Lambda}(\mathfrak{g}^*) \cong \Lambda(\mathfrak{g}_0^*) \; {}^g\!\otimes S(\mathfrak{g}_1^*)$.

In char. 0, $\mathrm{H}^\bullet(\mathfrak{g}, k)$ can be either f.d. or infinite-dimensional.

Positive characteristic (k algebraically closed, $p \geq 3$)

p-th powers in $S(\mathfrak{g}_{\bar{1}}^*) \subset \mathbf{\Lambda}(\mathfrak{g}^*)$ consist of cocycles, so get a map

$$\varphi : S(\mathfrak{g}_{\bar{1}}^*[p])^{(1)} \to \mathrm{H}^\bullet(\mathfrak{g}, k).$$

Study $|\mathfrak{g}| := \operatorname{Max}(\mathrm{H}^\bullet(\mathfrak{g}, k))$ via this map.

Theorem (DK)

Let $\mathfrak{g}$ be a f.d. Lie superalgebra. Let M be a finite-dimensional $\mathfrak{g}$-supermodule. Then there are homeomorphisms

$$|\mathfrak{g}| \cong \{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0\}$$
$$|\mathfrak{g}|_M \cong \left\{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0 \text{ and } M|_{\langle x \rangle} \text{ is not free}\right\} \cup \{0\}.$$

Since $[x, x] = 0$, have isomorphism of algebras $\langle x \rangle \cong k[x]/\langle x^2 \rangle$.

Identical in definition to varieties previously defined by Duflo and Serganova in characteristic 0 without using cohomology!

Cohomology of finite supergroup schemes

CFG for finite supergroup schemes

First step toward support varieties: cohomological finite generation

Drupieski (Adv. Math. 2016)

Let G be a finite supergroup scheme (equiv., a f.d. cocommutative Hopf superalgebra) over k and let M be a f.d. G-supermodule. Then $\mathrm{H}^\bullet(G,k)$ is a f.g. k-superalgebra and $\mathrm{H}^\bullet(G,M)$ is finite over $\mathrm{H}^\bullet(G,k)$.

Proved by way of cohomology calculations in the category of **strict polynomial superfunctors**, analogous to the argument for ordinary finite group schemes by Friedlander and Suslin.

Example of an ordinary strict polynomial functor

Suppose V has basis $\{u, v\}$ and W has basis $\{x, y\}$.

Then $S^2(V)$ has basis $\left\{u^2, uv, v^2\right\}$ and $S^2(W)$ has basis $\left\{x^2, xy, y^2\right\}$.

Let $\phi : V \to W$ be the linear map with associated matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

The linear map $S^2(\phi) : S^2(V) \to S^2(W)$ is defined for $f \in S^2(V)$ by

$$S^2(\phi)(f(u, v)) = f(\phi(u), \phi(v)).$$

The associated matrix for $S^2(\phi)$ is then

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & (ad + cb) & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

Strict polynomial superfunctors

Examples of strict polynomial superfunctors

Π parity flip functor	$(\Pi V)_{\bar 0} = V_{\bar 1}$, $(\Pi V)_{\bar 1} = V_{\bar 0}$
$\boldsymbol{\Gamma}^d(V) = (V^{\otimes d})^{\Sigma_d}$ super-symmetric tensors	$\boldsymbol{\Gamma}(V) = \Gamma(V_{\bar 0}) \otimes \Lambda(V_{\bar 1})$
$\boldsymbol{S}^d(V) = (V^{\otimes d})_{\Sigma_d}$ super-symmetric power	$\boldsymbol{S}(V) = S(V_{\bar 0}) \otimes \Lambda(V_{\bar 1})$
$\boldsymbol{\Lambda}^d(V)$ super-exterior power	$\boldsymbol{\Lambda}(V) = \Lambda(V_{\bar 0})\ {}^{g}\!\otimes S(V_{\bar 1})$
$\boldsymbol{A}^d(V)$ super-alternating tensors	$\boldsymbol{A}(V) = \Lambda(V_{\bar 0})\ {}^{g}\!\otimes \Gamma(V_{\bar 1})$
$\boldsymbol{I}^{(r)}(V) = V^{(r)}$ r-th Frobenius twist ($r \geq 1$)	$\boldsymbol{I}^{(r)} = {\boldsymbol{I}_0}^{(r)} \oplus {\boldsymbol{I}_1}^{(r)}$

