Support varieties for Lie superalgebras and finite graded group schemes

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- This is joint work with Jonathan Kujawa (University of Oklahoma).
- Work over a (algebraically closed) field k of characteristic $p \geq 0$.
- Denote the cohomological support variety of an A-module M by $|A|_M$.

What's so super about super linear algebra?

Something is "super" if it has a compatible $\mathbb{Z}/2\mathbb{Z}$-grading.

- Superspaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$, $W = W_{\overline{0}} \oplus W_{\overline{1}}$
- Induced gradings on tensor products, linear maps, etc.

$$(V \otimes W)_\ell = \bigoplus_{i+j=\ell} V_i \otimes W_j$$

$$\mathrm{Hom}_k(V,W)_\ell = \{f \in \mathrm{Hom}_k(V,W) : f(V_i) \subseteq W_{i+\ell}\}$$

- $V \otimes W \cong W \otimes V$ via the **supertwist** $v \otimes w \mapsto (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$

Define (Hopf) superalgebras and notions of (super)commutativity and (super)cocommutativity in terms of the "usual diagrams," but using the supertwist map whenever graded objects pass one another.

Examples of Hopf superalgebras

- Ordinary Hopf algebras (considered as purely even superalgebras).
- **$\mathbb{Z}$-graded Hopf algebras (as defined, e.g., by Milnor and Moore)**
- If $\mathfrak{g}$ is a (restricted) Lie superalgebra, then its (restricted) enveloping superalgebra $U(\mathfrak{g})$ (resp. $u(\mathfrak{g})$) is a Hopf superalgebra.

Recall that a **Lie superalgebra** is a superspace $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ equipped with an even map $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that for homogeneous $x, y, z \in \mathfrak{g}$,

- $[x, y] = -(-1)^{\overline{x}\cdot\overline{y}}[y, x]$
- $[x, [y, z]] = [[x, y], z] + (-1)^{\overline{x}\cdot\overline{y}}[y, [x, z]]$
- $[x, x] = 0$ if $x \in \mathfrak{g}_{\overline{0}}$ and $p = 2$
- $[x, [x, x]] = 0$ if $x \in \mathfrak{g}_{\overline{1}}$ and $p = 3$

Say that $\mathfrak{g}$ is a **restricted Lie superalgebra** if $\mathfrak{g}_{\overline{0}}$ is an ordinary restricted Lie algebra and $\mathfrak{g}_{\overline{1}}$ is a restricted $\mathfrak{g}_{\overline{0}}$-module under the adjoint action.

Classical correspondences:

affine group schemes $\leftrightarrow$ cocommutative Hopf algebras
finite group schemes $\leftrightarrow$ f.d. cocommutative Hopf algebras
height-one infinitesimal group schemes $\leftrightarrow$ f.d. restricted Lie algebras

Super correspondences:

affine supergroup schemes $\leftrightarrow$ cocommutative Hopf superalgebras
finite supergroup schemes $\leftrightarrow$ f.d. cocommutative Hopf superalgebras
height-one infinitesimal supergroup schemes $\leftrightarrow$ f.d. res. Lie superalgebras

Problem

Can we study support varieties for finite supergroup schemes?

Example: finite-dimensional exterior algebra

An exterior algebra $\Lambda(V)$ is a (super)commutative superalgebra.

$$ab = (-1)^{\overline{a}\cdot\overline{b}} ba$$

It is also a (super)cocommutative Hopf superalgebra.

$$\begin{aligned}
&\Delta(uv) \\
&= \Delta(u)\Delta(v) \\
&= (u \otimes 1 + 1 \otimes u)(v \otimes 1 + 1 \otimes v) \\
&= (u \otimes 1)(v \otimes 1) + (u \otimes 1)(1 \otimes v) + (1 \otimes u)(v \otimes 1) + (1 \otimes u)(1 \otimes v) \\
&= (uv \otimes 1) + (u \otimes v) - (v \otimes u) + (1 \otimes uv)
\end{aligned}$$

Theorem

Let V be a finite-dimensional vector space. Then $\mathrm{H}^{\bullet}(\Lambda(V),k) \cong S^{\bullet}(V^*)$.

