Cohomology rings for quantized enveloping algebras

Christopher Drupieski

University of Georgia

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Problem

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Progress around 1940:

- Results for classical groups by Pontrjagin, Cartan $\Rightarrow$ Brauer.
- Hopf: For all compact connected simple $G$, $H^\bullet(G, \mathbb{C})$ is an exterior algebra generated by elements in certain odd degrees.
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More progress around 1950:

- Koszul obtains Hopf’s results by working with the Lie algebra \( \mathfrak{g} \).
  
  Key tool is the Koszul complex \( \Lambda^\bullet(\mathfrak{g}^*) \) for computing \( H^\bullet(\mathfrak{g}, \mathbb{C}) \).
Theorem (Koszul)

\[ H^\bullet(g, \mathbb{C}) \cong \Lambda^\bullet(g^* \cdot g) \text{ is an exterior algebra over its primitive subspace.} \]

Example (Type $A_n$)

Let $g = \mathfrak{sl}_{n+1}$. Then $H^\bullet(g, \mathbb{C})$ is generated in degrees $3, 5, 7, \ldots, 2n + 1$. The degree $r$ generator is represented by the function

\[ \Phi_r(X_1, \ldots, X_r) = \sum_{\sigma \in S_r} (-1)^{\text{sgn}(\sigma)} \text{tr}(X_{\sigma(1)} \circ \cdots \circ X_{\sigma(r)}). \]
Degrees of generators for $H^\bullet(g, \mathbb{C})$:

<table>
<thead>
<tr>
<th>Type</th>
<th>Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_r$</td>
<td>$3, 5, 7, \ldots, 2r + 1$</td>
</tr>
<tr>
<td>$B_r$</td>
<td>$3, 7, 11, \ldots, 4r - 1$</td>
</tr>
<tr>
<td>$C_r$</td>
<td>$3, 7, 11, \ldots, 4r - 1$</td>
</tr>
<tr>
<td>$D_r$ ($r \geq 4$)</td>
<td>$3, 7, 11, \ldots, 4r - 5, 2r - 1$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$3, 9, 11, 15, 17, 23$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$3, 11, 15, 19, 23, 27, 35$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$3, 15, 23, 27, 35, 39, 47, 59$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$3, 11, 15, 23$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$3, 11$</td>
</tr>
</tbody>
</table>
The cohomology of $\mathfrak{g}$ is that of its universal enveloping algebra $U(\mathfrak{g})$. We can also consider the associated QEA $U_q(\mathfrak{g})$ over $k = \mathbb{C}(q)$.

**Problem**

_Can the cohomology calculations be generalized to the QEA $U_q = U_q(\mathfrak{g})$, despite the lack of an explicit projective resolution like the Koszul complex?_
The cohomology of \( \mathfrak{g} \) is that of its universal enveloping algebra \( U(\mathfrak{g}) \). We can also consider the associated QEA \( U_q(\mathfrak{g}) \) over \( k = \mathbb{C}(q) \).

**Problem**

*Can the cohomology calculations be generalized to the QEA \( U_q = U_q(\mathfrak{g}) \), despite the lack of an explicit projective resolution like the Koszul complex?*

Some evidence:

- There exists a sequence of degenerations \( U_q^{(1)}, U_q^{(2)}, \ldots, U_q^{(2N)} \) of \( U_q \) such that \( H^\bullet(U_q^{(2N)}, \mathbb{C}) \) is a twisted exterior algebra \( \Lambda_q^\bullet(\mathfrak{g}^*) \).
- Poincaré duality established for \( U_q \) and its specializations by Chemla (2004); also Brown and Zhang (2008), Kowalzig and Krähmer (2010).
Notation:

- $A$: the localization of $\mathbb{C}[q, q^{-1}]$ at the maximal ideal $(q - 1)$
- $U_A$: the $A$-subalgebra of $U_q$ generated by simple root vectors
- $U_A$ is an integral form for $U_q$: $U_A \otimes_A k = U_q$
- $U_1 := U_A \otimes_A \mathbb{C}_1 \cong U_A/(q - 1)U_A$, where $\mathbb{C}_1 = A/(q - 1)A$
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Fact:
$U_1$ is a central extension of $U(g)$ by the semisimple group ring $\mathbb{C}(\mathbb{Z}/2\mathbb{Z})^r$.

Consequence: $H^\bullet(U_1, \mathbb{C}) \cong H^\bullet(U(g), \mathbb{C})$.

Strategy:
Connect cohomology of $U_q$ to that of $U(g)$ via the integral from $U_A$. 
The cohomology ring $H^\bullet(U_q(\mathfrak{g}), k)$ is an exterior algebra over a graded subspace concentrated in the same odd degrees as for $U(\mathfrak{g})$.

Follows from showing that

$$H^\bullet(U_A, A) \text{ is } A\text{-free}$$
$$H^\bullet(U_1, \mathbb{C}) \cong H^\bullet(U_A, A) \otimes_A \mathbb{C}_1 \text{ and}$$
$$H^\bullet(U_q(\mathfrak{g}), k) = H^\bullet(U_A, A) \otimes_A k.$$
Short exact sequence from the universal coefficient theorem:

\[ 0 \to H^n(U_A, A) \otimes_A \mathbb{C}_1 \xrightarrow{i} H^n(U(g), \mathbb{C}) \to \text{Tor}_{1}^{A}(H^{n+1}(U_A, A), \mathbb{C}_1) \to 0. \]

The following are equivalent:

- The map \(i\) is surjective.
- The Tor group is zero.
- \(H^{n+1}(U_A, A)\) is \(A\)-free.

