Some quantum analogues of results from Lie algebra cohomology

Christopher M. Drupieski
University of Georgia

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Motivating Problem

Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra.

- Compute the Lie algebra cohomology ring $H^\bullet(\mathfrak{g}, \mathbb{C}) = \operatorname{Ext}^\bullet_{U(\mathfrak{g})}(\mathbb{C}, \mathbb{C})$.
- Given f.d. irreducible $\mathfrak{g}$-modules V and W, compute $\operatorname{Ext}^\bullet_{U(\mathfrak{g})}(V, W)$.

- Lie group analogue solved using topological methods by 1940.
- Purely algebraic proofs appear by 1950, make critical use of the Koszul complex $\Lambda^\bullet(\mathfrak{g}^*)$ for Lie algebra cohomology.

$\Lambda^n(\mathfrak{g}^*) \cong$ space of n-multilinear alternating maps on $\mathfrak{g}$

Theorem (Chevalley–Eilenberg 1946, Koszul 1950)

Let $V \not\cong W$ be finite-dimensional irreducible $\mathfrak{g}$-modules.

- $\operatorname{Ext}^\bullet_{U(\mathfrak{g})}(V, W) = 0$.
- $\operatorname{Ext}^\bullet_{U(\mathfrak{g})}(\mathbb{C}, \mathbb{C}) = \mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C}) \cong \Lambda^\bullet(\mathfrak{g}^*)^{\mathfrak{g}}$ is an exterior algebra.

Example (Type A_n, $\mathfrak{g} = \mathfrak{sl}_{n+1}$)

$\Lambda^\bullet(\mathfrak{g}^*)^{\mathfrak{g}} \cong \operatorname{Hom}_{\mathfrak{g}}(\Lambda^\bullet(\mathfrak{g}), \mathbb{C})$ is generated in degrees $3, 5, 7, \ldots, 2n+1$.
The degree r generator is represented by the function

$$\Phi_r(X_1, \ldots, X_r) = \sum_{\sigma \in S_r} (-1)^{\operatorname{sgn}(\sigma)} \operatorname{tr}(X_{\sigma(1)} \circ \cdots \circ X_{\sigma(r)}).$$

Degrees of generators for $H^\bullet(\mathfrak{g}, \mathbb{C})$:

Type	Degrees
A_r	$3, 5, 7, \dots, 2r+1$
B_r	$3, 7, 11, \dots, 4r-1$
C_r	$3, 7, 11, \dots, 4r-1$
D_r $(r \geq 4)$	$3, 7, 11, \dots, 4r-5, 2r-1$
E_6	$3, 9, 11, 15, 17, 23$
E_7	$3, 11, 15, 19, 23, 27, 35$
E_8	$3, 15, 23, 27, 35, 39, 47, 59$
F_4	$3, 11, 15, 23$
G_2	$3, 11$

Problem

Consider the quantized enveloping algebra $U_q = U_q(\mathfrak{g})$ associated to $\mathfrak{g}$.

- If $V \ncong W$ are irreducible U_q-modules, is $\operatorname{Ext}^\bullet_{U_q}(V, W) = 0$?
- Is $\operatorname{Ext}^\bullet_{U_q}(\mathbb{C}(q), \mathbb{C}(q)) = \mathrm{H}^\bullet(U_q, \mathbb{C}(q))$ an exterior algebra?

Complete reducibility of finite-dimensional representations still holds, but we have no quantum version of the Koszul complex $\Lambda^\bullet(\mathfrak{g}^*)$, so no hope of ripping off the classical arguments.

Theorem

Let $V \not\cong W$ be f.d. irreducible U_q-modules. Then $\operatorname{Ext}^\bullet_{U_q}(V, W) = 0$.

Strategy (borrowed from Kumar's work on Kac–Moody Lie algebras):

- Turn Ext-vanishing condition into a Tor-vanishing condition.
- Reduce to vanishing of Tor between certain Verma modules.

