Second cohomology for finite groups of Lie type

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August 4, 2012

References:

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Ground rules:

- k - algebraically closed field of characteristic $p > 0$
- G - simple, simply-connected algebraic group scheme over k
- T - maximal torus of G
- B - Borel subgroup of G containing T
- U - unipotent radical of B
- $F : G \to G$ - standard Frobenius morphism on G
- $G(\mathbb{F}_q) = G^{F^r}$ - finite subgroup of $\mathbb{F}_q$-rational points in G, $q = p^r$
- $G_r = \ker(F^r)$ - scheme-theoretic r-th Frobenius kernel of G

Example: The Special Linear Group

- $G = SL_n(k)$
- T - diagonal matrices in G
- B - lower triangular matrices in G
- U - lower triangular unipotent matrices in G
- $F : (a_{ij}) \mapsto (a_{ij}^p)$
- $G(\mathbb{F}_q) = SL_n(\mathbb{F}_q)$
- For each commutative k-algebra A,

$$(SL_n)_r(A) = \left\{ (a_{ij}) \in SL_n(A) : (a_{ij}^{p^r}) = \text{ the identity matrix} \right\}.$$

$(SL_n)_r(A)$ is a nontrivial group if and only if A contains nilpotents.

The Goal

Find $\mathrm{H}^1(G(\mathbb{F}_q), V)$ and $\mathrm{H}^2(G(\mathbb{F}_q), V)$ for V an irreducible $G(\mathbb{F}_q)$-module.

Subgoals (i.e., what people have actually managed to do):

- Compute for V in various classes of irreducible $G(\mathbb{F}_q)$-modules
- Determine sufficient conditions for the cohomology groups to vanish
- Compute under restrictions on p and q (specific small values, or $\gg 0$)

Refined Goal

Relate $\mathrm{H}^1(G(\mathbb{F}_q), V)$ and $\mathrm{H}^2(G(\mathbb{F}_q), V)$ to rational cohomology for G.

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Why this is reasonable and desirable:

- The irreducible $kG(\mathbb{F}_q)$-modules all lift to rational G-modules.
- More machinery available for dealing with rational G-cohomology.
- Rational G-modules carry more information: Every rational G-module decomposes into simultaneous eigenspaces (weight spaces) for T.

Example: Adjoint representation of $SL_3(\mathbb{F}_4)$ on $\mathfrak{sl}_3$

Adjoint representation $\mathfrak{sl}_3$ - traceless 3×3 matrices with coefficients in k.

Basis of eigenvectors for the conjugation action of T:

$$\{E_{ij}, E_{ii} - E_{i+1,i+1} : 1 \leq i, j \leq n, i \neq j\}$$

If $n = 3$, then $T(\mathbb{F}_4)$ can't distinguish the eigenvalues of E_{12} and E_{23}. In fact, all root spaces look the same to $T(\mathbb{F}_4)$ up to twisting by $\mathrm{Gal}(\mathbb{F}_4)$.

Important and popular facts:

$$
\begin{aligned}
\mathrm{H}^i(G(\mathbb{F}_q),V) &\hookrightarrow \mathrm{H}^i(B(\mathbb{F}_q),V) = \mathrm{H}^i(U(\mathbb{F}_q),V)^{T(\mathbb{F}_q)} \\
\mathrm{H}^i(G,V) &\cong \mathrm{H}^i(B,V) = \mathrm{H}^i(U,V)^T \\
& \quad\ \mathrm{H}^i(B_r,V) = \mathrm{H}^i(U_r,V)^{T_r}
\end{aligned}
$$

Cline, Parshall, Scott (1975, 1977), Jones (1975)

Computed, for all p and q, the dimension of $\mathrm{H}^1(G(\mathbb{F}_q), L(\lambda))$ for λ a nonzero minimal dominant weight, i.e., a minuscule weight or a maximal short root.

- $L(\lambda)$ is the head of the Weyl module $V(\lambda)$.
- Lower bound: $\dim \operatorname{rad}_G V(\lambda) \leq \dim \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda))$
- Upper bound in terms of spaces of cocycles for root subgroups:

$$\sum_{\alpha \in \Delta} \dim Z^1(U_\alpha(\mathbb{F}_q), L(\lambda))^{T(\mathbb{F}_q)} - (\dim L(\lambda)^{T(\mathbb{F}_q)} - \dim L(\lambda)^{B(\mathbb{F}_q)})$$

For λ a nonzero minimal dominant weight, $\dim \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) \leq 1$, except for type D_{2n} with $p = 2$, where the dimension is sometimes 2.