The cohomology ring is **graded-(super)commutative** in the sense that

$$ab = (-1)^{\deg(a)\cdot\deg(b)+\overline{a}\cdot\overline{b}}ba.$$

Theorem (Aramova–Avramov–Herzog, 2000)

Let M be a finite-dimensional $\Lambda(V)$-supermodule. Then

$$|\Lambda(V)|_M \cong \left\{ v \in V : M|_{\langle v \rangle} \text{ is not free} \right\}.$$

In the theorem, $\langle v \rangle$ refers to an algebra isomorphic to $\Lambda(v) \cong k[v]/\langle v^2 \rangle$.

In characteristic 0, this is most of the complete picture!

Suppose k is an algebraically closed field of characteristic 0.

Theorem (Kostant)

Let A be a cocommutative Hopf superalgebra over k. Then $A \cong kG\#U(\mathfrak{g})$ for some Lie superalgebra $\mathfrak{g}$ over k, some group G, and some representation $\pi : G \to \mathrm{Aut}(\mathfrak{g})$.

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Corollary

Let A be a finite-dimensional cocommutative Hopf superalgebra over k. Then $A \cong kG\#\Lambda(V)$ for some finite group G and some finite-dimensional purely odd kG-module V.

Given $kG\#\Lambda(V)$ as in the corollary, we denote the corresponding finite supergroup scheme by $G \ltimes V$.

Theorem

Let $G \ltimes V$ be a finite k-supergroup scheme. Let M and N be $G \ltimes V$-supermodules. Then $\operatorname{Ext}^\bullet_{G \ltimes V}(M, N) \cong \operatorname{Ext}^\bullet_{\Lambda(V)}(M, N)^G$. In particular,

$$\mathrm{H}^\bullet(G \ltimes V, k) \cong \mathrm{H}^\bullet(\Lambda(V), k)^G \cong S^\bullet(V^*)^G.$$

Proof

Apply the LHS spectral sequence for the normal Hopf sub-superalgebra $\Lambda(V)$, and use the fact that kG is a semisimple algebra (characteristic 0).

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Corollary

Let $G \ltimes V$ be a finite k-supergroup scheme, and let M be a finite-dimensional $G \ltimes V$-supermodule. Then

$$|G \ltimes V| \cong V/G, \text{ the quotient of } V \text{ by } G$$
$$|G \ltimes V|_M \cong \left\{ [v] \in V/G : M|_{\langle v \rangle} \text{ is not free} \right\}.$$

Cohomology of Lie superalgebras

$\mathrm{H}^\bullet(\mathfrak{g}, k)$ is the cohomology ring of the enveloping superalgebra $U(\mathfrak{g})$.

- $\mathrm{H}^\bullet(\mathfrak{g}, k)$ can be computed via the super Koszul resolution $(\mathbf{\Lambda}(\mathfrak{g}^*), \partial)$
- As a superalgebra, $\mathbf{\Lambda}(\mathfrak{g}^*) \cong \Lambda(\mathfrak{g}_{\overline{0}}^*) \,{}^{g}\!\otimes S(\mathfrak{g}_{\overline{1}}^*)$.
- $\partial : \mathbf{\Lambda}^1(\mathfrak{g}^*) \to \mathbf{\Lambda}^2(\mathfrak{g}^*)$ is dual to the Lie bracket $\mathbf{\Lambda}^2(g) \to \mathbf{\Lambda}^1(\mathfrak{g}) = \mathfrak{g}$.
- $\mathrm{H}^\bullet(\mathfrak{g}, M)$ can be computed as the cohomology of $(M \otimes \mathbf{\Lambda}(\mathfrak{g}^*), \partial_M)$
- $\partial_M : M \to M \otimes \mathbf{\Lambda}^1(\mathfrak{g}^*)$ is dual to the $\mathfrak{g}$-action $\mathfrak{g} \otimes M \to M$.

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Results in characteristic zero

- $\mathrm{H}^\bullet(\mathfrak{g}, k)$ can be either finite-dimensional or infinite-dimensional
- If $\mathfrak{g} = \mathfrak{g}_{\bar{1}}$, then $U(\mathfrak{g}) = \Lambda(\mathfrak{g})$ and $\mathrm{H}^\bullet(\mathfrak{g}, k) \cong S(\mathfrak{g}^*)$.
- If $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $m \geq n$, then $\mathrm{H}^\bullet(\mathfrak{g}, k) \cong \mathrm{H}^\bullet(\mathfrak{gl}_m, k)$ [Fuks–Leites]

Boe–Kujawa–Nakano (2009, 2010, 2011, 2012)

- Support varieties in terms of relative cohomology $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\overline{0}}; k)$
- Work in the category $\mathcal{F}$ of f.d. $\mathfrak{g}$-supermodules that are s.s. over $\mathfrak{g}_{\overline{0}}$
- Variety theory can measure *defect* of $\mathfrak{g}$ and *atypicality* of modules. Dimension does not equal complexity.

