Cohomology rings for quantized enveloping algebras

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More progress around 1950:

- Koszul obtains Hopf's results by working with the Lie algebra $\mathfrak{g}$. Key tool is the Koszul complex $\Lambda^\bullet(\mathfrak{g}^*)$ for computing $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C})$.

Theorem (Koszul)

$H^\bullet(\mathfrak{g},\mathbb{C}) \cong \Lambda^\bullet(\mathfrak{g}^*)^{\mathfrak{g}}$ *is an exterior algebra over its primitive subspace.*

Example (Type A_n)

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Then $H^\bullet(\mathfrak{g},\mathbb{C})$ is generated in degrees $3, 5, 7, \ldots, 2n+1$. The degree r generator is represented by the function

$$\Phi_r(X_1,\ldots,X_r) = \sum_{\sigma\in S_r} (-1)^{\mathrm{sgn}(\sigma)} \operatorname{tr}(X_{\sigma(1)} \circ \cdots \circ X_{\sigma(r)}).$$

Degrees of generators for $H^\bullet(\mathfrak{g}, \mathbb{C})$:

Type	Degrees
A_r	$3, 5, 7, \ldots, 2r+1$
B_r	$3, 7, 11, \ldots, 4r-1$
C_r	$3, 7, 11, \ldots, 4r-1$
D_r $(r \geq 4)$	$3, 7, 11, \ldots, 4r-5, 2r-1$
E_6	$3, 9, 11, 15, 17, 23$
E_7	$3, 11, 15, 19, 23, 27, 35$
E_8	$3, 15, 23, 27, 35, 39, 47, 59$
F_4	$3, 11, 15, 23$
G_2	$3, 11$

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Some evidence:

- There exists a sequence of degenerations $U_q^{(1)}, U_q^{(2)}, \ldots, U_q^{(2N)}$ of U_q such that $\mathrm{H}^\bullet(U_q^{(2N)}, \mathbb{C})$ is a twisted exterior algebra $\Lambda_q^\bullet(\mathfrak{g}^*)$.
- Poincaré duality established for U_q and its specializations by Chemla (2004); also Brown and Zhang (2008), Kowalzig and Krähmer (2010).

Notation:

- A: the localization of $\mathbb{C}[q, q^{-1}]$ at the maximal ideal $(q-1)$
- U_{A}: the A-subalgebra of U_q generated by simple root vectors
- U_{A} is an integral form for U_q: $U_{\mathsf{A}} \otimes_{\mathsf{A}} k = U_q$
- $U_1 := U_{\mathsf{A}} \otimes_{\mathsf{A}} \mathbb{C}_1 \cong U_{\mathsf{A}}/(q-1)U_{\mathsf{A}}$, where $\mathbb{C}_1 = \mathsf{A}/(q-1)\mathsf{A}$

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Fact:
U_1 is a central extension of $U(\mathfrak{g})$ by the semisimple group ring $\mathbb{C}(\mathbb{Z}/2\mathbb{Z})^r$.
Consequence: $\mathrm{H}^\bullet(U_1, \mathbb{C}) \cong \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$.

Strategy:
Connect cohomology of U_q to that of $U(\mathfrak{g})$ via the integral from U_{A}.

Theorem

The cohomology ring $\mathrm{H}^\bullet(U_q(\mathfrak{g}), k)$ *is an exterior algebra over a graded subspace concentrated in the same odd degrees as for* $U(\mathfrak{g})$.

Follows from showing that

$$\begin{aligned} \mathrm{H}^\bullet(U_\mathrm{A}, \mathrm{A}) &\text{ is A-free} \\ \mathrm{H}^\bullet(U_1, \mathbb{C}) &\cong \mathrm{H}^\bullet(U_\mathrm{A}, \mathrm{A}) \otimes_\mathrm{A} \mathbb{C}_1 \text{ and} \\ \mathrm{H}^\bullet(U_q(\mathfrak{g}), k) &= \mathrm{H}^\bullet(U_\mathrm{A}, \mathrm{A}) \otimes_\mathrm{A} k. \end{aligned}$$

Short exact sequence from the universal coefficient theorem:

$$0 \to \mathrm{H}^n(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \xrightarrow{i} \mathrm{H}^n(U(\mathfrak{g}), \mathbb{C}) \to \mathrm{Tor}_1^\mathsf{A}(\mathrm{H}^{n+1}(U_\mathsf{A}, \mathsf{A}), \mathbb{C}_1) \to 0.$$

The following are equivalent:

- The map i is surjective.
- The Tor group is zero.
- $\mathrm{H}^{n+1}(U_\mathsf{A}, \mathsf{A})$ is A-free.

Also:

- If $\mathrm{H}^n(U(\mathfrak{g}), \mathbb{C}) = 0$, then $\mathrm{H}^n(U_\mathsf{A}, \mathsf{A}) = 0$, because $\mathrm{H}^n(U_\mathsf{A}, \mathsf{A})$ is a finitely-generated A-module and A is a local ring.

$$0 \to \mathrm{H}^n(U_{\mathrm{A}}, \mathrm{A}) \otimes_{\mathrm{A}} \mathbb{C}_1 \xrightarrow{i} \mathrm{H}^n(U(\mathfrak{g}), \mathbb{C}) \to \mathrm{Tor}_1^{\mathrm{A}}(\mathrm{H}^{n+1}(U_{\mathrm{A}}, \mathrm{A}), \mathbb{C}_1) \to 0$$

Example (Type A_2)

- $\mathrm{H}^\bullet(\mathfrak{sl}_3, \mathbb{C})$ is generated in degrees $3, 5$, nonzero in degrees $0, 3, 5, 8$.
- $\mathrm{H}^n(\mathfrak{sl}_3, \mathbb{C}) = 0$ for $n = 4, 6$, so i is surjective for $n = 3, 5$.
- Then $i : \mathrm{H}^\bullet(U_{\mathrm{A}}, \mathrm{A}) \otimes_{\mathrm{A}} \mathbb{C}_1 \to \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$ is an isomorphism and $\mathrm{H}^\bullet(U_{\mathrm{A}}, \mathrm{A})$ is a free A-module.

