Support varieties for Lie superalgebras and finite supergroup schemes

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What does it mean to be "super"?

Something is "super" if it has a compatible $\mathbb{Z}/2\mathbb{Z}$-grading.

- Superspaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$, $W = W_{\overline{0}} \oplus W_{\overline{1}}$
- Induced gradings on tensor products, linear maps, etc.

$$(V \otimes W)_\ell = \bigoplus_{i+j=\ell} V_i \otimes W_j$$

$$\mathrm{Hom}_k(V,W)_\ell = \{f \in \mathrm{Hom}_k(V,W) : f(V_i) \subseteq W_{i+\ell}\}$$

- $V \otimes W \cong W \otimes V$ via the **supertwist** $v \otimes w \mapsto (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$

Define (Hopf) superalgebras and 'super' notions of (co)commutativity in terms of the "usual diagrams," but use the supertwist whenever objects pass one another. In other words, insert $\pm$ signs as a topologist would.

Examples of Hopf superalgebras

- Ordinary Hopf algebras (considered as purely even superalgebras).
- Exterior algebra $\Lambda(V) = \bigoplus_{i \geq 0} \Lambda^i(V)$ (reduce $\mathbb{Z}$-grading modulo 2)
- **$\mathbb{Z}$-graded Hopf algebras in the sense of Milnor and Moore**
- If $\mathfrak{g}$ is a (restricted) Lie superalgebra, then its (restricted) enveloping superalgebra is a (finite-dimensional) Hopf superalgebra.

Classical correspondences:

affine group schemes $\leftrightarrow$ cocommutative Hopf algebras
finite group schemes $\leftrightarrow$ f.d. cocommutative Hopf algebras
height-one infinitesimal group schemes $\leftrightarrow$ f.d. restricted Lie algebras

Super correspondences:

affine supergroup schemes $\leftrightarrow$ cocommutative Hopf superalgebras
finite supergroup schemes $\leftrightarrow$ f.d. cocommutative Hopf superalgebras
height-one infinitesimal supergroup schemes $\leftrightarrow$ f.d. res. Lie superalgebras

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Suppose k is an algebraically closed field of characteristic 0.

Theorem (Kostant)

Let A be a cocommutative Hopf superalgebra over k. Then $A \cong U(\mathfrak{g})\#kG$ for some Lie superalgebra $\mathfrak{g}$ over k and some subgroup $G \leq \mathrm{Aut}(\mathfrak{g})$.

Corollary

Let A be a finite-dimensional cocommutative Hopf superalgebra over k. Then $A \cong \Lambda(V)\#kG$ for some finite group G and some finite-dimensional purely odd kG-module V.

Given $\Lambda(V)\#kG$ as in the corollary, we denote the corresponding finite supergroup scheme by $V \rtimes G$.

Suppose k is an algebraically closed field of characteristic 0.

Theorem

Let $V \rtimes G$ be a finite k-supergroup scheme. Let M and N be $V \rtimes G$-supermodules. Then $\operatorname{Ext}^{\bullet}_{V \rtimes G}(M,N) \cong \operatorname{Ext}^{\bullet}_{\Lambda(V)}(M,N)^G$. In particular,

$$\mathrm{H}^{\bullet}(V \rtimes G, k) \cong \mathrm{H}^{\bullet}(\Lambda(V), k)^G \cong S^{\bullet}(V^*)^G.$$

Corollary

Let $V \rtimes G$ be a finite k-supergroup scheme, and let M be a finite-dimensional $V \rtimes G$-supermodule. Then $|V \rtimes G| \cong V/G$, the quotient of V by G, and

$$|V \rtimes G|_M \cong \left\{ [v] \in V/G : M|_{\langle v \rangle} \text{ is not free} \right\}.$$

The subalgebra $\langle v \rangle$ generated by v looks like $k[v]/(v^2)$.

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Cohomology of Lie superalgebras

- $\mathrm{H}^\bullet(\mathfrak{g}, k)$ is the cohomology ring of the enveloping superalgebra $U(\mathfrak{g})$.
- $\mathrm{H}^\bullet(\mathfrak{g}, k)$ can be computed via the super Koszul resolution $(\boldsymbol{\Lambda}(\mathfrak{g}^*), \partial)$
- As a superalgebra, $\boldsymbol{\Lambda}(\mathfrak{g}^*) \cong \Lambda(\mathfrak{g}_{\overline{0}}^*) \,{}^{g}\!\otimes\, S(\mathfrak{g}_{\overline{1}}^*)$.

Results in characteristic zero

- $\mathrm{H}^\bullet(\mathfrak{g}, k)$ can be either finite-dimensional or infinite-dimensional
  - If $\mathfrak{g} = \mathfrak{g}_{\overline{1}}$, then $U(\mathfrak{g}) = \Lambda(\mathfrak{g})$ and $\mathrm{H}^\bullet(\mathfrak{g}, k) \cong S(\mathfrak{g}^*)$.
  - If $\mathfrak{g} = \mathfrak{gl}(m|n)$, then $\mathrm{H}^\bullet(\mathfrak{g}, k)$ is a f.d. exterior algebra [Fuks–Leites]
- Boe–Kujawa–Nakano: varieties using relative cohomology
- Duflo–Serganova: associated varieties X_M (no cohomology)

From now on, assume that k is algebraically closed of characteristic $p \geq 3$.