Avrunin (1978)

Suppose for all weights μ of $T(\mathbb{F}_q)$ in V and for all $\alpha, \beta \in \Phi$ that $\alpha \not\equiv \mu$ and $(\alpha, \beta) \not\equiv \mu \mod \operatorname{Gal}(\mathbb{F}_q)$. Then $\mathrm{H}^2(G(\mathbb{F}_q), V) = 0$.

Proof

Look at a central series for $U(\mathbb{F}_q)$ where the factors are products of root subgroups to analyze the weights of $T(\mathbb{F}_q)$ in $\mathrm{H}^2(U(\mathbb{F}_q), V)$. Use this to deduce that $\mathrm{H}^2(U(\mathbb{F}_q), V)^{T(\mathbb{F}_q)} = 0$, and hence $\mathrm{H}^2(G(\mathbb{F}_q), V) = 0$. □

Corollary (Avrunin)

Suppose $q > 4$. Let $\lambda \in X(T)_+$ be a nonzero minimal dominant weight. Then $\mathrm{H}^2(G(\mathbb{F}_q), L(\lambda)) = 0$, except maybe type A_2, $q = 5$, $\lambda \in \{\omega_1, \omega_2\}$.

Cline, Parshall, Scott, van der Kallen (1977)

Let V be a finite-dimensional rational G-module, and let $i \in \mathbb{N}$. Then for all sufficiently large e and q, the restriction map is an isomorphism

$$\mathrm{H}^i(G, V^{(e)}) \xrightarrow{\sim} \mathrm{H}^i(G(\mathbb{F}_q), V^{(e)}).$$

$$\begin{array}{ccc} \mathrm{H}^i(G,V) & \xrightarrow{\sim} & \mathrm{H}^i(B,V) \\ \downarrow & & \downarrow \\ \mathrm{H}^i(G(\mathbb{F}_q),V) & \hookrightarrow & \mathrm{H}^i(B(\mathbb{F}_q),V). \end{array}$$

So for H^1 and H^2, we can get answers for $G(\mathbb{F}_q)$ in terms of G-cohomology if we take q large, and if we sometimes also replace V by $V^{(1)}$ or $V^{(2)}$.

Consider $\operatorname{ind}_{G(\mathbb{F}_q)}^G(-)$. There exists a short exact sequence

$$0 \to k \to \operatorname{ind}_{G(\mathbb{F}_q)}^G(k) \to N \to 0.$$

Let M be a rational G-module. Obtain the new short exact sequence

$$0 \to M \to \operatorname{ind}_{G(\mathbb{F}_q)}^G(M) \to M \otimes N \to 0.$$

Now using $\operatorname{Ext}_G^n(k, \operatorname{ind}_{G(\mathbb{F}_q)}^G(M)) \cong \operatorname{Ext}_{G(\mathbb{F}_q)}^n(k, M)$, we get:

Long exact sequence for restriction

$$\begin{array}{llllll} 0 & \to & \operatorname{Hom}_G(k,M) & \stackrel{\text{res}}{\to} & \operatorname{Hom}_{G(\mathbb{F}_q)}(k,M) & \to & \operatorname{Hom}_G(k, M\otimes N) \\ & \to & \operatorname{Ext}_G^1(k,M) & \stackrel{\text{res}}{\to} & \operatorname{Ext}_{G(\mathbb{F}_q)}^1(k,M) & \to & \operatorname{Ext}_G^1(k, M\otimes N) \\ & \to & \operatorname{Ext}_G^2(k,M) & \stackrel{\text{res}}{\to} & \operatorname{Ext}_{G(\mathbb{F}_q)}^2(k,M) & \to & \operatorname{Ext}_G^2(k, M\otimes N) \\ & \to & \dots \end{array}$$

Bendel, Nakano, Pillen (2010)

$\text{ind}_{G(\mathbb{F}_q)}^{G}(k)$ admits a filtration by G-submodules with sections of the form

$$H^0(\mu) \otimes H^0(\mu^*)^{(r)} \quad \mu \in X(T)_+.$$

Corollary: $N = \text{coker}(k \to \text{ind}_{G(\mathbb{F}_q)}^{G}(k))$ admits such a filtration with $\mu \neq 0$.

