Support varieties for infinitesimal supergroups

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Work over an algebraically closed field k of characteristic $p \geq 3$.

Let A be (something like) a Hopf algebra over k.

Suppose $\mathrm{H}^\bullet(A, k) = \mathrm{Ext}_A^\bullet(k, k)$ is finitely generated as a k-algebra.

Cohomological spectrum and support varieties

The **cohomological spectrum** of A is the affine algebraic variety

$$|A| = \mathrm{MaxSpec}\left(\mathrm{H}^\bullet(A, k)\right).$$

Given an A-module M, let $I_A(M)$ be the kernel of the map

$$\mathrm{H}^\bullet(A, k) = \mathrm{Ext}_A^\bullet(k, k) \xrightarrow{-\otimes M} \mathrm{Ext}_A^\bullet(M, M).$$

The **cohomological support variety** associated to M is

$$|A|_M = \mathrm{MaxSpec}\left(\mathrm{H}^\bullet(A, k)/I_A(M)\right),$$

a closed subvariety of the cohomological spectrum.

Language of group schemes vs. Hopf algebras

- finite group scheme $G \leftrightarrow$ f.d. cocommutative Hopf algebra kG
- infinitesimal group scheme $G \leftrightarrow$ f.d. cocommutative Hopf algebra kG such that the dual algebra $(kG)^* = k[G]$ is local

It is an open question whether $\mathrm{H}^\bullet(A, k)$ is finitely-generated for all finite-dimensional Hopf algebras, but finite generation has been verified in a number of cases, including:

- group algebras of finite groups (Golod, Venkov, Evens 1961)
- **finite group schemes** (Friedlander–Suslin 1997)
- **finite supergroup schemes** (Drupieski 2016)

In these contexts, support varieties will have sensible properties.

Suslin–Friedlander–Bendel (1997)

Let G be an infinitesimal k-group scheme of height $\leq r$. Then there exists a homeomorphism

$$|kG| \simeq V_r(G) := \mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, G).$$

Example

For $GL_{n(r)}$, the r-th Frobenius kernel of GL_n, one has

$$V_r(GL_{n(r)}) \cong \left\{(\alpha_0, \dots, \alpha_{r-1}) \in \mathfrak{gl}_n^{\times r} : \alpha_i^p = 0, [\alpha_i, \alpha_j] = 0, \forall i, j\right\},$$

the variety of r-tuples of commuting p-nilpotent matrices.

If $\nu : \mathbb{G}_{a(r)} \to G$ is a one-parameter subgroup, and if M is a rational G-module, then M pulls back to a rational $\mathbb{G}_{a(r)}$-module, $\nu^*(M)$.

Equivalently, $\nu^*(M)$ is a module over the group algebra

$$k\mathbb{G}_{a(r)} = \left(k[T]/(T^{p^r})\right)^{\#} = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p).$$

Suslin–Friedlander–Bendel (1997)

Let G be infinitesimal of height $\leq r$. If M is a finite-dimensional rational G-module, then $|kG| \simeq V_r(G)$ restricts to

$$\begin{aligned} |kG|_M &\simeq V_r(G)_M \\ &= \left\{\nu \in V_r(G) : \nu^*(M) \text{ is not free over } k[u_{r-1}]/(u_{r-1}^p)\right\}. \end{aligned}$$

Since $|kG|_M = \{0\}$ iff M is projective, this means projectivity can be detected by restrictions to algebras of the form $k[u]/(u^p)$.

(How) can this be generalized to infinitesimal supergroup schemes?

The Hopf superalgebra $\mathbb{P}_r$

$\mathbb{P}_r = k[u_0, \ldots, u_{r-1}, v]/\langle u_0^p, \ldots, u_{r-2}^p, u_{r-1}^p + v^2 \rangle$, where

- $u_0, \ldots, u_{r-1}$ are of even superdegree, v is of odd superdegree,
- coproducts for $u_0, \ldots, u_{r-1}$ look like they do in $k\mathbb{G}_{a(r)}$,
- u_0, u_{r-1}^p, and v are primitive

Lemma

Let G be a finite k-supergroup scheme. Then

$$V_r(G) := \mathrm{Hom}_{Hopf}(\mathbb{P}_r, kG),$$

admits the structure of an affine algebraic variety over k.

$$\mathbb{P}_r = k[u_0, \ldots, u_{r-1}, v]/\langle u_0^p, \ldots, u_{r-2}^p, u_{r-1}^p + v^2 \rangle$$

$$V_r(G) = \mathrm{Hom}_{Hopf}(\mathbb{P}_r, kG)$$

Remarks

1. If G is purely even, then $V_r(G) \cong \mathrm{Hom}_{Grp}(\mathbb{G}_{a(r)}, G)$ as in SFB.
2. Suppose G is infinitesimal of height one, so that $kG = V(\mathfrak{g})$, where $\mathfrak{g} = \mathrm{Lie}(G)$. Then

$$V_1(G) \cong \left\{ (x, y) \in \mathfrak{g}_{\overline{0}} \times \mathfrak{g}_{\overline{1}} : [x, y] = 0 \text{ and } x^{[p]} + \tfrac{1}{2}[y, y] = 0 \right\}$$