Duflo–Serganova (arXiv 2005)

- Given a $\mathfrak{g}$-supermodule M, defined the *associated variety*

$$X_M = \left\{ x \in \mathfrak{g}_{\overline{1}} : [x, x] = 0 \text{ and } M|_{\langle x \rangle} \text{ is not free} \right\}$$

- Relatively simple $GL_m \times GL_n$ orbit structure when $\mathfrak{g} = \mathfrak{gl}(m|n)$.
- Varieties detect projectivity in $\mathcal{F}$.
- Not defined in terms of cohomology.

Assumption

For the rest of this talk, assume that k is of characteristic $p \geq 3$.

- p-th powers in $S(\mathfrak{g}_{\bar{1}}^*) \subset \mathbf{\Lambda}(\mathfrak{g}^*)$ consist of cocycles. Induced map

$$\varphi : S(\mathfrak{g}_{\bar{1}}^*[p])^{(1)} \to \mathrm{H}^\bullet(\mathfrak{g}, k).$$

- Induced map of varieties $|\mathfrak{g}|_M \to \mathcal{X}_{\mathfrak{g}}(M)$ is a homeomorphism.

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Theorem

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra. Let M be a finite-dimensional $\mathfrak{g}$-supermodule. Then

$$\mathcal{X}_\mathfrak{g}(M) \cong \left\{ x \in \mathfrak{g}_{\bar{1}} : [x,x] = 0 \text{ and } M|_{\langle x \rangle} \text{ is not free} \right\}.$$

Proof

Modify Jantzen's arguments for restricted Lie algebras in characteristic 2.

First step toward support varieties: cohomological finite generation (CFG)

Theorem (D 2014)

Let G be a finite k-supergroup scheme (equivalently, a finite-dimensional cocommutative Hopf superalgebra). Then $\mathrm{H}^\bullet(G, k)$ is a finitely-generated k-superalgebra.

Remark

If A is a Hopf superalgebra, then $A\#(\mathbb{Z}/2\mathbb{Z})$ is an ordinary Hopf algebra, and $\mathrm{H}^\bullet(A\#(\mathbb{Z}/2\mathbb{Z}), k) \cong \mathrm{H}^\bullet(A, k)_{\overline{0}}$.

Can view the theorem as a generalization of the Friedlander–Suslin CFG result in multiple ways (to a strictly larger class of ordinary Hopf algebras, or to Hopf algebra objects in another symmetric monoidal category).

- $\mathcal{V}$ category of finite-dimensional superspaces
- $V^{\otimes d}$ is naturally a right $\mathfrak{S}_d$-module (signed place permutations)
- $\boldsymbol{\Gamma}^d(V) = (V^{\otimes d})^{\mathfrak{S}_d}$
- $\boldsymbol{\Gamma}^d(\mathcal{V})$: category with the same objects as $\mathcal{V}$, but with morphisms

$$\mathrm{Hom}_{\boldsymbol{\Gamma}^d(\mathcal{V})}(V, W) = \boldsymbol{\Gamma}^d[\mathrm{Hom}_k(V, W)] \cong \mathrm{Hom}_{k\mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d}).$$

Strict polynomial superfunctors (Axtell 2013)

The category $\mathcal{P}_d$ of homogeneous degree-d strict polynomial superfunctors is the category of functors $F : \boldsymbol{\Gamma}^d\mathcal{V} \to \mathcal{V}$ such that for each $V, W \in \mathcal{V}$,

$$F_{V,W} : \mathrm{Hom}_{k\mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d}) \to \mathrm{Hom}_k(F(V), F(W))$$

is an even k-linear map.