- Involution $\omega : U_q \to U_q$ that interchanges positive and negative roots and acts like the antipode S on U_q^0. Then for V, W f.d. irreducible,

$$\mathrm{Ext}^\bullet_{U_q}(V, W) \cong \mathrm{Ext}^\bullet_{U_q}(V, {}^\omega(W^*)) \cong \mathrm{Tor}^{U_q}_\bullet(({}^\omega V)^{S^{-1}}, W)^*$$

- Since $V \not\cong W$, they're in different blocks of the BGG category $\mathcal{O}^q$. Now show: If M and N are modules in different blocks of $\mathcal{O}^q$, then

$$\mathrm{Tor}^{U_q}_\bullet(({}^\omega M)^{S^{-1}}, N) = 0.$$

- Use long exact sequences for Tor to reduce to the case when M and N are Verma modules in different blocks of $\mathcal{O}^q$.

Let $M(\lambda)$, $M(\mu)$ be Verma modules in different blocks of $\mathcal{O}^q$. So $\lambda \neq \mu$.

As a $U_q(\mathfrak{b}^+)$-module, ${}^\omega M(\lambda) \cong U_q(\mathfrak{b}^+) \otimes_{U_q^0} k_{-\lambda}$. Then

$$\begin{aligned}
\operatorname{Tor}_\bullet^{U_q}(({}^\omega M(\lambda))^{S^{-1}}, M(\mu)) &= \operatorname{Tor}_\bullet^{U_q}(({}^\omega M(\lambda))^{S^{-1}}, U_q \otimes_{U_q(\mathfrak{b}^+)} k_\mu) \\
&\cong \operatorname{Tor}_\bullet^{U_q(\mathfrak{b}^+)}(({}^\omega M(\lambda))^{S^{-1}}, k_\mu) \\
&\cong \operatorname{Tor}_\bullet^{U_q(\mathfrak{b}^+)}(k, {}^\omega M(\lambda) \otimes k_\mu) \\
&\cong \operatorname{Tor}_\bullet^{U_q(\mathfrak{b}^+)}(k, U_q(\mathfrak{b}^+) \otimes_{U_q^0} k_{\mu-\lambda}) \\
&\cong \operatorname{Tor}_\bullet^{U_q^0}(k, k_{\mu-\lambda})
\end{aligned}$$

Exercise: $\operatorname{Tor}_\bullet^{U_q^0}(k, k_{\mu-\lambda})^* = \operatorname{Ext}^\bullet_{U_q^0}(k, k_{\lambda-\mu}) = 0$. (Use $U_q^0 \cong k\mathbb{Z}^n$.)

Theorem

The cohomology ring $H^\bullet(U_q(\mathfrak{g}), \mathbb{C}(q))$ is an exterior algebra over a graded subspace concentrated in the same odd degrees as for $U(\mathfrak{g})$.

Notation

- A: the localization of $\mathbb{C}[q, q^{-1}]$ at the maximal ideal $(q-1)$
- U_A: the A-subalgebra of U_q generated by simple root vectors
- U_A is an integral form for U_q: $U_A \otimes_A \mathbb{C}(q) = U_q$
- $U_1 := U_A \otimes_A \mathbb{C}_1 \cong U_A/(q-1)U_A$, where $\mathbb{C}_1 = A/(q-1)A$

- Fact: U_1 is a central extension of $U(\mathfrak{g})$ by $\mathbb{C}(\mathbb{Z}/2\mathbb{Z})^r$.
- Consequence: $H^\bullet(U_1, \mathbb{C}) \cong H^\bullet(U(\mathfrak{g}), \mathbb{C})$.

- Split exact sequence from the universal coefficient theorem:

$$0 \to \mathrm{H}^n(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \xrightarrow{i} \mathrm{H}^n(U(\mathfrak{g}), \mathbb{C}) \to \mathrm{Tor}_1^\mathsf{A}(\mathrm{H}^{n+1}(U_\mathsf{A}, \mathsf{A}), \mathbb{C}_1) \to 0.$$

- $\mathrm{H}^\bullet(U_q, \mathbb{C}(q)) \cong \mathrm{H}^\bullet(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}(q)$, so

$$\dim \mathrm{H}^\bullet(U_q, \mathbb{C}(q)) \leq \dim \mathrm{H}^\bullet(U_q, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \leq \dim \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C}).$$

- Key step in the argument is to show that the latter two dimensions are equal, that is, that i is an isomorphism. Since A is a local PID, this is equivalent to showing that $\mathrm{H}^\bullet(U_\mathsf{A}, \mathsf{A})$ is A-free. From this it quickly follows that $\mathrm{H}^\bullet(U_\mathsf{A}, \mathsf{A})$ and $\mathrm{H}^\bullet(U_q, \mathbb{C}(q))$ are exterior algebras.

