Cohomology rings of infinitesimal unipotent algebraic and quantum groups

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Standard notation:

- k an algebraically closed field of characteristic p,
- G a simple, simply–connected algebraic group (e.g., $G = SL_n(\overline{\mathbb{F}}_p)$)
- $T \subset G$ a maximal torus,
- Φ the root system of T in G,
- h the Coxeter number of Φ,
- $B \subset G$ a Borel subgroup corresponding to Φ^+,
- $U \subset B$ the unipotent radical of B,
- $U_1 \subset B_1$ the first Frobenius kernels of U and B.
- $\mathfrak{n} = \text{Lie}(U)$, the nilradical of $\mathfrak{b} = \text{Lie}(B)$.

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Equivalently: What is the ring structure of $\mathrm{H}^\bullet(u(\mathfrak{n}), k)$?
$u(\mathfrak{n})$ = restricted enveloping algebra of $\mathfrak{n}$.

Theorem (Friedlander–Parshall, 1986)

Suppose $p > h$. Then there exists a filtration on $\mathrm{H}^\bullet(U_1, k)$ such that

$$\operatorname{gr} \mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{n}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{n}, k).$$

$\mathrm{H}^\bullet(\mathfrak{n}, k) =$ ordinary Lie algebra cohomology.
$S^\bullet(\mathfrak{n}^*)^{(1)} =$ polynomial ring generated in degree two.
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In $\mathrm{H}^\bullet(U_1, k)$:
$(a \otimes b)(c \otimes d) = (ac \otimes bd)$ + terms with higher degree polynomial part.

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1. F-P Spectral sequence:

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2. Explicit computation of $\mathrm{H}^\bullet(\mathfrak{n}, k)$ as a B-module: If $p > h$, then

$$\mathrm{H}^\bullet(\mathfrak{n}, k) = \bigoplus_{w \in W} w \cdot 0. \qquad \text{(Kostant's Theorem)}$$

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Can we "ungrade" the ring isomorphism, i.e., is the vector space isomorphism $\mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{n}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{n}, k)$ *also a ring isomorphism?*

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Theorem (Crane, UVA Ph.D. thesis, 1983)

Suppose $G = SL_n$ *and* $p > h = n + 1$. *Then, as a ring,*

$$\mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{n}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{n}, k).$$

Theorem (DNN)

Suppose $p > 2(h-1)$. Then, as a ring,

$$\mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{n}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{n}, k).$$

Proof.

Look at the weight of $x_1 x_2$, for $x_i \in \mathrm{H}^\bullet(U_1, k)_{w_i \cdot 0} \cong \mathrm{H}^\bullet(\mathfrak{n}, k)_{w_i \cdot 0}$.

Proof.

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- Either $x_1x_2 = 0$ in $\mathrm{H}^\bullet(U_1, k)$, or x_1x_2 has T-weight

$$w_1 \cdot 0 + w_2 \cdot 0 = w_3 \cdot 0 + p\sigma$$

  for some $w_3 \in W$ and $\sigma \in \mathbb{N}\Phi^-$.

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- Suppose $\sigma \neq 0$. Then for some $y, y', w_1', w_2' \in W$,

  $$y(w_1' \cdot 0) + y'(w_2' \cdot 0) = p\widetilde{\sigma} \in X(T)_+ \cap \mathbb{Z}\Phi. \qquad (1)$$

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- Then $2p \leq p(\widetilde{\sigma}, \alpha_0^\vee) \leq 4(\rho, \alpha_0^\vee) = 4(h-1)$.
- So $p > 2(h-1)$ implies $\sigma = 0$. □

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Example (Type B_2, $p = 5$)

Let α, β be simple with α long. Note that $h = 4 < p < 6 = 2(h-1)$.

$$s_\beta s_\alpha \cdot 0 + s_\beta s_\alpha \cdot 0 = s_\alpha s_\beta \cdot 0 + 5(-\beta)$$

Corresponds to squaring an element in $\mathrm{H}^2(\mathfrak{n}, \mathbb{F}_5)$ of weight $s_\beta s_\alpha \cdot 0$.

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Though all elements of $\mathrm{H}^\bullet(\mathfrak{n}, \mathbb{F}_5)$ square to zero, we have verified using MAGMA that this vector does NOT square to zero in $\mathrm{H}^\bullet(U_1, \mathbb{F}_5)$. So the ring isomorphism need not hold for $h < p < 2(h-1)$.

