Support varieties and modules of finite projective dimension for Lie superalgebras

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Motivating Question

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k (e.g., the Lie algebra of a linear algebraic group).

What does the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, k) = \mathrm{Ext}^\bullet_{\mathfrak{g}}(k, k)$ look like?

What does its maximal ideal spectrum $\mathrm{Max}(\mathrm{H}^\bullet(\mathfrak{g}, k))$ look like?

Classical Theorem

Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra. Then $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C})$ is an exterior algebra generated in odd degrees.

More generally:

Elementary result

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k. Then $\mathrm{H}^\bullet(\mathfrak{g}, k)$ is finite-dimensional, and $\mathrm{H}^i(\mathfrak{g}, k) = 0$ for $i > \dim_k(\mathfrak{g})$.

So $\mathrm{Max}(\mathrm{H}^\bullet(\mathfrak{g}, k))$ is not very interesting in this situation.

Different Source of Motivation

Let $k = \overline{k}$ of characteristic $p > 0$.

Let $\mathfrak{g}$ be a finite-dimensional **restricted** Lie algebra over k. So $\mathfrak{g}$ is equipped with a semilinear map $x \mapsto x^{[p]}$ such that $\mathrm{ad}(x)^p = \mathrm{ad}(x^{[p]})$.

Let $V(\mathfrak{g})$ be the restricted enveloping algebra of $\mathfrak{g}$ (a f.d. Hopf algebra).

Friedlander–Parshall (1980s), Suslin–Friedlander–Bendel (1997)

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra over k. Then

$$\mathrm{Max}\left(\mathrm{H}^\bullet(V(\mathfrak{g}), \mathbb{C})\right) \simeq \mathcal{N}_p(\mathfrak{g}) = \left\{x \in \mathfrak{g} : x^{[p]} = 0\right\}.$$

$\mathcal{N}_p(\mathfrak{g})$ is the **restricted nullcone** of $\mathfrak{g}$.

If $\mathfrak{g} = \mathfrak{gl}_n(k)$, then $x^{[p]} = x^p$, and $\mathcal{N}_p(\mathfrak{g})$ is the variety of p-nilpotent matrices. If $p > n$, then $\mathcal{N}_p(\mathfrak{g})$ is all nilpotent matrices in $\mathfrak{g}$.

Support varieties

Let A be a Hopf algebra over k. Then $\mathrm{H}^\bullet(A,k)$ is graded-commutative. Suppose $\mathrm{H}^\bullet(A,k)$ is finitely-generated as a k-algebra.

Cohomological spectrum and support varieties

The **cohomological spectrum** of A is the affine algebraic variety

$$|A| = \mathrm{Max}\left(\mathrm{H}^\bullet(A,k)\right).$$

Given an A-module M, let $I_A(M)$ be the kernel of the (k-algebra) map

$$\mathrm{H}^\bullet(A,k) = \mathrm{Ext}_A^\bullet(k,k) \xrightarrow{-\otimes M} \mathrm{Ext}_A^\bullet(M,M).$$

The **cohomological support variety** associated to M is

$$|A|_M = \mathrm{Max}\left(\mathrm{H}^\bullet(A,k)/I_A(M)\right),$$

which is a closed conical subvariety of $|A|$.

Friedlander–Parshall, Suslin–Friedlander–Bendel

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra over k, and let M be a finite-dimensional restricted $\mathfrak{g}$-module. Then

$$|V(\mathfrak{g})| \simeq \mathcal{N}_p(\mathfrak{g}) = \left\{x \in \mathfrak{g} : x^{[p]} = 0\right\},$$
$$|V(\mathfrak{g})|_M \simeq \left\{x \in \mathcal{N}_p(\mathfrak{g}) : M|_{\langle x\rangle} \text{ is not free}\right\} \cup \{0\}.$$

Moreover, $|V(\mathfrak{g})|_M = \{0\}$ if and only if M is projective for $V(\mathfrak{g})$.

For $x \in \mathcal{N}_p(\mathfrak{g})$, $M|_{\langle x\rangle}$ is restriction to subalgebra of the form $k[x]/(x^p)$.

General result

Let A be a finite-dimensional Hopf algebra over k. Suppose $\mathrm{H}^\bullet(A, k)$ is a finitely-generated k-algebra, and suppose for all f.d. A-modules M that $\mathrm{Ext}_A^\bullet(M, M)$ is a finite $\mathrm{H}^\bullet(A, k)$-module. Then

$$|A|_M = \{0\} \iff M \text{ is projective.}$$

Lie superalgebras

A **Lie superalgebra** (for simplicity, over a field of characteristic $\neq 2$) is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ equipped with an even bilinear map $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that for all homogeneous elements $x, y, z \in \mathfrak{g}$,

1. $[x, y] = -(-1)^{\bar{x}\cdot\bar{y}}[y, x]$
2. $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x}\cdot\bar{y}}[y, [x, z]]$
3. $[x, x] = 0$ if $x \in \mathfrak{g}_{\bar{0}}$
4. $[[y, y], y] = 0$ if $y \in \mathfrak{g}_{\bar{1}}$

So in particular, $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module.