Non-example: $V \mapsto V_{\bar 0}$ (incompatible with composing odd maps)

Main calculation: structure of the extension algebra

$$\mathrm{Ext}^{\bullet}_{\mathcal{P}}(I^{(r)}, I^{(r)}) = \begin{pmatrix} \mathrm{Ext}^{\bullet}_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) & \mathrm{Ext}^{\bullet}_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}) \\ \mathrm{Ext}^{\bullet}_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}) & \mathrm{Ext}^{\bullet}_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)}) \end{pmatrix}$$

Cohomology of strict polynomial superfunctors

Drupieski (2016)

$\mathrm{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)})$ is generated as an algebra by extension classes

- $e_i' \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}({I_0}^{(r)}, {I_0}^{(r)})$ and $e_i'' \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}({I_1}^{(r)}, {I_1}^{(r)})$ $\quad (1 \le i \le r)$
- $c_r \in \mathrm{Ext}^{p^r}_{\mathcal{P}}({I_1}^{(r)}, {I_0}^{(r)})$ and $c_r^\Pi \in \mathrm{Ext}^{p^r}_{\mathcal{P}}({I_0}^{(r)}, {I_1}^{(r)})$

These generators satisfy:

- $(e_i')^p = 0 = (e_i'')^p$ if $1 \le i \le r-1$.
- $(e_r')^p = c_r \circ c_r^\Pi$ and $(e_r'')^p = c_r^\Pi \circ c_r$.
- The e_i', e_i'' generate a commutative subalgebra.
- The e_i' restrict to Friedlander and Suslin's extension classes
- Have $e_i' \circ c_r = \pm c_r \circ e_i''$. But is it $+$ or $-$? (It is $+$ for $i = r$.)

Support varieties for restricted Lie superalgebras

Restricted Lie superalgebras

Theorem

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie superalgebra. Then

$$V_{\mathfrak{g}}(k) \cong \{x + y \mid x \in \mathfrak{g}_{\bar{0}}, y \in \mathfrak{g}_{\bar{1}}, [x, y] = 0, x^{[p]} = y^2\}$$

where $y^2 := \frac{1}{2}[y, y]$.

- Relations come from the functor cohomology calculations
- Sufficiency comes from explicit calculations for the restricted subalgebra generated by x and y, using an “explicit” projective resolution constructed by Iwai–Shimada and May.
- Agrees with results of Nakano & Palmieri (1998) for finite-dimensional subalgebras of the Steenrod algebra
- Support variety $V_{\mathfrak{g}}(M)$ of a nontrivial supermodule M?

Arbitrary infinitesimal supergroup schemes

Now let $G \subset GL(m|n)$ be a height-r infinitesimal supergroup scheme.

Possible description for $|G|$ à la Suslin–Friedlander–Bendel?

Set of all r-tuples $(x_0, \ldots, x_{r-1}, y)$ such that

- $x_i \in \mathfrak{g}_{\overline{0}}$ for $0 \leq i \leq r-1$, and $y \in \mathfrak{g}_{\overline{1}}$
- Entries pairwise commute
- $x_i^{[p]} = 0$ for $0 \leq i \leq r-2$
- $x_{r-1}^{[p]} = y^2$

Open and ongoing topics

- Completely identify the spectrum of $\mathrm{H}^\bullet(G, k)$ or $\mathrm{H}^\bullet(V(\mathfrak{g}), k)$
- Rank variety description for support varieties?
- Super one-parameter subgroups?
    - Restrictions to subalgebras of the form $k[u, v]/\langle u^p + v^2, u^{p^s} \rangle$?
- Super Π-points?