The Lie superalgebra of transpositions

arXiv:2310.01555

Christopher Drupieski (DePaul University)
Jonathan Kujawa (University of Oklahoma)

AMS Fall Southeastern Sectional Meeting, October 13–15, 2023

Special Session on Representation Theory of Finite and Algebraic Groups

MathOverflow: August 8, 2022

mathoverflow

Home

PUBLIC

Questions

Tags

Users

Unanswered

Looking for your Teams?

What is the Lie superalgebra generated by permutations?

Asked 1 year, 2 months ago Modified 8 days ago Viewed 268 times

7

Consider the group algebra of the symmetric group $\mathbb{C}S_n$. Then there is a corresponding Lie algebra $\mathfrak{L}(S_n)$ defined by

$$[\sigma, \tau] = \sigma \circ \tau - \tau \circ \sigma,$$

where $\sigma, \tau \in S_n$. The structure of $\mathfrak{L}(S_n)$ in terms of simple factors has been considered in this post. One can also ask the same question for the Lie subalgebra of $\mathfrak{L}(S_n)$ generated by transpositions, which was considered in this post.

Now, since there is a $\mathbb{Z}_2$ grading of $\mathbb{C}S_n$, one can also define a Lie superalgebra $s\mathfrak{L}(S_n)$ on it by replacing the commutators with anti-commutators

$$\{\sigma, \tau\} = \sigma \circ \tau + \tau \circ \sigma,$$

for all $\sigma, \tau \in S_n^{(1)}$, where $S_n^{(1)}$ is the odd part of the symmetric group, and all other commutators remain unchanged. Now we have similar questions: what is the structure of $s\mathfrak{L}(S_n)$ in terms of simple Lie superalgebras? What is the subalgebra of $s\mathfrak{L}(S_n)$ generated by transpositions?

My attempt is for $n = 3$, $s\mathfrak{L}(S_n) \cong \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(1|0) \oplus \mathfrak{gl}(0|1)$, while the subalgebra generated by transpositions is $\mathfrak{sl}(1|1) \oplus \mathfrak{gl}(1|0) \oplus \mathfrak{gl}(0|1)$. I think in general $s\mathfrak{L}(S_n)$ should be very similar to $\mathfrak{L}(S_n)$, but it might be much harder to determine the subalgebra generated by transpositions.

rt.representation-theory | lie-algebras | permutations | symmetric-groups | lie-superalgebras

Share Cite Edit Follow Flag

edited Oct 4 at 16:50
Jules Lamers

asked Aug 8, 2022 at 20:26
WunderNatur

Question

Considering the group algebra $\mathbb{C}S_n$ of the symmetric group S_n as a superalgebra (by considering the even permutations in S_n to be of even superdegree and the odd permutations in S_n to be of odd superdegree), and considering $\mathbb{C}S_n$ as a Lie superalgebra via the super commutator,

$$[x, y] = xy - (-1)^{\bar{x}\cdot\bar{y}}yx,$$

what is the structure of $\mathbb{C}S_n$ as a Lie superalgebra, and what is the structure of the Lie subsuperalgebra of $\mathbb{C}S_n$ generated by the transpositions?

Classical Artin–Wedderburn Theory

Structure of finite-dimensional semisimple algebras over $\mathbb{C}$

Let A be a finite-dimensional associative semisimple algebra over $\mathbb{C}$, and let $V_1, \ldots, V_m$ be a complete set of pairwise non-isomorphic simple A-modules. Then as a $\mathbb{C}$-algebra,

$$A \cong \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_m).$$

In particular, A is a direct sum of simple $\mathbb{C}$-algebras.

The group algebra of the symmetric group S_n

Given a partition $\lambda \vdash n$, let S^λ be the corresponding simple Specht module for $\mathbb{C}S_n$. Then

$$\mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} \mathrm{End}(S^\lambda).$$

Superalgebra

The prefix super indicates that an object is grade by $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Denote the decomposition of a vector superspace (over $\mathbb{C}$) into its homogeneous (even and odd) components by $V = V_{\bar{0}} \oplus V_{\bar{1}}$.

Write $\bar{v} \in \mathbb{Z}_2$ to denote the superdegree of an element $v \in V_{\bar{0}} \cup V_{\bar{1}}$.

If V and W are vector superspaces, then $\mathrm{Hom}(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)$ inherits a $\mathbb{Z}_2$-grading: $\mathrm{Hom}(V, W)_{\bar{j}} = \left\{ f \in \mathrm{Hom}(V, W) : f(V_{\bar{i}}) \subseteq W_{\bar{i}+\bar{j}} \right\}$.

