Cohomology of Finite Groups: An introduction to the Algebra VRG

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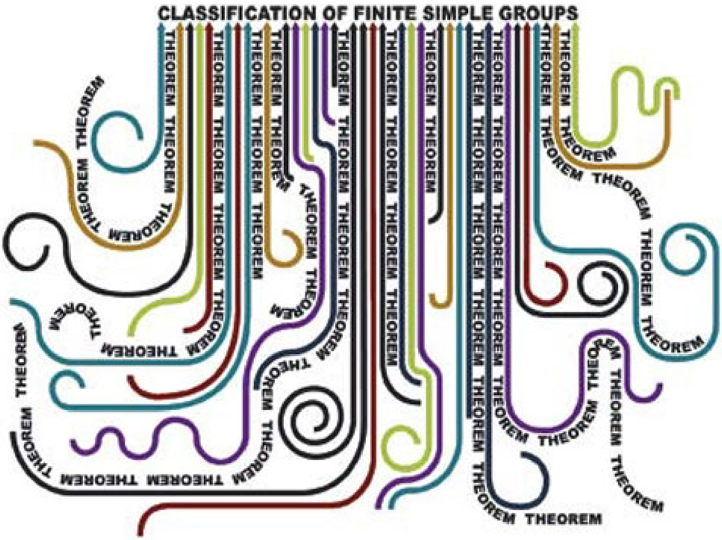
THE ENORMOUS THEOREM



http://plus.maths.org/issue41/features/elwes/

simple groups : groups :: atoms : molecules

Every finite simple group is isomorphic to

- A cyclic group of prime power order, or
- An alternating group of degree at least 5, or
- A finite simple group of Lie type, or
- One of the 26 sporadic simple groups.

Some finite simple groups fall into more than one category.

The finite groups of Lie type comprise the bulk of the list.

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First, what is an algebraic group?

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Definition

A affine algebraic group is a group that is simultaneously an affine algebraic variety, that is, the zero set of a collection of polynomials.

Example (Special Linear Group)

The special linear group $SL_n(\overline{\mathbb{F}}_p)$ consists of all $n \times n$ matrices with coefficients in $\overline{\mathbb{F}}_p$ that have determinant 1. It is the zero set of the polynomial

$$\left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) X_{1,\sigma(1)} X_{2,\sigma(2)} \cdots X_{n,\sigma(n)} \right) - 1.$$

Taking coefficients in the finite field $\mathbb{F}_q$, $q = p^r$, we get the group

$$SL_n(\mathbb{F}_q) \leq SL_n(\overline{\mathbb{F}}_p)$$

The finite groups of Lie type all arise more-or-less in this fashion.

$SL_n(\mathbb{F}_q)$ need not be simple (constant diagonal matrices in center).

The quotient $PSL_n(\mathbb{F}_q)$ is simple if $q \neq 2, 3$.

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Example (Niles' least favorite example)

Let $V = (\overline{\mathbb{F}}_p)^n$ (column vectors).
Then $SL_n(\overline{\mathbb{F}}_p)$ acts on V by matrix multiplication.
The finite group $SL_n(\mathbb{F}_q)$ acts on V by restriction.

Example (Niles' favorite example)

Let V be a vector space with basis $\{v_1, v_2, v_3, v_4\}$. The group S_4 acts on V by permuting the basis vectors in the obvious way.

$$(1,2,3).(3v_1 + 7v_2 - 6v_3 + 2v_4) = 3v_2 + 7v_3 - 6v_1 + 2v_4$$

~~What is cohomology?~~ What is cohomology good for?

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- Explain how simple groups fit together into larger groups.
- Explain how group representations fit together.

Results so far of the Algebra Research Group:

- Studied cohomology for finite groups of Lie type.

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- Lots of computer calculations to handle exceptional cases.

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- Try to compute higher cohomology groups $\mathrm{H}^2(G(\mathbb{F}_q), V)$.
- Unanswered questions about when $\mathrm{H}^1(G(\mathbb{F}_q), V) \neq 0$.