Support schemes for infinitesimal unipotent supergroups

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Let k be a field of characteristic $p \geq 3$.

Let A be (something like) a Hopf algebra over k.

Cohomological spectrum and support varieties

The **cohomological spectrum** of A is the affine scheme

$$|A| = \operatorname{Spec}\left(\mathrm{H}^\bullet(A, k)\right).$$

Given an A-module M, let $I_A(M)$ be the kernel of the (k-algebra) map

$$\mathrm{H}^\bullet(A, k) = \operatorname{Ext}_A^\bullet(k, k) \xrightarrow{-\otimes M} \operatorname{Ext}_A^\bullet(M, M).$$

The **cohomological support scheme** associated to M is

$$|A|_M = \operatorname{Spec}\left(\mathrm{H}^\bullet(A, k)/I_A(M)\right),$$

a closed subscheme of the cohomological spectrum.

Superalgebra

- super $\equiv$ graded by $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$
- topologist's sign convention when odd degree symbols pass

Infinitesimal unipotent supergroups

kG is the group algebra of a **finite supergroup (scheme)** G if:

- kG is a finite-dimensional cocommutative Hopf superalgebra.

Then G is **infinitesimal** if

- the dual Hopf superalgebra $k[G] = kG^{\#}$ is local,

and G is **unipotent** if

- kG is a local k-algebra.

General examples of Hopf superalgebras

- Any ordinary Hopf algebra, considered as a purely even Hopf superalgebra, i.e., $A = A_{\bar{0}}$.
- Any $\mathbb{Z}$-graded Hopf algebra (in the sense of Milnor and Moore), the $\mathbb{Z}_2$-grading obtained by reducing the $\mathbb{Z}$-grading modulo 2
- e.g., exterior algebra $\Lambda(V)$ with $V = V_{\bar{1}}$ is both commutative and cocommutative in the super sense
- (restricted) enveloping algebra of (restricted) Lie superalgebra

Special examples of Hopf superalgebras

– $\mathbb{G}_{a(r)}$ the r-th infinitesimal Frobenius kernel of the additive group

$$k\mathbb{G}_{a(r)} = \left(k[T]/(T^{p^r})\right)^{\#} = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p)$$

– $\mathbb{P}_r = k[u_0, \ldots, u_{r-1}, v]/(u_0^p, \ldots, u_{r-2}^p, u_{r-1}^p + v^2)$, where

- $u_0, \ldots, u_{r-1}$ are of even superdegree, v is of odd superdegree,
- coproducts for $u_0, \ldots, u_{r-1}$ look like they do in $k\mathbb{G}_{a(r)}$,
- u_{r-1}^p and v are primitive.

– In particular, $\mathbb{P}_1 = k[u, v]/(u^p + v^2)$ (hypersurface ring)

$$\mathbb{P}_r = k[u_0, \ldots, u_{r-1}, v]/(u_0^p, \ldots, u_{r-2}^p, u_{r-1}^p + v^2)$$

Lemma

Let G be a finite k-supergroup scheme. Then the functor from commutative k-algebras to sets,

$$V_r(G) : A \mapsto V_r(G)(A) = \mathrm{Hom}_{sHopf/A}(\mathbb{P}_r \otimes_k A, kG \otimes_k A),$$

admits the structure of an affine scheme of finite type over k.

Remark

If G is an ordinary finite group scheme, i.e., if kG is purely even, then

$$V_r(G) = \mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, G),$$

the functor of infinitesimal one-parameter subgroups in G of height $\leq r$, as defined by Suslin, Friedlander, and Bendel (1997).

$$\mathbb{P}_1 = k[u,v]/(u^p + v^2), \quad \mathbb{P}_r = k[u_0,\ldots,u_{r-1},v]/(u_0^p,\ldots,u_{r-2}^p,u_{r-1}^p + v^2)$$

Superalgebra map $\iota : \mathbb{P}_1 \hookrightarrow \mathbb{P}_r$ defined by $\iota(u) = u_{r-1}$ and $\iota(v) = v$.

