Some graded analogues of one-parameter subgroups and applications to the cohomology of $GL_{m|n(r)}$

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Work over a field k of characteristic $p > 0$. (Assume $k = \overline{k}$ and $p \geq 3$.)

$\mathbb{G}_a$ – additive group scheme $\qquad \mathbb{G}_a(A) = (A, +)$

$\mathbb{G}_{a(r)}$ – r-th Frobenius kernel of $\mathbb{G}_a$ $\qquad \mathbb{G}_{a(r)}(A) = \{t \in A : t^{p^r} = 0\}$

Infinitesimal one-parameter subgroups

Given an affine group scheme G, an infinitesimal one-parameter subgroup of height $\leq r$ in G is a group scheme homomorphism

$$\nu : \mathbb{G}_{a(r)} \to G.$$

The set of all such homomorphisms is denoted $V_r(G)$.

$$V_r(G) = \mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, G)$$

The set $V_r(G)$ admits the structure of an affine variety.

Let A be an augmented k-algebra. Suppose $\mathrm{H}^\bullet(A,k) = \mathrm{Ext}_A^\bullet(k,k)$ is "commutative" and finitely generated.

Cohomological spectrum and support varieties

The **cohomological spectrum** of A is the affine algebraic variety

$$|A| = \mathrm{MaxSpec}\left(\mathrm{H}^\bullet(A,k)\right).$$

Given an A-module M, let $I_A(M)$ be the kernel of the map

$$\mathrm{H}^\bullet(A,k) = \mathrm{Ext}_A^\bullet(k,k) \xrightarrow{-\otimes M} \mathrm{Ext}_A^\bullet(M,M).$$

The **cohomological support variety** associated to M is

$$|A|_M = \mathrm{MaxSpec}\left(\mathrm{H}^\bullet(A,k)/I_A(M)\right),$$

a closed subvariety of the cohomological spectrum.

In reasonable settings, support varieties detect projectivity.

Suslin–Friedlander–Bendel (1997)

Let G be an infinitesimal group scheme of height $\leq r$, and let kG be its group ring ($kG = k[G]^{\#}$). Then there exists a homeomorphism

$$|kG| \cong V_r(G) = \mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, G)$$

For $G = GL_{n(r)}$, the r-th Frobenius kernel of GL_n, one has

$$V_r(GL_{n(r)}) = \left\{(\alpha_0, \ldots, \alpha_{r-1}) \in \mathfrak{gl}_n^{\times r} : \alpha_i^p = 0, [\alpha_i, \alpha_j] = 0, \forall i, j\right\}.$$

If G is an arbitrary infinitesimal group scheme of height $\leq r$, then there exists a closed embedding $G \hookrightarrow GL_{n(r)}$ for some n.

If $\nu : \mathbb{G}_{a(r)} \to G$ is a one-parameter subgroup, and if M is a rational G-module, then M pulls back to a rational $\mathbb{G}_{a(r)}$-module, $\nu^*(M)$.

Equivalently, $\nu^*(M)$ is a module over the group algebra

$$k\mathbb{G}_{a(r)} = k[\mathbb{G}_{a(r)}]^{\#} = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p).$$

Suslin–Friedlander–Bendel (1997)

Let G be infinitesimal of height $\leq r$. If M is a finite-dimensional rational G-module, then

$$|kG|_M \cong \left\{\nu \in V_r(G) : \nu^*(M) \text{ is not free over } k[u_{r-1}]/(u_{r-1}^p)\right\}.$$

So, for example, the projectivity of M can be detected by restrictions along various ν to algebras of the form $k[u]/(u^p)$.

Our motivating question

(How) can this be generalized to supergroups?

Wikipedia definition of a supergroup

A **supergroup** is a music group whose members are already successful as solo artists or as part of other groups or well known in other musical professions.

What do we mean by “super”?

On object is “super” if it is appropriately graded by $\mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$.

- Super vector spaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$
- $V \otimes W \cong W \otimes V$ via the **supertwist** $v \otimes w \mapsto (-1)^{\overline{v} \cdot \overline{w}} w \otimes v$

Define (Hopf) superalgebras and ‘super’ (co)commutativity in terms of the “usual diagrams,” but use the supertwist when objects pass.