If V is a vector superspace, then $\Pi(V)$ is its parity shift:

$$\Pi(V)_{\bar{0}} = V_{\bar{1}} \quad \text{and} \quad \Pi(V)_{\bar{1}} = V_{\bar{0}}.$$

Consider $\mathbb{C}$ as a superspace in even superdegree, and write

$$\mathbb{C}^{m|n} = \mathbb{C}^m \oplus \Pi(\mathbb{C}^n).$$

(Semi)simple superalgebras

Unless specified otherwise, all superalgebras are associative, and all superalgebras and supermodules are finite-dimensional over $\mathbb{C}$.

Definition

A superalgebra A is simple if it has no nontrivial superideals.

Definition

A superalgebra A is semisimple if every A-supermodule V is a (direct) sum of simple A-supermodules.

Simple superalgebras and simple supermodules come in two flavors.

Type M simple superalgebras

If $V = \mathbb{C}^{m|n}$, then $\mathrm{End}(V) \cong M(m|n)$ is a simple superalgebra, where

$$M(m|n) := \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] : \begin{array}{ll} A \in M_m(\mathbb{C}), & B \in M_{m\times n}(\mathbb{C}), \\ C \in M_{n\times m}(\mathbb{C}), & D \in M_n(\mathbb{C}). \end{array} \right\}.$$

As an ungraded associative algebra, $M(m|n) \cong \mathfrak{gl}(m+n)$.

Type Q simple superalgebras

If $V = \mathbb{C}^{n|n}$ with odd involution $J : V \to V$, then

$$Q(V) = \{\theta \in \mathrm{End}(V) : J \circ \theta = \theta \circ J\}$$

is a simple superalgebra. One has $Q(V) \cong Q(n)$, where

$$Q(n) := \left\{ \left[\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right] : A \in M_n(\mathbb{C}), B \in M_n(\mathbb{C}) \right\}.$$

As an ungraded associative algebra, $Q(n) \cong \mathfrak{gl}(n) \oplus \mathfrak{gl}(n)$ via the map

$$\left[\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right] \mapsto (A + B, A - B).$$

Two types of simple supermodules

Definition

Let V be a simple A-supermodule.

Say that V is absolutely irreducible (or of Type M) if V is simple as an ungraded A-module.

Say that V is self-associate (or of Type Q) if V is reducible as an ungraded A-module.

Let $\pi_V : V \to V$ be the parity automorphism, $\pi_V(v) = (-1)^{\overline{v}} \cdot v$.

Lemma

Let V be a self-associate simple A-supermodule. Then there exists a (ungraded) simple A-submodule U of V such that

$$V = U \oplus \pi_V(U),$$

with $U \ncong \pi_V(U)$ as ungraded A-modules, and

$$V_{\overline{0}} = \{u + \pi_V(u) : u \in U\}, \quad V_{\overline{1}} = \{u - \pi_V(u) : u \in U\}.$$

An odd involution $J : V \to V$ is defined by

$$J(u \pm \pi_V(u)) = u \mp \pi_V(u).$$

Super Artin–Wedderburn Theorem

Let A be a finite-dimensional associative superalgebra A.

If $\{V_1, \dots, V_n\}$ is a complete set of simple A-supermodules (up to homogeneous isomorphism), such that $V_1, \dots, V_m$ are absolutely irreducible and $V_{m+1}, \dots, V_n$ are self-associate, then

$$A \cong \left[\bigoplus_{i=1}^{m} \mathrm{End}(V_i)\right] \oplus \left[\bigoplus_{i=m+1}^{n} Q(V_i)\right].$$

Lemma

Let A be a finite-dimensional associative superalgebra. Then A is semisimple as a superalgebra if and only if A is semisimple as an ordinary ungraded algebra.

The group algebra of the symmetric group, as a superalgebra

The symmetric group S_n is a supergroup, with

- $(S_n)_{\overline{0}} = A_n$, the alternating group.
- $(S_n)_{\overline{1}} = S_n \backslash A_n$, the set of odd permutations.

This extends to a $\mathbb{Z}_2$-grading on the group algebra $\mathbb{C}S_n$, with

- $(\mathbb{C}S_n)_{\overline{0}} = \mathbb{C}A_n$, the group algebra of the alternating group.

Simple supermodules for the symmetric group

Let $\mathcal{P}(n) = \{\lambda : \lambda \vdash n\}$.

Given $\lambda \vdash n$, let λ' be the conjugate (transpose) partition.