This argument also applies for types $A_1, B_2, E_7, E_8, F_4, G_2$.

$$0 \to \mathrm{H}^n(U_{\mathrm{A}}, \mathrm{A}) \otimes_{\mathrm{A}} \mathbb{C}_1 \xrightarrow{i} \mathrm{H}^n(U(\mathfrak{g}), \mathbb{C}) \to \mathrm{Tor}_1^{\mathrm{A}}(\mathrm{H}^{n+1}(U_{\mathrm{A}}, \mathrm{A}), \mathbb{C}_1) \to 0$$

Example (Type A_3)

$\mathrm{H}^\bullet(\mathfrak{sl}_4, \mathbb{C})$ is generated in degrees $3, 5, 7$, nonzero in degrees

$$0, 3, 5, 7, 8, 10, 12, 15.$$

Commutative square induced by the inclusion of Dynkin diagrams:

$$\begin{array}{ccc} \mathrm{H}^8(U_{\mathrm{A}}(\mathfrak{sl}_4), \mathrm{A}) \otimes_{\mathrm{A}} \mathbb{C}_1 & \xrightarrow{\sim} & \mathrm{H}^8(U(\mathfrak{sl}_4), \mathbb{C}) \\ \downarrow \mathrm{res} \otimes_{\mathrm{A}} 1 & & \downarrow \mathrm{res} \\ \mathrm{H}^8(U_{\mathrm{A}}(\mathfrak{sl}_3), \mathrm{A}) \otimes_{\mathrm{A}} \mathbb{C}_1 & \xrightarrow{\sim} & \mathrm{H}^8(U(\mathfrak{sl}_3), \mathbb{C}). \end{array}$$

So $\mathrm{res} : \mathrm{H}^8(U_{\mathrm{A}}(\mathfrak{sl}_4), \mathrm{A}) \to \mathrm{H}^8(U_{\mathrm{A}}(\mathfrak{sl}_3), \mathrm{A})$ is onto by Nakayama's Lemma, and hence $\mathrm{H}^8(U_{\mathrm{A}}(\mathfrak{sl}_4), \mathrm{A})$ is A-free of the same rank by dim. comparison.

To handle the higher rank cases and type E_6, must understand the Lie algebra cohomology restriction map $\mathrm{res} : \mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{g}', \mathbb{C})$ when the inclusion $\mathfrak{g}' \subset \mathfrak{g}$ corresponds to an inclusion of Dynkin diagrams.

There exists a natural map $\rho : S^\bullet(\mathfrak{g}^*)^\mathfrak{g} \to \Lambda^{\bullet-1}(\mathfrak{g}^*)^\mathfrak{g}$ with image the space of primitive elements in $\Lambda^\bullet(\mathfrak{g}^*)^\mathfrak{g}$ and kernel $(S^\bullet(\mathfrak{g}^*)^\mathfrak{g})^2$.

Use the isomorphism $S^\bullet(\mathfrak{g}^*)^\mathfrak{g} \cong S^\bullet(\mathfrak{h}^*)^W$ to turn the cohomological restriction map into a problem about the restriction of polynomial invariants for the Weyl group. (Polynomial restriction maps from E_7 to E_6 and E_6 to D_5 computed by Toda and Watanabe in mid 1970s.)

Theorem

$H^\bullet(U_q, k)$ *is an exterior algebra and* $\dim H^\bullet(U_q, k) = \dim H^\bullet(U(\mathfrak{g}), \mathbb{C})$.

- $\dim H^3(U_q(\mathfrak{g}), k) = 1$ (Alternate proof à la the Killing form?)
- For $\varepsilon \in \mathbb{C}^\times$ not a small root of unity, $\dim H^\bullet(U_\varepsilon, \mathbb{C}) \geq \dim H^\bullet(U_q, k)$, with equality for almost all $\varepsilon \in \mathbb{C}^\times$. In particular, equality holds for all $\varepsilon \in \mathbb{C}$ that are transcendental over $\mathbb{Q}$.
- Conjecture: Equality holds if ε is not a root of unity, or if ε is a root of unity of sufficiently large order (say, $\varepsilon^\ell = 1$ with $\ell > h$).

Theorem

Let $\varepsilon \in \mathbb{C}$ *be a primitive p-th root of unity with p prime and* $p > 3(h-1)$*. Then* $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$ *is an exterior algebra generated in odd degrees.*

Theorem

Let $\varepsilon \in \mathbb{C}$ be a primitive p-th root of unity with p prime and $p > 3(h-1)$. Then $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$ is an exterior algebra generated in odd degrees.

Proof sketch.

For $p > 3(h-1)$, the mod-p Lie algebra cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}_{\mathbb{F}_p}, \mathbb{F}_p)$ is an exterior algebra (Friedlander and Parshall, 1986).

Let $\mathcal{Z}$ be the localization of $\mathbb{Z}[\varepsilon]$ at the kernel of the map $\mathbb{Z}[\varepsilon] \to \mathbb{F}_p$ sending $\varepsilon \mapsto 1$. Then $\mathcal{Z}$ is a local ring with quotient field $\mathbb{Q}(\varepsilon)$ and residue field $\mathbb{F}_p$. Now use $U_{\mathcal{Z}}$ to compare cohomology for $U(\mathfrak{g}_{\mathbb{F}_p})$ and U_ε. □