Examples of Hopf superalgebras

– Ordinary Hopf algebras (as purely even superalgebras)

– $\mathbb{Z}$-graded Hopf algebras in the sense of Milnor and Moore

– Enveloping superalgebras of (restricted) Lie superalgebras

Exterior algebra of a vector space V

Reduce the natural $\mathbb{Z}$-grading on $\Lambda(V)$ modulo 2 to view $\Lambda(V)$ as a superspace. Then $\Lambda(V)$ is a commutative superalgebra:

$$ab = (-1)^{\overline{a}\cdot\overline{b}} ba$$

It is also a cocommutative Hopf superalgebra:

$$\begin{aligned}
\Delta(v^2) &= \Delta(v)\Delta(v) \\
&= (v \otimes 1 + 1 \otimes v)(v \otimes 1 + 1 \otimes v) \\
&= (v^2 \otimes 1) + (v \otimes 1)(1 \otimes v) + (1 \otimes v)(v \otimes 1) + (1 \otimes v^2) \\
&= (v^2 \otimes 1) + (v \otimes v) - (v \otimes v) + (1 \otimes v^2) \\
&= 0
\end{aligned}$$

The super sign convention is precisely what is needed to ensure that the coproduct preserves the defining algebra relation.

Defining correspondences

$$\text{affine supergroup schemes} \xleftrightarrow{\text{anti}} \text{commutative Hopf superalgebras}$$

$$G \longleftrightarrow k[G]$$

$$\text{finite supergroup schemes} \longleftrightarrow \text{f.d. cocommut. Hopf superalgebras}$$

$$G \longleftrightarrow kG = k[G]^{\#}$$

Cautionary example

Let $A = k[u,v]/\langle u^p, v^2\rangle$ with $\overline{u} = \overline{0}$ and $\overline{v} = \overline{1}$.

Then A is a Hopf superalgebra with u and v both primitive. (In fact, A is the restricted enveloping algebra of a res. Lie superalgebra.)

Define M to be the A-supermodule with homogeneous basis

$$\{x_0, \dots, x_{p-1}, y_0, \dots, y_{p-1}\}, \quad x_i \text{ even}, \quad y_i \text{ odd},$$

such that $u.x_i = x_{i+1}$, $u.y_i = y_{i+1}$, $v.x_i = y_{i+1}$, and $v.y_i = x_{i+p-1}$.

Claim: M is projective over all proper cyclic subalgebras of A, but is not projective over A itself. *So in contrast to the classical theory, need more than just cyclic subalgebras to detect projectivity.*

What supergroups play the role of $\mathbb{G}_{a(r)}$ in the super theory? These will be our graded analogues of one-parameter subgroups.

Let $f = T^{p^t} + \sum_{i=1}^{t-1} a_i T^{p^i} \in k[T]$ be a p-polynomial (no linear term).

Let $\eta \in k$ be a scalar.

The infinitesimal multiparameter supergroup $\mathbb{M}_{r;f,\eta}$

$$k\mathbb{M}_{r;f,\eta} = k[u_0, \ldots, u_{r-1}, v]/\langle u_0^p, \ldots, u_{r-2}^p, u_{r-1}^p + v^2, f(u_{r-1}) + \eta u_0 \rangle$$

$u_0, \ldots, u_{r-1}$ are even; their coproducts look like they do in $k\mathbb{G}_{a(r)}$

u_{r-1}^p is primitive

v is an odd primitive generator

hence $u_{r-1}^p + v^2$ and $f(u_{r-1}) + \eta u_0$ are primitive

Our result: use these supergroups to generalize the SFB calculation

$$|GL_n(r)| \cong V_r(GL_{n(r)}) = \mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, GL_{n(r)}).$$

Let $m, n \in \mathbb{Z}_{\geq 0}$, and let $A = A_{\overline{0}} \oplus A_{\overline{1}}$ be a commutative superalgebra.

$\mathrm{Mat}_{m|n}(A)_{\overline{0}}$ consists of all block matrices

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

with $T_1 \in M_{m\times m}(A_{\overline{0}})$, $T_2 \in M_{m\times n}(A_{\overline{1}})$, $T_3 \in M_{n\times m}(A_{\overline{1}})$, $T_4 \in M_{n\times n}(A_{\overline{0}})$, while in $\mathrm{Mat}_{m|n}(A)_{\overline{1}}$ the parities of the entries are reversed.

General linear supergroup $GL_{m|n}$ and its Frobenius kernels

For each commutative superalgebra A, $GL_{m|n}(A)$ is the group of all invertible matrices in $\mathrm{Mat}_{m|n}(A)_{\overline{0}}$.

$GL_{m|n(r)}$ is the scheme-theoretic kernel of the map that raises each individual matrix entry to the p^r-th power.