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Can we also generalize to quantum groups another calculation of F-P:

Theorem (Friedlander–Parshall, 1986)

Suppose $\lambda \in C_{\mathbb{Z}}$. Then, as a graded T-module and as a $\mathrm{H}^\bullet(B_1, k)$-module,

$$\mathrm{H}^\bullet(U_1, L(\lambda)) \cong S^\bullet(\mathfrak{n}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}, L(\lambda)).$$

Let q be an indeterminate. Set $\mathfrak{g} = \mathrm{Lie}(G)$.

Definition

The quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is a $\mathbb{C}(q)$-algebra defined by generators and relations similar to those defining the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Let $\zeta \in \mathbb{C}$ be a primitive ℓ-th root of unity, $\mathsf{A} = \mathbb{Z}[q, q^{-1}]$.

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- $u_\zeta(\mathfrak{n}) \cong \mathcal{U}_\zeta(\mathfrak{n})//Z$, where Z is generated by $\left\{E_\alpha^\ell : \alpha \in \Phi^+\right\}$.

1. No analogue of the F-P spectral sequence for quantum groups. We do have the LHS spectral sequence:

$$E_2^{i,j}(L^\zeta(\lambda)) = \mathrm{H}^i(u_\zeta(\mathfrak{n}), \mathrm{H}^j(Z, L^\zeta(\lambda))) \Rightarrow \mathrm{H}^{i+j}(\mathcal{U}_\zeta(\mathfrak{n}), L^\zeta(\lambda))$$

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2. Have a computation for the target:

Theorem (UGA VIGRE Algebra Group, 2008)

Suppose $\ell > h$ and $\lambda \in C_{\mathbb{Z}}$. Then $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{n}), L^\zeta(\lambda)) \cong \bigoplus_{w \in W} w \cdot \lambda$.

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3. $\mathcal{U}_\zeta(\mathfrak{n})$ and $u_\zeta(\mathfrak{n})$ are not Hopf algebras, so we don't automatically have nice product structures on the LHS spectral sequence.
4. The Borel subalgebras $u_\zeta(\mathfrak{b}) \cong u_\zeta^0 \otimes u_\zeta(\mathfrak{n})$ and $\mathcal{U}_\zeta(\mathfrak{b}) \cong u_\zeta^0 \otimes \mathcal{U}_\zeta(\mathfrak{n})$ **are** Hopf algebras, as is $Z \subset \mathcal{U}_\zeta(\mathfrak{b})$.

Work one u_ζ^0-weight space at a time.

$$\bigoplus_{\mu \in X} \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{n}), L^\zeta(\lambda))_{w\cdot\lambda+\ell\mu} \cong \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{b}), L^\zeta(\lambda) \otimes -w\cdot\lambda)$$

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LHS Spectral sequence for the Borel subalgebras:

$$E_2^{i,j} = \mathrm{H}^i(u_\zeta(\mathfrak{b}), \mathrm{H}^j(Z, L^\zeta(\lambda) \otimes -w \cdot \lambda) \Rightarrow \mathrm{H}^{i+j}(\mathcal{U}_\zeta(\mathfrak{b}), L^\zeta(\lambda) \otimes -w \cdot \lambda).$$

This LHS spectral sequence **is** compatible with cup products.

Theorem (DNN)

Suppose ℓ is odd, coprime to 3 if Φ has type G_2, and $\ell > 2(h-1)$. Then there exists a ring isomorphism

$$\begin{aligned} \mathrm{H}^\bullet(u_\zeta(\mathfrak{n}), \mathbb{C}) &\cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{n}), \mathbb{C}) \\ &\cong S^\bullet(\mathfrak{n}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{n}), \mathbb{C}). \end{aligned}$$

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Theorem (DNN)

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$$\begin{aligned} \mathrm{H}^\bullet(u_\zeta(\mathfrak{n}), L^\zeta(\lambda)) &\cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{n}), L^\zeta(\lambda)) \\ &\cong S^\bullet(\mathfrak{n}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{n}), L^\zeta(\lambda)) \end{aligned}$$

as a weight module and as a module for $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$.