Then $\text{Ext}_G^i(k, L(\lambda) \otimes N) = 0$ if it is zero for each section, i.e., if for $\mu \neq 0$,

$$\text{Ext}_G^i(V(\mu)^{(r)}, L(\lambda) \otimes H^0(\mu)) = 0.$$

30,000 ft (9,144 m) view of our strategy

$$\mathrm{H}^i(G(\mathbb{F}_q), L(\lambda))$$

$\Updownarrow$ Induction

$$\mathrm{H}^i(G, \mathrm{ind}_{G(\mathbb{F}_q)}^G L(\lambda))$$

$\Uparrow$ Filtrations

$$\mathrm{Ext}_G^i(V(\mu)^{(r)}, L(\lambda) \otimes H^0(\mu))$$

$\Uparrow$ Spectral Sequences

$$\mathrm{Ext}_{G/G_r}^i(V(\mu)^{(r)}, \mathrm{Ext}_{G_r}^j(k, L(\lambda) \otimes H^0(\mu)))$$

$\Uparrow$ Spectral Sequences

$$R^i \mathrm{ind}_{B/B_r}^{G/G_r} \mathrm{Ext}_{B_r}^j(k, L(\lambda) \otimes \mu)$$

$\Uparrow$ Weight combinatorics

$$\mathrm{Ext}_{U_r}^j(k, L(\lambda))$$

Isomorphism theorem for first cohomology

Let $\lambda \in X_r(T)$. Suppose $\operatorname{Ext}^1_{U_r}(k, L(\lambda))$ is semisimple as a B/U_r-module, and that $\operatorname{Ext}^1_{U_r}(k, L(\lambda))^{T(\mathbb{F}_q)} = \operatorname{Ext}^1_{U_r}(k, L(\lambda))^T$. Then

$$\mathrm{H}^1(G, L(\lambda)) \cong \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)).$$

Isomorphism theorem for first cohomology

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$$\mathrm{H}^1(G, L(\lambda)) \cong \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)).$$

Isomorphism theorem for second cohomology

Let $\lambda \in X_r(T)$. Suppose $\operatorname{Ext}^1_{U_r}(k, L(\lambda))$ is semisimple as a B/U_r-module, that $\operatorname{Ext}^i_{U_r}(k, L(\lambda))^{T(\mathbb{F}_q)} = \operatorname{Ext}^i_{U_r}(k, L(\lambda))^T$ for $i \in \{1, 2\}$, and that

$$p^r > \max\left\{-(\nu, \gamma^\vee) : \gamma \in \Delta,\ \operatorname{Ext}^1_{U_r}(k, L(\lambda))_\nu \neq 0\right\}.$$

Then $\mathrm{H}^2(G, L(\lambda)) \cong \mathrm{H}^2(G(\mathbb{F}_q), L(\lambda))$.

Theorem 3.2.4. *Suppose $\lambda \in X(T)_+$ is a dominant root or is less than or equal to a fundamental weight. Assume that $p > 5$ if Φ is of type E_8 or G_2, and $p > 3$ otherwise. Then as a B/U_r-module, $\operatorname{Ext}^1_{U_r}(L(\lambda), k) = \operatorname{soc}_{B/U_r} \operatorname{Ext}^1_{U_r}(L(\lambda), k)$, that is,*

$$\operatorname{Ext}^1_{U_r}\left(L(\lambda), k\right) \cong \bigoplus_{\alpha \in \Delta} -s_\alpha \cdot \lambda \oplus \bigoplus_{\substack{\alpha \in \Delta \\ 0<n<r}} -\left(\lambda - p^n\alpha\right) \oplus \bigoplus_{\substack{\sigma \in X(T)_+ \\ \sigma < \lambda}} (-\sigma)^{\oplus m_\sigma}$$

where $m_\sigma = \dim \operatorname{Ext}^1_G(L(\lambda), H^0(\sigma))$.

- Determine the socle using Andersen's results on $\operatorname{Ext}^1_B(L(\lambda), \mu)$.
- Get an injection $\operatorname{Ext}^1_{U_r}(L(\lambda), k) \hookrightarrow Q$ into the injective hull of the socle. Then show that $\operatorname{soc}_{B/U_r} \operatorname{Ext}^1_{U_r}(L(\lambda), k) = \operatorname{Ext}^1_{U_r}(L(\lambda), k)$ by showing that no weight from the second socle layer of Q can be a weight of $\operatorname{Ext}^1_{U_r}(L(\lambda), k)$.

First Cohomology Main Theorem

Let $\lambda \in X(T)_+$ be a fundamental dominant weight. Assume $q > 3$ and

$$\begin{array}{ll} p > 2 & \text{if } \Phi \text{ has type } A_n,\ D_n; \\ p > 3 & \text{if } \Phi \text{ has type } B_n,\ C_n,\ E_6,\ E_7,\ F_4,\ G_2; \\ p > 5 & \text{if } \Phi \text{ has type } E_8. \end{array}$$

Then $\dim \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) = \dim \mathrm{H}^1(G, L(\lambda)) \leq 1$.