Ambient superscheme (analogue of the commuting variety for GL_n)

$$V_r(GL_{m|n})(A) = \Big\{(\alpha_0, \ldots, \alpha_{r-1}, \beta) \in (\mathrm{Mat}_{m|n}(A)_{\bar{0}})^{\times r} \times \mathrm{Mat}_{m|n}(A)_{\bar{1}} : \\ [\alpha_i, \alpha_j] = [\alpha_i, \beta] = 0 \text{ for all } 0 \leq i, j \leq r-1, \\ \alpha_i^p = 0 \text{ for all } 0 \leq i \leq r-2, \text{ and } \alpha_{r-1}^p + \beta^2 = 0\Big\}.$$

Lemma

Homomorphisms $\rho : \mathbb{M}_{r;f,\eta} \otimes_k A \to GL_{m|n} \otimes_k A$ correspond to points in the closed sub-superscheme

$$V_{r;f,\eta}(GL_{m|n})(A) = \\ \left\{(\alpha_0, \ldots, \alpha_{r-1}, \beta) \in V_r(GL_{m|n})(A) : f(\alpha_{r-1}) + \eta\alpha_0 = 0\right\}.$$

Note that $V_r(GL_{m|n})(k) = \bigcup_{f,\eta} V_{r;f,\eta}(GL_{m|n})(k)$.

Following SFB to relate $V_r(GL_{m|n})$ to $|GL_{m|n(r)}|$

1. Explicitly calculated the images of certain “universal extension classes” under the maps in cohomology

$$\rho^*_{(\underline{\alpha}|\beta)} : \mathrm{H}^\bullet(GL_{m|n(r)}, k) \to \mathrm{H}^\bullet(\mathbb{M}_{r;f,\eta}, k)$$

corresponding to homomorphisms $\rho_{(\underline{\alpha}|\beta)} : \mathbb{M}_{r;f,\eta} \to GL_{m|n(r)}$.

The cohomology rings $\mathrm{H}^\bullet(\mathbb{M}_{r;f,\eta}, k)$ have nice descriptions

– For $f = T^p$ and $\eta = 0$, one has

$$\mathrm{H}^\bullet(\mathbb{M}_{r;T^p,0}, k) \cong k[x_1, \ldots, x_r, y] \otimes \Lambda(\lambda_1, \ldots, \lambda_r),$$

– If $f = T^{p^s}$ with $s \geq 2$, then

$$\mathrm{H}^\bullet(\mathbb{M}_{r;T^{p^s},0}, k) \cong k[x_1, \ldots, x_r, y, w_s]/\langle x_r - y^2 \rangle \otimes \Lambda(\lambda_1, \ldots, \lambda_r).$$

Following SFB to relate $V_r(GL_{m|n})$ to $|GL_{m|n(r)}|$

2. Then able to construct algebra homomorphisms

$$k[V_r(GL_{m|n})] \xrightarrow{\overline{\phi}} H(GL_{m|n(r)}, k) \xrightarrow{\psi_{r;f,\eta}} k[V_{r;f,\eta}(GL_{m|n})]$$

   such that $H(G,k) := \mathrm{H}^{even}(G,k)_{\overline{0}} \oplus \mathrm{H}^{odd}(G,k)_{\overline{1}}$ is finite over $\overline{\phi}$.

3. Get induced morphisms of varieties

$$\Theta_{r;f,\eta} : V_{r;f,\eta}(GL_{m|n})(k) \xrightarrow{\Psi_{r;f,\eta}} \left|GL_{m|n(r)}\right| \xrightarrow{\Phi} V_r(GL_{m|n})(k)$$

   Showed that $\Theta_{r;f,\eta}$ is the Frobenius morphism composed with the natural inclusion. The $V_{r;f,\eta}(GL_{m|n})(k)$ cover $V_r(GL_{m|n})(k)$, so...

Corollary

There is a finite surjective morphism of varieties

$$\left|GL_{m|n(r)}\right| \to V_r(GL_{m|n})(k).$$

Open Problems

Problem

What is the correct coordinate-free approach?

— SFB looked at $\mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, G)$

— $\mathrm{Hom}_{Grp}(\mathbb{M}_{r;T^{p^s},0}, \mathbb{M}_{r;T^{p^s},0})$ already seems too big

Problem

Can you detect projectivity of modules, or nilpotence of cohomology classes, by looking at restrictions to multiparameter supergroups...