Examples of strict polynomial superfunctors

- Π parity flip functor $(\Pi V)_{\overline{0}} = V_{\overline{1}}$, $(\Pi V)_{\overline{1}} = V_{\overline{0}}$
- $\boldsymbol{T}^d(V) = V^{\otimes d}$ tensor power
- $\boldsymbol{\Gamma}^d(V) = (V^{\otimes d})^{\Sigma_d}$ super-symmetric tensors $\boldsymbol{\Gamma}(V) = \Gamma(V_{\overline{0}}) \otimes \Lambda(V_{\overline{1}})$
- $\boldsymbol{S}^d(V) = (V^{\otimes d})_{\Sigma_d}$ super-symmetric power $\boldsymbol{S}(V) = S(V_{\overline{0}}) \otimes \Lambda(V_{\overline{1}})$
- $\boldsymbol{\Lambda}^d(V)$ super-exterior power $\boldsymbol{\Lambda}(V) = \Lambda(V_{\overline{0}}) \,{}^{g}\!\otimes S(V_{\overline{1}})$
- $\boldsymbol{A}^d(V)$ super-alternating tensors $\boldsymbol{A}(V) = \Lambda(V_{\overline{0}}) \,{}^{g}\!\otimes \Gamma(V_{\overline{1}})$
- $\boldsymbol{I}^{(r)}(V) = V^{(r)}$ r-th Frobenius twist ($r \geq 1$) $\boldsymbol{I}^{(r)} = \boldsymbol{I}_0{}^{(r)} \oplus \boldsymbol{I}_1{}^{(r)}$
- Non-example: $V \mapsto V_{\overline{0}}$ (incompatible with composition of odd maps)

- SPSFs can restrict to ordinary SPFs in two different ways
- Ordinary SPFs in general don't seem lift to SPSFs
- Frobenius twists of SPFs lift to SPSFs in several different ways

Calculate the structure of the extension algebra

$$\operatorname{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}^{(r)}, \boldsymbol{I}^{(r)}) = \begin{pmatrix} \operatorname{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_0^{(r)}, \boldsymbol{I}_0^{(r)}) & \operatorname{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_1^{(r)}, \boldsymbol{I}_0^{(r)}) \\ \operatorname{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_0^{(r)}, \boldsymbol{I}_1^{(r)}) & \operatorname{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_1^{(r)}, \boldsymbol{I}_1^{(r)}) \end{pmatrix}$$

Theorem (D 2015)

$\mathrm{Ext}_{\mathcal{P}}^{\bullet}(\boldsymbol{I}^{(r)}, \boldsymbol{I}^{(r)})$ is generated as an algebra by extension classes

- $\boldsymbol{e}_i \in \mathrm{Ext}_{\mathcal{P}}^{2p^{i-1}}({\boldsymbol{I}_0}^{(r)}, {\boldsymbol{I}_0}^{(r)})$ and $\boldsymbol{e}_i^{\Pi} \in \mathrm{Ext}_{\mathcal{P}}^{2p^{i-1}}({\boldsymbol{I}_1}^{(r)}, {\boldsymbol{I}_1}^{(r)})$ $(1 \leq i \leq r)$
- $\boldsymbol{c}_r \in \mathrm{Ext}_{\mathcal{P}}^{p^r}({\boldsymbol{I}_1}^{(r)}, {\boldsymbol{I}_0}^{(r)})$ and $\boldsymbol{c}_r^{\Pi} \in \mathrm{Ext}_{\mathcal{P}}^{p^r}({\boldsymbol{I}_0}^{(r)}, {\boldsymbol{I}_1}^{(r)})$

These generators satisfy:

- $(\boldsymbol{e}_i)^p = 0 = (\boldsymbol{e}_i^{\Pi})^p$ if $1 \leq i \leq r-1$.
- $(\boldsymbol{e}_r)^p = \boldsymbol{c}_r \circ \boldsymbol{c}_r^{\Pi}$ and $(\boldsymbol{e}_r^{\Pi})^p = \boldsymbol{c}_r^{\Pi} \circ \boldsymbol{c}_r$.
- The $\boldsymbol{e}_i, \boldsymbol{e}_i^{\Pi}$ generate a commutative subalgebra.
- $(\boldsymbol{c}_r \circ \boldsymbol{c}_r^{\Pi})$ and $(\boldsymbol{c}_r^{\Pi} \circ \boldsymbol{c}_r)$ each generate polynomial subalgebras.
- The $\boldsymbol{e}_i$ restrict to Friedlander and Suslin's universal extension classes
- $\boldsymbol{c}_r$ generates $\mathrm{Ext}_{\mathcal{P}}^{\bullet}({\boldsymbol{I}_1}^{(r)}, {\boldsymbol{I}_0}^{(r)})$ over the matrix ring
- $\boldsymbol{c}_r^{\Pi}$ generates $\mathrm{Ext}_{\mathcal{P}}^{\bullet}({\boldsymbol{I}_0}^{(r)}, {\boldsymbol{I}_1}^{(r)})$ over the matrix ring
- Have $\boldsymbol{e}_i \circ \boldsymbol{c}_r = \pm(\boldsymbol{c}_r \circ {\boldsymbol{e}_i}^{\Pi})$. But is it $+$ or $-$?