The spaces are nonzero (and one-dimensional) in the following cases:

- Φ has type E_7, $p = 7$, and $\lambda = \omega_6$; and
- Φ has type C_n, $n \geq 3$, and $\lambda = \omega_j$ with $\frac{j}{2}$ a nonzero term in the p-adic expansion of $n+1$, but not the last term in the expansion.

Second Cohomology Main Theorem A

Suppose $p > 3$ and $q > 5$. Let $\lambda \in X(T)_+$ be less than or equal to a fundamental dominant weight. Assume also that λ is not a dominant root. Then $\mathrm{H}^2(G, L(\lambda)) \cong \mathrm{H}^2(G(\mathbb{F}_q), L(\lambda))$.

Corollary

Suppose p, q, λ are as above. Then $\mathrm{H}^2(G(\mathbb{F}_q), L(\lambda)) = 0$ except possibly in a small number of explicit cases in exceptional types, and except possibly in type C_n when $\lambda = \omega_j$ with j even and $p \leq n$.

Second Cohomology Main Theorem B

Let $p > 3$ and $q > 5$. Let $\lambda = \widetilde{\alpha}$ be the highest root. Assume $p \nmid n+1$ in type A_n, and $p \nmid n-1$ in type B_n. Then $L(\lambda) = H^0(\lambda) = \mathfrak{g}$, and

$$\mathrm{H}^2(G(\mathbb{F}_q), \mathfrak{g}) = k.$$

Also have $\mathrm{H}^2(SL_3(\mathbb{F}_5), L(\omega_1)) = \mathrm{H}^2(SL_3(\mathbb{F}_5), L(\omega_2)) = k$.

Different strategy in these cases for analyzing the long exact sequence:

$$\begin{array}{llllll}
\to & \mathrm{Ext}^1_G(k, L(\lambda)) & \stackrel{\mathrm{res}}{\to} & \mathrm{Ext}^1_{G(\mathbb{F}_q)}(k, L(\lambda)) & \to & \mathrm{Ext}^1_G(k, L(\lambda) \otimes N) \\
\to & \mathrm{Ext}^2_G(k, L(\lambda)) & \stackrel{\mathrm{res}}{\to} & \mathrm{Ext}^2_{G(\mathbb{F}_q)}(k, L(\lambda)) & \to & \mathrm{Ext}^2_G(k, L(\lambda) \otimes N) \\
\to & \mathrm{Ext}^3_G(k, L(\lambda)) & \to & \cdots & &
\end{array}$$

Our original commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^1(G, L(\lambda)) & \xrightarrow{\sim} & \mathrm{H}^1(B, L(\lambda)) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) & \hookrightarrow & \mathrm{H}^1(B(\mathbb{F}_q), L(\lambda)). \end{array}$$

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New diagram:

$$\begin{array}{ccccc} \mathrm{H}^1(G, L(\lambda)) & & \xrightarrow{\sim} & & \mathrm{H}^1(B, L(\lambda)) \\ \downarrow & & & & \downarrow \\ \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) & \hookrightarrow & \mathrm{H}^1(U(\mathbb{F}_q), L(\lambda))^{T(\mathbb{F}_q)} & & \mathrm{H}^1(U_1, L(\lambda))^{T(\mathbb{F}_q)}. \end{array}$$

Lemma

Suppose $p > 2$ and $\lambda \in X_1(T)$. Then $\mathrm{H}^1(B, L(\lambda)) \hookrightarrow \mathrm{H}^1(U_1, L(\lambda))^{T(\mathbb{F}_q)}$.

Proof

LHS spectral sequence for B/B_1 combined with $\operatorname{Ext}^1_B(k, L(\lambda))$. □

$$\begin{array}{ccccc}
\mathrm{H}^1(G, L(\lambda)) & & \xrightarrow{\sim} & & \mathrm{H}^1(B, L(\lambda)) \\
\downarrow & & & & \downarrow \\
\mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) & \hookrightarrow & \mathrm{H}^1(U(\mathbb{F}_q), L(\lambda))^{T(\mathbb{F}_q)} & & \mathrm{H}^1(U_1, L(\lambda))^{T(\mathbb{F}_q)}.
\end{array}$$

Recall:

- $U(\mathbb{F}_q)$ is filtered by its lower central series.
- $kU(\mathbb{F}_q)$ is filtered by the powers of its augmentation ideal.