Also:

- If \(H^n(U(g), \mathbb{C}) = 0\), then \(H^n(U_A, A) = 0\), because \(H^n(U_A, A)\) is a finitely-generated \(A\)-module and \(A\) is a local ring.
$0 \to H^n(U_A, A) \otimes_A C_1 \overset{i}{\to} H^n(U(g), \mathbb{C}) \to \text{Tor}^A_1(H^{n+1}(U_A, A), C_1) \to 0$

**Example (Type $A_2$)**

- $H^\bullet(\mathfrak{sl}_3, \mathbb{C})$ is generated in degrees $3, 5$, nonzero in degrees $0, 3, 5, 8$.
- $H^n(\mathfrak{sl}_3, \mathbb{C}) = 0$ for $n = 4, 6$, so $i$ is surjective for $n = 3, 5$.
- Then $i : H^\bullet(U_A, A) \otimes_A C_1 \to H^\bullet(U(g), \mathbb{C})$ is an isomorphism and $H^\bullet(U_A, A)$ is a free $A$-module.

This argument also applies for types $A_1, B_2, E_7, E_8, F_4, G_2$. 
0 \rightarrow H^n(U_A, A) \otimes_A C_1 \xrightarrow{i} H^n(U(g), \mathbb{C}) \rightarrow \text{Tor}_1^A(H^{n+1}(U_A, A), C_1) \rightarrow 0

**Example (Type $A_3$)**

$H^\bullet(\mathfrak{sl}_4, \mathbb{C})$ is generated in degrees 3, 5, 7, nonzero in degrees 0, 3, 5, 7, 8, 10, 12, 15.

Commutative square induced by the inclusion of Dynkin diagrams:

$H^8(U_A(\mathfrak{sl}_4), A) \otimes_A C_1 \xrightarrow{\sim} H^8(U(\mathfrak{sl}_4), \mathbb{C})$

\[\text{res} \otimes_A 1 \quad \downarrow \text{res}\]

$H^8(U_A(\mathfrak{sl}_3), A) \otimes_A C_1 \xrightarrow{\sim} H^8(U(\mathfrak{sl}_3), \mathbb{C})$.

So res : $H^8(U_A(\mathfrak{sl}_4), A) \rightarrow H^8(U_A(\mathfrak{sl}_3), A)$ is onto by Nakayama’s Lemma, and hence $H^8(U_A(\mathfrak{sl}_4), A)$ is A-free of the same rank by dim. comparison.
To handle the higher rank cases and type $E_6$, must understand the Lie algebra cohomology restriction map $\text{res} : H^\bullet(g, \mathbb{C}) \to H^\bullet(g', \mathbb{C})$ when the inclusion $g' \subset g$ corresponds to an inclusion of Dynkin diagrams.

There exists a natural map $\rho : S^\bullet(g^*)^g \to \Lambda^{\bullet-1}(g^*)^g$ with image the space of primitive elements in $\Lambda^\bullet(g^*)^g$ and kernel $(S^\bullet(g^*)^g)^2$.

Use the isomorphism $S^\bullet(g^*)^g \cong S^\bullet(\mathfrak{h}^*)^W$ to turn the cohomological restriction map into a problem about the restriction of polynomial invariants for the Weyl group. (Polynomial restriction maps from $E_7$ to $E_6$ and $E_6$ to $D_5$ computed by Toda and Watanabe in mid 1970s.)
Theorem

\[ H^\bullet(U_q, k) \text{ is an exterior algebra and } \dim H^\bullet(U_q, k) = \dim H^\bullet(U(g), \mathbb{C}). \]

- \( \dim H^3(U_q(g), k) = 1 \) (Alternate proof à la the Killing form?)
- For \( \varepsilon \in \mathbb{C}^\times \) not a small root of unity, \( \dim H^\bullet(U_{\varepsilon}, \mathbb{C}) \geq \dim H^\bullet(U_q, k) \), with equality for almost all \( \varepsilon \in \mathbb{C}^\times \). In particular, equality holds for all \( \varepsilon \in \mathbb{C} \) that are transcendental over \( \mathbb{Q} \).
- Conjecture: Equality holds if \( \varepsilon \) is not a root of unity, or if \( \varepsilon \) is a root of unity of sufficiently large order (say, \( \varepsilon^\ell = 1 \) with \( \ell > h \)).
Theorem

Let $\varepsilon \in \mathbb{C}$ be a primitive $p$-th root of unity with $p$ prime and $p > 3(h - 1)$. Then $H^\bullet(\mathbf{U}_\varepsilon, \mathbb{C})$ is an exterior algebra generated in odd degrees.

Proof sketch.

For $p > 3(h - 1)$, the mod-$p$ Lie algebra cohomology ring $H^\bullet(\mathfrak{g} \mathbb{F}_p, \mathbb{F}_p)$ is an exterior algebra (Friedlander and Parshall, 1986). Let $\mathbb{Z}$ be the localization of $\mathbb{Z}[\varepsilon]$ at the kernel of the map $\mathbb{Z}[\varepsilon] \to \mathbb{F}_p$ sending $\varepsilon \mapsto 1$. Then $\mathbb{Z}$ is a local ring with quotient field $\mathbb{Q}(\varepsilon)$ and residue field $\mathbb{F}_p$. Now use $\mathbf{U}_\mathbb{Z}$ to compare cohomology for $\mathbf{U}(\mathfrak{g} \mathbb{F}_p)$ and $\mathbf{U}_\varepsilon$. 

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For $p > 3(h - 1)$, the mod-$p$ Lie algebra cohomology ring $H^\bullet(g_{\mathbb{F}_p}, \mathbb{F}_p)$ is an exterior algebra (Friedlander and Parshall, 1986).

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