$\mathfrak{g}$ is a **restricted** Lie superalgebra if additionally $\mathfrak{g}_{\bar{0}}$ is a restricted Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a restricted $\mathfrak{g}_{\bar{0}}$-module.

Superized Motivating Question

Let $\mathfrak{g}$ be a finite-dimensional Lie **super**algebra over a field k.

What does its maximal ideal spectrum $\mathrm{Max}(\mathrm{H}^\bullet(\mathfrak{g}, k))$ look like?

One extreme

Suppose $\mathfrak{g} = \mathfrak{g}_{\bar{1}}$ is a purely odd abelian Lie superalgebra. Then $U(\mathfrak{g}) = \Lambda(\mathfrak{g})$, and $\mathrm{H}^\bullet(\mathfrak{g}, k) = \mathrm{H}^\bullet(\Lambda(\mathfrak{g}), k) \cong S(\mathfrak{g}^*)$.

Another extreme in characteristic 0

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, so that $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}_m \oplus \mathfrak{gl}_n$. If $m \geq n$, then the inclusion $\mathfrak{gl}_m \subseteq \mathfrak{g}$ induces $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{gl}_m, \mathbb{C})$. In particular, $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C})$ is a finite-dimensional exterior algebra.

So in characteristic 0, the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, k)$ may not lead to an interesting support variety theory.

Two alternate approaches in characteristic 0

1. Boe, Kujawa, & Nakano (2009–2017): Developed extensive variety theory based on the relative cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ when $\mathfrak{g}$ is a classical Lie superalgebra.
2. Duflo & Serganova (2005): For a (f.d.) $\mathfrak{g}$-supermodule M, defined (without reference to cohomology) the associated variety

$$\mathcal{X}_{\mathfrak{g}}(M) = \left\{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0 \text{ and } M|_{\langle x \rangle} \text{ is not free}\right\} \cup \{0\},$$

a subvariety of the odd nullcone

$$\mathcal{X}_{\mathfrak{g}}(k) = \{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0\}.$$

These variety theories capture (complementary) aspects of module theory in the category $\mathcal{F}$ of finite-dimensional $\mathfrak{g}$-supermodules that are completely reducible over $\mathfrak{g}_{\bar{0}}$.

Things change in characteristic $p \geq 3$

The superexterior algebra $\Lambda_s(\mathfrak{g}^*)$ is the free anti-(super)commutative algebra generated by $\mathfrak{g}^*$. As a superspace,

$$\Lambda_s(\mathfrak{g}^*) \cong \Lambda(\mathfrak{g}^*_{\bar{0}}) \otimes S(\mathfrak{g}^*_{\bar{1}}).$$

As an algebra, the generators in $\Lambda(\mathfrak{g}^*_{\bar{0}})$ and $S(\mathfrak{g}^*_{\bar{1}})$ anti-commute.

The Lie bracket on $\mathfrak{g}$ is a map $\Lambda^2_s(\mathfrak{g}) \to \mathfrak{g}$. Its transpose defines a map $\mathfrak{g}^* \to \Lambda^2_s(\mathfrak{g}^*)$, which extends to a derivation $\partial : \Lambda_s(\mathfrak{g}^*) \to \Lambda_s(\mathfrak{g}^*)$. Then

$$\mathrm{H}^\bullet(\mathfrak{g}, k) = \mathrm{H}^\bullet(\Lambda_s(\mathfrak{g}^*), \partial).$$

Characteristic $p \geq 3$

The p-th powers in $S(\mathfrak{g}^*_{\bar{1}})$ are cocycles for ∂. Get an algebra map

$$\varphi : S(\mathfrak{g}^*_{\bar{1}})^{(1)} \to \mathrm{H}^\bullet(\mathfrak{g}, k).$$

Let $k = \overline{k}$ of characteristic $p \geq 3$.

$|U(\mathfrak{g})| = \mathrm{Max}(\mathrm{H}^\bullet(\mathfrak{g}, k))$

$\mathcal{X}_{\mathfrak{g}}(k) = \{x \in \mathfrak{g}_{\overline{1}} : [x, x] = 0\}$ odd nullcone

$\mathcal{X}_{\mathfrak{g}}(M) = \left\{x \in \mathfrak{g}_{\overline{1}} : [x, x] = 0 \text{ and } M|_{\langle x \rangle} \text{ is not free}\right\} \cup \{0\}$

For $x \in \mathcal{X}_{\mathfrak{g}}(k)$, $M|_{\langle x \rangle}$ is restriction to subalgebra of the form $k[x]/(x^2)$.