- Super Koszul complex $\mathbf{\Lambda}(\mathfrak{g}^*) \cong \Lambda(\mathfrak{g}_{\overline{0}}^*) \; {}^{g}\!\otimes \; S(\mathfrak{g}_{\overline{1}}^*)$
- p-th powers in $S(\mathfrak{g}_{\overline{1}}^*) \subset \mathbf{\Lambda}(\mathfrak{g}^*)$ consist of cocycles. Induced map

$$\varphi : S(\mathfrak{g}_{\overline{1}}^*[p])^{(1)} \to \mathrm{H}^\bullet(\mathfrak{g}, k).$$

Theorem

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra. Let M be a finite-dimensional $\mathfrak{g}$-supermodule. Then there are homeomorphisms

$$|\mathfrak{g}| \cong \{x \in \mathfrak{g}_{\overline{1}} : [x,x] = 0\}$$
$$|\mathfrak{g}|_M \cong \left\{x \in \mathfrak{g}_{\overline{1}} : [x,x] = 0 \text{ and } M|_{\langle x \rangle} \text{ is not free}\right\} \cup \{0\}.$$

Proof

Modify Jantzen's arguments for restricted Lie algebras in characteristic 2.

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First step toward support varieties: cohomological finite generation (CFG)

Theorem (Drupieski 2016)

Let G be a finite supergroup scheme over k and let M be a finite-dimensional G-supermodule. Then $\mathrm{H}^\bullet(G, k)$ is a finitely-generated k-superalgebra and $\mathrm{H}^\bullet(G, M)$ is finite over $\mathrm{H}^\bullet(G, k)$.

Proved by way of cohomology calculations in the category of strict polynomial superfunctors (SPSFs), analogous to the argument for ordinary finite group schemes using ordinary strict polynomial functors (SPFs) given by Friedlander and Suslin.

Examples of strict polynomial superfunctors

- Π parity flip functor $\qquad (\Pi V)_{\overline{0}} = V_{\overline{1}},\ (\Pi V)_{\overline{1}} = V_{\overline{0}}$
- $\boldsymbol{\Gamma}^d(V) = (V^{\otimes d})^{\Sigma_d}$ super-symmetric tensors $\qquad \boldsymbol{\Gamma}(V) = \Gamma(V_{\overline{0}}) \otimes \Lambda(V_{\overline{1}})$
- $\boldsymbol{S}^d(V) = (V^{\otimes d})_{\Sigma_d}$ super-symmetric power $\qquad \boldsymbol{S}(V) = S(V_{\overline{0}}) \otimes \Lambda(V_{\overline{1}})$
- $\boldsymbol{\Lambda}^d(V)$ super-exterior power $\qquad \boldsymbol{\Lambda}(V) = \Lambda(V_{\overline{0}}) \ {}^{g}\!\otimes S(V_{\overline{1}})$
- $\boldsymbol{A}^d(V)$ super-alternating tensors $\qquad \boldsymbol{A}(V) = \Lambda(V_{\overline{0}}) \ {}^{g}\!\otimes \Gamma(V_{\overline{1}})$
- $\boldsymbol{I}^{(r)}(V) = V^{(r)}$ r-th Frobenius twist ($r \geq 1$) $\qquad \boldsymbol{I}^{(r)} = \boldsymbol{I}_0{}^{(r)} \oplus \boldsymbol{I}_1{}^{(r)}$
- Non-example: $V \mapsto V_{\overline{0}}$ (incompatible with composition of odd maps)

- SPSFs can restrict to ordinary SPFs in two different ways
- Ordinary SPFs in general don't seem lift to SPSFs
- Frobenius twists of SPFs lift to SPSFs in several different ways

$$\mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}^{(r)}, \boldsymbol{I}^{(r)}) = \begin{pmatrix} \mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_0^{(r)}, \boldsymbol{I}_0^{(r)}) & \mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_1^{(r)}, \boldsymbol{I}_0^{(r)}) \\ \mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_0^{(r)}, \boldsymbol{I}_1^{(r)}) & \mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}_1^{(r)}, \boldsymbol{I}_1^{(r)}) \end{pmatrix}$$

Theorem (Drupieski 2016)

$\mathrm{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}^{(r)}, \boldsymbol{I}^{(r)})$ is generated as an algebra by extension classes

- $\boldsymbol{e}_i' \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}(\boldsymbol{I}_0{}^{(r)}, \boldsymbol{I}_0{}^{(r)})$ and $\boldsymbol{e}_i'' \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}(\boldsymbol{I}_1{}^{(r)}, \boldsymbol{I}_1{}^{(r)})$ $\quad (1 \leq i \leq r)$
- $\boldsymbol{c}_r \in \mathrm{Ext}^{p^r}_{\mathcal{P}}(\boldsymbol{I}_1{}^{(r)}, \boldsymbol{I}_0{}^{(r)})$ and $\boldsymbol{c}_r^\Pi \in \mathrm{Ext}^{p^r}_{\mathcal{P}}(\boldsymbol{I}_0{}^{(r)}, \boldsymbol{I}_1{}^{(r)})$