The support set $V_r(G)_M$

Let G be a finite k-supergroup scheme and M a finite-dimensional kG-supermodule. Set

$$V_r(G)_M = \left\{\nu \in V_r(G) = \mathrm{Hom}_{sHopf}(\mathbb{P}_r, kG) : \mathrm{projdim}_{\mathbb{P}_1}(\iota^*\nu^*M) = \infty\right\}.$$

Proposition

$V_r(G)_M$ is a Zariski closed conical subset of $V_r(G)$.

Key ingredient of the proof.

Use explicit $\mathbb{P}_1$-projective resolution of k constructed via a matrix factorization to show that $\mathrm{projdim}_{\mathbb{P}_1}(M) = \infty$ if and only if a certain cup product in cohomology is nonzero. □

$$\mathrm{H}^\bullet(G,k) = \mathrm{H}^\bullet(kG,k), \quad |G| = \mathrm{Spec}\left(\mathrm{H}^\bullet(G,k)\right)$$

Theorem (Drupieski–Kujawa, to be posted on arXiv soon)

Let G be an **infinitesimal unipotent** k-supergroup scheme of height $\leq r$. Then there is a natural k-algebra map $\psi : H(G,k) \to k[V_r(G)]$, which which defines a universal homeomorphism of schemes

$$|G| \simeq V_r(G) = \mathrm{Hom}_{sHopf}(\mathbb{P}_r, kG).$$

This restricts for each finite-dimensional kG-supermodule M to a homeomorphism

$$|G|_M \simeq \left\{\nu \in V_r(G) : \mathrm{projdim}_{\mathbb{P}_1}(\iota^*\nu^*M) = \infty\right\}.$$

Proof uses a detection theorem of Benson–Iyengar–Krause–Pevtsova.

If G is a purely even, theorem reduces to results of SFB (1997).

Toward the non-unipotent case

Suitably interpreted, our previous work on $GL_{m|n(r)}$ shows that $\left|GL_{m|n(r)}\right|$ identifies up to a finite morphism with $V_r(GL_{m|n(r)})(k)$.

So perhaps $V_r(G)$ is the correct ambient scheme to consider even for non-unipotent G?

$V_r(G)_M$ also makes sense, and is closed in $V_r(G)$, for non-unipotent G.

Let $f = T^{p^t} + \sum_{i=1}^{t-1} a_i T^{p^i} \in k[T]$ be a p-polynomial (no linear term).

Let $\eta \in k$ be a scalar.

The infinitesimal multiparameter supergroup $\mathbb{M}_{r;f,\eta}$

$$k\mathbb{M}_{r;f,\eta} = \mathbb{P}_r / \langle f(u_{r-1}) + \eta u_0 \rangle$$

Proposition

Every finite-dimensional Hopf quotient of $\mathbb{P}_r$ is of the form

- $k\mathbb{G}_{a(s)}$ for some $0 \leq s \leq r$,
- $k\mathbb{G}_a^- = k[v]/\langle v^2 \rangle$, or
- $k\mathbb{M}_{s;f,\eta}$ for some $1 \leq s \leq r$ and some f, η as above.

Benson–Iyengar–Krause–Pevtsova

For **unipotent** finite supergroup schemes, projectivity of modules and nilpotence in cohomology are detected (after field extension) by restriction to **'elementary'** subsupergroup schemes.

The *infinitesimal* elementary k-supergroup schemes are

- $\mathbb{G}_{a(r)}$ for $r \geq 0$,
- $\mathbb{G}_{a(r)} \times \mathbb{G}_a^-$ for $r \geq 0$,
- $\mathbb{M}_{r;T^{p^s},0}$ for $r, s \geq 1$,
- $\mathbb{M}_{r,T^{p^s},\eta}$ for $r \geq 2$, $s \geq 1$, and $0 \neq \eta \in k$.

The group algebras of these each occur as Hopf quotients of $\mathbb{P}_r$.

Roughly: $\mathbb{M}_{r;f,\eta}$ is unipotent if the polynomial f is a single monomial.

Question

For arbitrary infinitesimal supergroups, is projectivity of modules and nilpotence in cohomology detected (after field extension) by restriction to finite-dimensional Hopf superalgebra quotients of $\mathbb{P}_r$?

Seems likely that the hardest part of extending the identification $|G|_M \simeq V_r(G)_M$ to the non-unipotent case will be in answering the previous question.