Drupieski–Kujawa (J. Algebra 2019)

The map $\varphi : S(\mathfrak{g}_{\overline{1}}^*)^{(1)} \to \mathrm{H}^\bullet(\mathfrak{g}, k)$ induces a homeomorphism

$$\mathcal{X}_{\mathfrak{g}}(k) \simeq |U(\mathfrak{g})| \,.$$

By naturality, get for each finite-dimensional $\mathfrak{g}$-module M

$$\mathcal{X}_{\mathfrak{g}}(M) \subseteq |U(\mathfrak{g})|_M \,.$$

Is the inclusion $\mathcal{X}_{\mathfrak{g}}(M) \subseteq |U(\mathfrak{g})|_M$ an equality?

Main Theorem [Alg. Number Th. 15 (2021), 1157–1180]

Let M be a finite-dimensional $\mathfrak{g}$-supermodule. Then

$$|U(\mathfrak{g})|_M \simeq \mathcal{X}_{\mathfrak{g}}(M) = \left\{ x \in \mathfrak{g}_{\overline{1}} : [x,x] = 0 \text{ and } M|_{\langle x \rangle} \text{ is not free} \right\} \cup \{0\}.$$

Main ideas of the proof:

- Clifford filtration to pass to a simpler graded Lie superalgebra $\widetilde{\mathfrak{g}}$ and an associated graded module $\widetilde{M}$.
- Action of $U(\widetilde{\mathfrak{g}})$ on $\widetilde{M}$ factors through a p-nilpotent restricted Lie superalgebra, or equivalently, a height-1 **unipotent** supergroup scheme. Support varieties for finite **unipotent** supergroups are understood by work of Benson–Iyengar–Krause–Pevtsova and Drupieski–Kujawa.

Clifford filtration

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra.

The **Clifford filtration** on $\mathfrak{g}$ is the Lie superalgebra filtration

$$0 = F^0\mathfrak{g} \subseteq F^1\mathfrak{g} \subseteq F^2\mathfrak{g} = \mathfrak{g}$$

defined by $F^1\mathfrak{g} = \mathfrak{g}_{\bar{1}}$. The associated graded algebra $\widetilde{\mathfrak{g}}$ satisfies:

- $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_1 \oplus \widetilde{\mathfrak{g}}_2$
- $\widetilde{\mathfrak{g}}_1 \cong \mathfrak{g}_{\bar{1}}$ and $\widetilde{\mathfrak{g}}_2 \cong \mathfrak{g}_{\bar{0}}$ as superspaces
- $\widetilde{\mathfrak{g}}_2$ is central in $\widetilde{\mathfrak{g}}$
- The Lie bracket $[\cdot,\cdot] : \widetilde{\mathfrak{g}}_1 \otimes \widetilde{\mathfrak{g}}_1 \to \widetilde{\mathfrak{g}}_2$ identifies with the original Lie bracket $[\cdot,\cdot] : \mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$. In particular, $\mathcal{X}_{\mathfrak{g}}(k) \cong \mathcal{X}_{\widetilde{\mathfrak{g}}}(k)$.

So $\widetilde{\mathfrak{g}}$ is simpler in structure, but it still retains information relevant to the varieties $\mathcal{X}_{\mathfrak{g}}(k)$ and $\mathcal{X}_{\mathfrak{g}}(M)$.

Relating $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$

Clifford filtration spectral sequence

Let M be a finite-dimensional $\mathfrak{g}$-supermodule, and let N be a finitely-generated $\mathfrak{g}$-supermodule, equipped with 'standard' filtrations. Then there exists a spectral sequence

$$E_1^{i,j}(M,N) = \mathrm{Ext}_{\widetilde{\mathfrak{g}}}^{i+j}(\widetilde{M},\widetilde{N})_{-i} \Rightarrow \mathrm{Ext}_{\mathfrak{g}}^{i+j}(M,N),$$

where $\widetilde{M}$ and $\widetilde{N}$ are the associated graded $\widetilde{\mathfrak{g}}$-supermodules.

Studying this spectral sequence, we deduce

$$\mathcal{X}_{\mathfrak{g}}(M) \subseteq |U(\mathfrak{g})|_M \subseteq |U(\widetilde{\mathfrak{g}})|_{\widetilde{M}} \stackrel{?}{=} \mathcal{X}_{\widetilde{\mathfrak{g}}}(\widetilde{M})$$

Also show $\mathrm{Ext}_{\mathfrak{g}}^{\bullet}(M,N)$ is finite over $\mathrm{H}^{\bullet}(\mathfrak{g},k)$ (for f.g. N, not just f.d.).