Let $\overline{\mathcal{P}}(n)$ be a fixed set of representatives for the relation $\lambda \sim \lambda'$.

Let $E_n = \left\{\lambda \in \overline{\mathcal{P}}(n) : \lambda \neq \lambda'\right\}$ and $F_n = \left\{\lambda \in \overline{\mathcal{P}}(n) : \lambda = \lambda'\right\}$.

What do YOU think the simple $\mathbb{C}S_n$-supermodules look like?

Simple supermodules for the symmetric group

Let $\mathcal{P}(n) = \{\lambda : \lambda \vdash n\}$.

Given $\lambda \vdash n$, let λ' be the conjugate (transpose) partition.

Let $\overline{\mathcal{P}}(n)$ be a fixed set of representatives for the relation $\lambda \sim \lambda'$.

Let $E_n = \left\{\lambda \in \overline{\mathcal{P}}(n) : \lambda \neq \lambda'\right\}$ and $F_n = \left\{\lambda \in \overline{\mathcal{P}}(n) : \lambda = \lambda'\right\}$.

What do YOU think the simple $\mathbb{C}S_n$-supermodules look like?

Simple supermodules for $\mathbb{C}S_n$ (up to parity shift)

Simple $\mathbb{C}S_n$-supermodules are indexed by the set $\overline{\mathcal{P}}(n)$.

$$W^\lambda = \begin{cases} S^\lambda & \text{if } \lambda \in F_n \text{ (Type Q, absolutely irreducible case)} \\ S^\lambda \oplus S^{\lambda'} & \text{if } \lambda \in E_n \text{ (Type M, self-associate case)} \end{cases}$$

Structure of simple supermodules for the symmetric group

Type M simple supermodules $W^\lambda = S^\lambda \oplus S^{\lambda'} \quad (\lambda \neq \lambda')$

The odd involution $J : W^\lambda \to W^\lambda$ can be interpreted as an even isomorphism of $\mathbb{C}S_n$-supermodules

$$W^\lambda \cong \Pi(W^\lambda).$$

Type Q simple supermodules $W^\lambda = S^\lambda \quad (\lambda = \lambda')$

As a $\mathbb{C}A_n$-module,

$$S^\lambda = S^{\lambda^+} \oplus S^{\lambda^-},$$

These are the homogeneous subspaces of W^λ. Consequently, W^λ is not even isomorphic to $\Pi(W^\lambda)$ because $S^{\lambda^+} \ncong S^{\lambda^-}$ as $\mathbb{C}A_n$-modules.

“Multiplicity free” restriction

Restriction to $\mathbb{C}S_{n-1}$ in terms of Young lattice ordering $\mu \prec \lambda$:

$$W^\lambda\!\downarrow_{\mathbb{C}S_{n-1}} \cong \begin{cases} \left[\bigoplus_{\substack{\mu\prec\lambda \\ \mu\neq\mu'}} W^\mu\right] \oplus \left[\bigoplus_{\substack{\mu\prec\lambda \\ \mu=\mu'}} W^\mu \oplus \Pi(W^\mu)\right] & \text{if } \lambda \in E_n, \\ \bigoplus_{\substack{\mu\prec\lambda \\ \mathrm{cont}(\lambda/\mu)\geq 0}} W^\mu & \text{if } \lambda \in F_n. \end{cases}$$

Super Artin–Wedderburn Theorem for the symmetric group

From the classification of the simple supermodules, get isomorphisms of associative superalgebras

$$\begin{aligned} \mathbb{C}S_n &\cong \left[\bigoplus_{\lambda\in E_n} Q(W^\lambda)\right] \oplus \left[\bigoplus_{\lambda\in F_n} \mathrm{End}(W^\lambda)\right] \\ &\cong \left[\bigoplus_{\lambda\in E_n} Q(f^\lambda)\right] \oplus \left[\bigoplus_{\lambda\in F_n} M(\tfrac{1}{2}f^\lambda, \tfrac{1}{2}f^\lambda)\right] \end{aligned}$$

where $f^\lambda = \mathrm{dim}(S^\lambda)$. Then as a Lie superalgebra,

$$\mathbb{C}S_n \cong \left[\bigoplus_{\lambda\in E_n} \mathfrak{q}(f^\lambda)\right] \oplus \left[\bigoplus_{\lambda\in F_n} \mathfrak{gl}(\tfrac{1}{2}f^\lambda, \tfrac{1}{2}f^\lambda)\right]$$