Let $G \subset GL(m|n)$ be an infinitesimal supergroup scheme of height $\leq r$.

Evaluation and restriction maps

$$\begin{aligned} \mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}^{(r)}, \boldsymbol{I}^{(r)}) &\to \mathrm{Ext}^\bullet_{GL(m|n)}((k^{m|n})^{(r)}, (k^{m|n})^{(r)}) \\ &\cong \mathrm{Ext}^\bullet_{GL(m|n)}(k, \mathfrak{gl}(m|n)^{(r)}) \\ &\to \mathrm{Ext}^\bullet_G(k, \mathfrak{gl}(m|n)^{(r)}) \\ &\cong \mathrm{Hom}_k(\mathfrak{gl}(m|n)^{*(r)}, \mathrm{H}^\bullet(G,k)) \end{aligned}$$

For $r = 1$, the strict polynomial superfunctor extension classes give rise to a superalgebra homomorphism

$$\varphi : S(\mathfrak{gl}(m|n)^*_{\overline{0}}[2])^{(1)} \otimes S(\mathfrak{gl}(m|n)^*_{\overline{1}}[p])^{(1)} \to \mathrm{H}^\bullet(G,k).$$

Induced finite map of varieties $|G| \to \mathfrak{gl}(m|n)$ with image $V_G(k)$.

Change to the language of finite-dimensional restricted Lie superalgebras.

Theorem

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie superalgebra. Then

$$V_{u(\mathfrak{g})}(k) \cong \{x + y \mid x \in \mathfrak{g}_{\overline{0}}, y \in \mathfrak{g}_{\overline{1}}, [x, y] = 0, x^{[p]} = y^2\}$$

where $y^2 := \frac{1}{2}[y, y]$.

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Main ideas of the proof

Relations come from the extension classes for polynomial superfunctors.

- Sufficiency of the relations comes from explicit calculations for the restricted subalgebra generated by x and y, using an "explicit" projective resolution constructed by Iwai–Shimada and May.
- $x^{[p]} = y^2$ comes from $(\boldsymbol{e}_1)^p = \boldsymbol{c}_1 \circ \boldsymbol{c}_1^{\Pi}$ and $(\boldsymbol{e}_1^{\Pi})^p = \boldsymbol{c}_1^{\Pi} \circ \boldsymbol{c}_1$
- $[x, y] = 0$ "comes from" $\boldsymbol{e}_1 \circ \boldsymbol{c}_1 = \boldsymbol{c}_1 \circ \boldsymbol{e}_1^{\Pi}$ and $\boldsymbol{e}_1^{\Pi} \circ \boldsymbol{c}_1^{\Pi} = \boldsymbol{c}_1^{\Pi} \circ \boldsymbol{e}_1$

Rank varieties

- Given a $u(\mathfrak{g})$-supermodule M, what is $V_{u(\mathfrak{g})}(M)$?
- If $x + y \in V_{u(\mathfrak{g})}(M)$ and x and y are both nonzero, what does this mean about the restriction $M|_{\langle x,y \rangle_p}$?

Now let $G \subset GL(m|n)$ be a height-r infinitesimal supergroup scheme. The polynomial superfunctor classes give rise to a homomorphism

$$\left[\bigotimes_{i=1}^{r} S(\mathfrak{gl}(m|n)^*_{\overline{0}}[2p^{i-1}])^{(r)}\right] \otimes S(\mathfrak{gl}(m|n)^*_{\overline{1}}[p^r])^{(r)} \to \mathrm{H}^\bullet(G,k)$$

over whose image $\mathrm{H}^\bullet(G,k)$ is finite.

Possible description for $|G|$?

Set of all r-tuples $(x_1, \dots, x_{r-1}, x_r + y)$ such that

- $x_i \in \mathfrak{g}_{\overline{0}}$ for $1 \leq i \leq r$
- $y \in \mathfrak{g}_{\overline{1}}$.
- Entries pairwise commute, and $[x_r, y] = 0$.
- $x_i^{[p]} = 0$ for $1 \leq i \leq r-1$.
- $x_r^{[p]} = y^2$.