These generators satisfy:

- $(\boldsymbol{e}_i')^p = 0 = (\boldsymbol{e}_i'')^p$ if $1 \leq i \leq r-1$.
- $(\boldsymbol{e}_r')^p = \boldsymbol{c}_r \circ \boldsymbol{c}_r^\Pi$ and $(\boldsymbol{e}_r'')^p = \boldsymbol{c}_r^\Pi \circ \boldsymbol{c}_r$.
- The $\boldsymbol{e}_i', \boldsymbol{e}_i''$ generate a commutative subalgebra.
- The $\boldsymbol{e}_i'$ restrict to Friedlander and Suslin's universal extension classes
- Have $\boldsymbol{e}_i' \circ \boldsymbol{c}_r = \pm \boldsymbol{c}_r \circ \boldsymbol{e}_i''$. But is it $+$ or $-$? (It is $+$ for $i = r$.)

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Let $G \subset GL(m|n)$ be an infinitesimal supergroup scheme of height $\leq r$.

Evaluation and restriction maps

$$\begin{aligned}
\operatorname{Ext}^\bullet_{\mathcal{P}}(\boldsymbol{I}^{(r)}, \boldsymbol{I}^{(r)}) &\to \operatorname{Ext}^\bullet_{GL(m|n)}((k^{m|n})^{(r)}, (k^{m|n})^{(r)}) \\
&\cong \operatorname{Ext}^\bullet_{GL(m|n)}(k, \mathfrak{gl}(m|n)^{(r)}) \\
&\to \operatorname{Ext}^\bullet_{G}(k, \mathfrak{gl}(m|n)^{(r)}) \\
&\cong \operatorname{Hom}_k(\mathfrak{gl}(m|n)^{*(r)}, \mathrm{H}^\bullet(G,k))
\end{aligned}$$

For $r = 1$, the strict polynomial superfunctor extension classes give rise to a superalgebra homomorphism over which $\mathrm{H}^\bullet(G,k)$ is finite:

$$\varphi : S(\mathfrak{gl}(m|n)^*_{\overline{0}}[2])^{(1)} \otimes S(\mathfrak{gl}(m|n)^*_{\overline{1}}[p])^{(1)} \to \mathrm{H}^\bullet(G,k).$$

Induced finite map of varieties $|G| \to \mathfrak{gl}(m|n)$ with image $V_G(k)$.

Change to the language of finite-dimensional restricted Lie superalgebras.

Theorem

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie superalgebra. Then

$$V_{u(\mathfrak{g})}(k) \cong \{x + y \mid x \in \mathfrak{g}_{\overline{0}}, y \in \mathfrak{g}_{\overline{1}}, [x, y] = 0, x^{[p]} = y^2\}$$

where $y^2 := \frac{1}{2}[y, y]$.

- Relations come from the functor cohomology calculations
  - $[x, y] = 0$ comes from $\boldsymbol{e}_1' \circ \boldsymbol{c}_1 = \boldsymbol{c}_1 \circ \boldsymbol{e}_1''$ and $\boldsymbol{e}_1'' \circ \boldsymbol{c}_1^{\Pi} = \boldsymbol{c}_1^{\Pi} \circ \boldsymbol{e}_1'$.
  - $x^{[p]} = y^2$ comes from $(\boldsymbol{e}_1')^p = \boldsymbol{c}_1 \circ \boldsymbol{c}_1^{\Pi}$ and $(\boldsymbol{e}_1'')^p = \boldsymbol{c}_1^{\Pi} \circ \boldsymbol{c}_1$.
- Sufficiency of the conditions comes from explicit calculations for the restricted subalgebra generated by x and y, using an "explicit" projective resolution constructed by Iwai–Shimada and May.
- Support variety $V_{u(\mathfrak{g})}(M)$ of a nontrivial supermodule M ????

Now let $G \subset GL(m|n)$ be a height-r infinitesimal supergroup scheme. The polynomial superfunctor classes give rise to a homomorphism

$$\left[\bigotimes_{i=1}^{r} S(\mathfrak{gl}(m|n)_0^*[2p^{i-1}])^{(r)}\right] \otimes S(\mathfrak{gl}(m|n)_1^*[p^r])^{(r)} \to \mathrm{H}^\bullet(G,k)$$

over whose image $\mathrm{H}^\bullet(G,k)$ is finite.

Possible description for $|G|$ (à la Suslin–Friedlander–Bendel)?

Set of all r-tuples $(x_1,\dots,x_{r-1},x_r+y)$ such that

- $x_i \in \mathfrak{g}_{\overline{0}}$ for $1 \leq i \leq r$
- $y \in \mathfrak{g}_{\overline{1}}$.
- Entries pairwise commute, and $[x_r,y]=0$.
- $x_i^{[p]} = 0$ for $1 \leq i \leq r-1$.
- $x_r^{[p]} = y^2$.

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