For the r-th Frobenius kernel of $GL_{m|n}$ there is a natural identification

$$V_r(GL_{m|n(r)}) \cong \Big\{(\alpha_0, \dots, \alpha_{r-1}, \beta) \in \mathfrak{gl}(m|n)_{\bar{0}}^{\times r} \times \mathfrak{gl}(m|n)_{\bar{1}} : \\ [\alpha_i, \alpha_j] = [\alpha_i, \beta] = 0 \text{ for all } 0 \leq i, j \leq r-1, \\ \alpha_i^p = 0 \text{ for all } 0 \leq i \leq r-2, \text{ and } \alpha_{r-1}^p + \beta^2 = 0\Big\}.$$

Theorem (Drupieski–Kujawa 2017, reinterpreted)

There are morphisms of varieties

$$V_r(GL_{m|n(r)}) \xrightarrow{\Psi} \left|GL_{m|n(r)}\right| \xrightarrow{\Theta} V_r(GL_{m|n(r)})$$

with Θ finite such that $\Theta \circ \Psi$ is the r-th Frobenius twist map.

$\mathbb{P}_1 = k[u, v]/\langle u^p + v^2 \rangle$

$\mathbb{P}_r = k[u_0, \ldots, u_{r-1}, v]/\langle u_0^p, \ldots, u_{r-2}^p, u_{r-1}^p + v^2 \rangle$

Superalgebra map $\iota : \mathbb{P}_1 \hookrightarrow \mathbb{P}_r$ defined by $\iota(u) = u_{r-1}$ and $\iota(v) = v$.

The support set $V_r(G)_M$

Let G be a finite k-supergroup scheme and M a finite-dimensional kG-supermodule. Set

$$V_r(G)_M = \left\{ \nu \in V_r(G) : \operatorname{projdim}_{\mathbb{P}_1}(\iota^* \nu^* M) = \infty \right\}.$$

Proposition

$V_r(G)_M$ is a Zariski closed conical subvariety of $V_r(G)$.

Note: If V is a fin. dim. $k[u]/(u^p)$-module, then $\operatorname{projdim}(V) \in \{0, \infty\}$.
Follows that if G is purely even, our $V_r(G)$ identifies with that of SFB.

Theorem (Drupieski–Kujawa)

Let G be an **infinitesimal unipotent** k-supergroup scheme of height $\leq r$. Then there is a natural k-algebra map $\psi : H(G, k) \to k[V_r(G)]$, which induces a homeomorphism[1] of varieties

$$V_r(G) \simeq |G| \,.$$

This restricts for each finite-dimensional kG-supermodule M to a homeomorphism

$$V_r(G)_M \simeq |G|_M \,.$$

Conjecture

The above results hold without the assumption that G is unipotent.

[1]Specifically, an inseparable isogeny.

Let $f = T^{p^t} + \sum_{i=1}^{t-1} a_i T^{p^i} \in k[T]$ be a p-polynomial (no linear term).

Let $\eta \in k$ be a scalar.

The infinitesimal multiparameter supergroup $\mathbb{M}_{r;f,\eta}$

$$k\mathbb{M}_{r;f,\eta} = \mathbb{P}_r / \langle f(u_{r-1}) + \eta u_0 \rangle$$

Proposition

Every finite-dimensional Hopf quotient of $\mathbb{P}_r$ is of the form

- $k\mathbb{G}_{a(s)}$ for some $0 \leq s \leq r$,
- $k\mathbb{G}_a^- = k[v]/\langle v^2 \rangle$, or
- $k\mathbb{M}_{s;f,\eta}$ for some $1 \leq s \leq r$ and some f, η as above.

Benson–Iyengar–Krause–Pevtsova

For **unipotent** finite supergroup schemes, projectivity of modules and nilpotence in cohomology are detected (after field extension) by restriction to **'elementary'** subsupergroup schemes.

The *infinitesimal* elementary k-supergroup schemes are

- $\mathbb{G}_{a(r)}$ for $r \geq 0$,
- $\mathbb{G}_{a(r)} \times \mathbb{G}_a^-$ for $r \geq 0$,
- $\mathbb{M}_{r;T^{p^s},0}$ for $r, s \geq 1$,
- $\mathbb{M}_{r,T^{p^s},\eta}$ for $r \geq 2$, $s \geq 1$, and $0 \neq \eta \in k$.

The group algebras of these each occur as Hopf quotients of $\mathbb{P}_r$.

Roughly: $\mathbb{M}_{r;f,\eta}$ is unipotent if the polynomial f is a single monomial.

Question

For arbitrary infinitesimal supergroups, is projectivity of modules and nilpotence in cohomology detected (after field extension) by restriction to finite-dimensional Hopf superalgebra quotients of $\mathbb{P}_r$?