$$0 \to \mathrm{H}^n(U_{\mathsf{A}}, \mathsf{A}) \otimes_{\mathsf{A}} \mathbb{C}_1 \xrightarrow{i} \mathrm{H}^n(U(\mathfrak{g}), \mathbb{C}) \to \mathrm{Tor}_1^{\mathsf{A}}(\mathrm{H}^{n+1}(U_{\mathsf{A}}, \mathsf{A}), \mathbb{C}_1) \to 0$$

Example (Type A_2)

$\mathrm{H}^\bullet(\mathfrak{sl}_3, \mathbb{C})$ is generated in degrees $3, 5$, nonzero in degrees

$$0, 3, 5, 8.$$

$\mathrm{H}^n(U(\mathfrak{g}), \mathbb{C}) = 0$ for $n = 4, 6$, so i is surjective for $n = 3, 5$.
Then i is surjective for all n, so we get that $\mathrm{H}^\bullet(U_{\mathsf{A}}, \mathsf{A})$ is A-free.

This argument also applies for types $A_1, B_2, E_7, E_8, F_4, G_2$.

$$0 \to \mathrm{H}^n(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \xrightarrow{i} \mathrm{H}^n(U(\mathfrak{g}), \mathbb{C}) \to \mathrm{Tor}_1^\mathsf{A}(\mathrm{H}^{n+1}(U_\mathsf{A}, \mathsf{A}), \mathbb{C}_1) \to 0$$

Example (Type A_3)

$\mathrm{H}^\bullet(\mathfrak{sl}_4, \mathbb{C})$ is generated in degrees $3, 5, 7$, nonzero in degrees

$$0, 3, 5, 7, 8, 10, 12, 15.$$

Commutative square induced by the inclusion of Dynkin diagrams:

$$\begin{array}{ccc} \mathrm{H}^8(U_\mathsf{A}(\mathfrak{sl}_4), \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 & \xrightarrow{\sim} & \mathrm{H}^8(U(\mathfrak{sl}_4), \mathbb{C}) \\ \downarrow{\scriptstyle \mathrm{res} \otimes_\mathsf{A} 1} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{H}^8(U_\mathsf{A}(\mathfrak{sl}_3), \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 & \xrightarrow{\sim} & \mathrm{H}^8(U(\mathfrak{sl}_3), \mathbb{C}). \end{array}$$

So $\mathrm{res} : \mathrm{H}^8(U_\mathsf{A}(\mathfrak{sl}_4), \mathsf{A}) \to \mathrm{H}^8(U_\mathsf{A}(\mathfrak{sl}_3), \mathsf{A})$ is onto by Nakayama's Lemma, and hence $\mathrm{H}^8(U_\mathsf{A}(\mathfrak{sl}_4), \mathsf{A})$ is A-free of the same rank by dim. comparison.

Theorem

Let $U_q = U_q(\mathfrak{g})$ be the quantized enveloping algebra associated to $\mathfrak{g}$.

- If $V \not\cong W$ are irreducible U_q-modules, then $\operatorname{Ext}^\bullet_{U_q}(V, W) = 0$.
- $\operatorname{Ext}^\bullet_{U_q}(\mathbb{C}(q), \mathbb{C}(q)) = \mathrm{H}^\bullet(U_q, \mathbb{C}(q))$ is an exterior algebra, generated in the same odd degrees as $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C})$.

Further directions:

- Specialize parameter q to $\varepsilon \in \mathbb{C}$ and compute $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$?
  Have $\dim \mathrm{H}^\bullet(U_\varepsilon, \mathbb{C}) = \dim \mathrm{H}^\bullet(U_q, k)$ for almost all $\varepsilon \in \mathbb{C}^\times$.
  Can show true for some large roots of 1. For what $\varepsilon \in \mathbb{C}$ does it fail?
- Consider parabolic and Levi subalgebras $U_q(\mathfrak{p}_J)$ and $U_q(\mathfrak{l}_J)$ of $U_q(\mathfrak{g})$.
  Problem: $\mathfrak{l}_J = [\mathfrak{l}_J, \mathfrak{l}_J] \oplus \mathfrak{z}_J$, but no similar decomposition of $U_q(\mathfrak{l}_J)$.
  Also: $\mathrm{H}^\bullet(\mathfrak{l}_J, \mathbb{C})$ is generated by elements in adjacent degrees.