Relating support varieties for $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$

First show that $\mathcal{X}_{\widetilde{\mathfrak{g}}}(\widetilde{M}) = |U(\widetilde{\mathfrak{g}})|_{\widetilde{M}}$. Know the inclusion

$$\{x \in \widetilde{\mathfrak{g}}_1 : [x,x] = 0 \text{ and } \widetilde{M}|_{\langle x \rangle} \text{ is not free}\} \cup \{0\} \quad \subseteq \quad |U(\widetilde{\mathfrak{g}})|_{\widetilde{M}}.$$

Elements of $\widetilde{\mathfrak{g}}_2$ act nilpotently on $\widetilde{M}$ because $\widetilde{M}$ is finite-dimensional graded module concentrated in non-negative degrees. Replace $\widetilde{\mathfrak{g}}$ with a related p-nilpotent restricted Lie superalgebra $\widehat{\mathfrak{g}}$ acting on $\widetilde{M}$.

Then $U(\widehat{\mathfrak{g}}) \twoheadrightarrow V(\widehat{\mathfrak{g}})$ induces $|U(\widehat{\mathfrak{g}})| \to |V(\widehat{\mathfrak{g}})|$.

Drupieski–Kujawa (Adv. Math. 2019)

For any f.d. p-nilpotent res. Lie superalgebra $\widehat{\mathfrak{g}}$, and $\widehat{\mathfrak{g}}$-module $\widetilde{M}$,

$$|V(\widehat{\mathfrak{g}})|_{\widetilde{M}} \simeq \left\{(\alpha,\beta) \in \widehat{\mathfrak{g}}_{\overline{0}} \oplus \widehat{\mathfrak{g}}_{\overline{1}} : \alpha^{[p]} = \tfrac{1}{2}[\beta,\beta],\ \operatorname{projdim}_{\mathbb{P}}(\nu^*\widetilde{M}) = \infty\right\},$$

where $\mathbb{P} = k[u,v]/(u^p + v^2)$ and $\nu(u) = \alpha$, $\nu(v) = \beta$.

Use $|U(\widehat{\mathfrak{g}})|_{\widetilde{M}} \hookrightarrow |V(\widehat{\mathfrak{g}})|_{\widetilde{M}}$ has image in $\widehat{\mathfrak{g}}_1$ to show $\mathcal{X}_{\widehat{\mathfrak{g}}}(\widetilde{M}) = |U(\widehat{\mathfrak{g}})|_{\widetilde{M}}$.

Relating support varieties for $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$

Now in general one gets

$$\mathcal{X}_{\mathfrak{g}}(M) \subseteq |U(\mathfrak{g})|_M \subseteq |U(\widetilde{\mathfrak{g}})|_{\widetilde{M}} = \mathcal{X}_{\widetilde{\mathfrak{g}}}(\widetilde{M}).$$

To prove that $\mathcal{X}_{\mathfrak{g}}(M) = |U(\mathfrak{g})|_M$ for all $\mathfrak{g}$ and M, suffices to prove it for $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $M = k^{m|n}$. But even then $\mathcal{X}_{\mathfrak{g}}(M) \subsetneq \mathcal{X}_{\widetilde{\mathfrak{g}}}(\widetilde{M})$.

For $\mathfrak{g} = \mathfrak{gl}(m|n)$, the $GL_m \times GL_n$-orbit structure of $\mathcal{X}_{\mathfrak{g}}(k) = |U(\mathfrak{g})|$ is easy, and we just have to rule out the maximal orbits from $|U(\mathfrak{g})|_M$. For each maximal orbit, consider a *different* standard filtration on M, and show that a particular orbit representative is not in $\mathcal{X}_{\widetilde{\mathfrak{g}}}(\widetilde{M})$.

Applications

Tensor Product Property

Let M and N be finite-dimensional $\mathfrak{g}$-supermodules. Then

$$|U(\mathfrak{g})|_{M\otimes N} = |U(\mathfrak{g})|_M \cap |U(\mathfrak{g})|_N .$$

Theorem

Let M be a finite-dimensional $\mathfrak{g}$-supermodule. Then

$$\mathcal{X}_{\mathfrak{g}}(M) = \{0\} \iff \operatorname{projdim}_{U(\mathfrak{g})}(M) < \infty.$$

Proof: If $\mathcal{X}_{\mathfrak{g}}(M) = 0$, then for all finitely-generated N, one gets $\operatorname{Ext}^i_{\mathfrak{g}}(M, N)$ for $i \gg 0$. Now apply results of Avramov and Iyengar for Noether algebras.

Corollary (cf. Bøgvad 1984, Musson 2012)

$\mathcal{X}_{\mathfrak{g}}(k) = \{0\}$ if and only if $\operatorname{gldim}(U(\mathfrak{g})) < \infty$.