Lie subsuperalgebras

Given a Lie superalgebra $\mathfrak{g}$, let $\mathfrak{D}(\mathfrak{g})$ be its derived subsuperalgebra.

$$\begin{aligned}
\mathfrak{D}(\mathfrak{gl}(W^\lambda)) &= \mathfrak{sl}(W^\lambda) \\
&\cong \mathfrak{sl}(m|m) := \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathfrak{gl}(m|m) : \mathrm{tr}(A) - \mathrm{tr}(D) = 0 \right\} \\
\mathfrak{D}(\mathfrak{q}(W^\lambda)) &= \mathfrak{sq}(W^\lambda) \\
&\cong \mathfrak{sq}(n) := \left\{ \left[\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right] \in \mathfrak{q}(n) : \mathrm{tr}(B) = 0 \right\}
\end{aligned}$$

Lie subsuperalgebra generated by transpositions

Let $\mathfrak{g}_n \subset \mathbb{C}S_n$ be the Lie subsuperalgebra generated by transpositions.

Let $T_n = \sum_{1 \leq i < j \leq n} (i, j) \in \mathbb{C}S_n$ be the sum in $\mathbb{C}S_n$ of all transpositions.

Main Theorem

$$\mathfrak{g}_n = \mathfrak{D}(\mathbb{C}S_n) \oplus \mathbb{C} \cdot T_n,$$

where

$$\mathfrak{D}(\mathbb{C}S_n) \cong \Big[\bigoplus_{\lambda \in F_n} \mathfrak{sl}(W^\lambda) \Big] \oplus \Big[\bigoplus_{\lambda \in E_n} \mathfrak{sq}(W^\lambda) \Big]$$

$\mathfrak{g}_n \subseteq \mathfrak{D}(\mathbb{C}S_n) + \mathbb{C}T_n$ because $\mathfrak{g}_n$ is generated by T_n and the set

$$\left\{ \tau - \frac{2}{n(n-1)} \cdot T_n : \tau \text{ is a transposition} \right\}$$

Ideas behind the proof of the Main Theorem

$$\mathfrak{g}_n = \mathfrak{D}(\mathbb{C}S_n) \oplus \mathbb{C} \cdot T_n$$

Argue by induction on n to show $\mathfrak{D}(\mathbb{C}S_n) \subseteq \mathfrak{g}_n$.

Hard bit: Compute $\mathsf{im}\left(\mathfrak{g}_n \to \mathsf{End}(W^\lambda)\right)$

- Use description of the restriction $W^\lambda\downarrow_{\mathbb{C}S_{n-1}}$, and Gelfand–Zeitlin bases for the S^λ given by the simultaneous eigenvectors for the action of the Jucys–Murphy elements.

Deduce that $(\mathfrak{g}_n)_{\overline{0}}$ is a reductive Lie algebra.

Show that the semisimple Lie algebra $\mathfrak{D}((\mathfrak{g}_n)_{\overline{0}})$ is as large as we want it to be, and then use the action of this semisimple Lie algebra to deduce that all of $\mathfrak{D}(\mathbb{C}S_n)$ is contained in $\mathfrak{g}_n$.

L'algèbre de Lie des transpositions (J. Algebra 310 (2007) 742–774)

Marin studied the classical (non-super) analogue of this problem, motivated by the representation theory of the braid group.

Proposition 1. *L'algèbre de Lie $\mathfrak{g}_n$ est réductive, et son centre est de dimension* 1, *engendré par la somme T_n de toutes les transpositions. En conséquence $\mathfrak{g}_n \simeq \Bbbk \times \mathfrak{g}'_n$, et l'image de $\mathfrak{g}_n$ dans $\mathfrak{gl}(\lambda)$ est $\mathfrak{g}_\lambda \subset \mathfrak{sl}(\lambda)$ si T_n agit par* 0, *et $\Bbbk \times \mathfrak{g}_\lambda$ sinon.*

Théorème A. *Pour tout $n \geqslant 3$, ϕ_n est surjectif. En particulier,*

$$\mathfrak{g}'_n \simeq \mathfrak{sl}_{n-1}(\Bbbk) \times \left(\prod_{\lambda \in E_n/\sim} \mathfrak{sl}(\lambda) \right) \times \left(\prod_{\lambda \in F_n} \mathfrak{osp}(\lambda) \right)$$

et les représentations ρ_λ de $\mathfrak{g}'_n$ sont deux à deux non isomorphes.

Overall, Marin obtain a Lie algebra that is on the order of half the dimension of the Lie superalgebra we get.