Theorem (Lazard)

$\operatorname{gr} U(\mathbb{F}_q)$ is naturally a p-restricted Lie algebra over $\mathbb{F}_p$.

Theorem (Quillen)

There exists a natural isomorphism $\operatorname{gr} kU(\mathbb{F}_q) \cong u(\operatorname{gr} U(\mathbb{F}_q) \otimes_{\mathbb{F}_p} k)$.

Lin, Nakano (1999), Friedlander (2010)

There exists a natural isomorphism $\mathrm{gr}\, kU(\mathbb{F}_q) \cong u(\mathfrak{u}^{\oplus r})$.

If M is a rational B-module, then there exists a (weight) filtration on M such that $\mathrm{gr}\, M$ is naturally a $u(\mathfrak{u}^{\oplus r})$-module. The restriction of $\mathrm{gr}\, M$ to the first (or any) factor $\mathfrak{u} \subset \mathfrak{u}^{\oplus r}$ identifies with $M|_{\mathfrak{u}}$ (equivalently, with $M|_{U_1}$).

Consequence: There exists a May spectral sequence

$$E_1^{i,j} = \mathrm{H}^{i+j}(u(\mathfrak{u}^{\oplus r}), \mathrm{gr}\, M)_{(i)} \Rightarrow \mathrm{H}^{i+j}(U(\mathbb{F}_q), M).$$

Upshot: There exist vector space maps

$$\mathrm{H}^1(U(\mathbb{F}_q), M) \longrightarrow \mathrm{H}^1(u(\mathfrak{u}^{\oplus r}), \mathrm{gr}\, M)^{T(\mathbb{F}_q)} \xrightarrow{\text{res}} \mathrm{H}^1(U_1, M)^{T(\mathbb{F}_q)}.$$

Apply results of Parshall and Scott on filtered algebras, and spectral sequence and weight arguments, to conclude that the new diagram commutes and that the bottom row consists of injections:

$$\begin{array}{ccccc} \mathrm{H}^1(G, L(\lambda)) & & \xrightarrow{\sim} & & \mathrm{H}^1(U, L(\lambda))^T \\ \downarrow & & & & \downarrow \\ \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) & \hookrightarrow & \mathrm{H}^1(U(\mathbb{F}_q), L(\lambda))^{T(\mathbb{F}_q)} & \hookrightarrow & \mathrm{H}^1(U_1, L(\lambda))^{T(\mathbb{F}_q)} \end{array}$$

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\mathrm{H}^1(G, L(\lambda)) & & \xrightarrow{\quad\sim\quad} & & \mathrm{H}^1(U, L(\lambda))^T \\
\downarrow & & & & \big\downarrow \\
\mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) & \hookrightarrow & \mathrm{H}^1(U(\mathbb{F}_q), L(\lambda))^{T(\mathbb{F}_q)} & \hookrightarrow & \mathrm{H}^1(U_1, L(\lambda))^{T(\mathbb{F}_q)}
\end{array}$$

Theorem

Suppose $p > 2$, $q > 3$, and $\lambda \in X_1(T)$. Then

$$\dim \mathrm{H}^1(U_1, L(\lambda))^{T(\mathbb{F}_q)} = \dim \mathrm{H}^1(U_1, L(\lambda))^T = \dim \mathrm{H}^1(G, L(\lambda)).$$

Hence, $\mathrm{H}^1(G, L(\lambda)) \cong \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda))$.

Open Question about cohomology for Sp_{2n}

For $p \leq n$, what is $\mathrm{H}^2(G, L(\omega_j))$, and hence $\mathrm{H}^2(G(\mathbb{F}_q), L(\omega_j))$, for j even?

Open Question about cohomology for Sp_{2n}

For $p \leq n$, what is $\mathrm{H}^2(G, L(\omega_j))$, and hence $\mathrm{H}^2(G(\mathbb{F}_q), L(\omega_j))$, for j even?

Values of n and j for which $\mathrm{H}^2(Sp_{2n}, L(\omega_j))$ is 1-dimensional, $p = 3$.

n	j
6	6
7	6
8	
9	6
10	6
11	
12	6
13	6
14	

n	j
15	6, 8
16	6, 10
17	
18	6, 14
19	6, 16
20	18
21	6, 18
22	6, 18
23	18

n	j
24	6, 8, 18
25	6, 10, 18
26	
27	6, 14
28	6, 16
29	18
30	6, 18
31	6, 18
32	18

n	j
33	6, 8, 18
34	6, 10, 18
35	
36	6, 14
37	6, 16
38	18
39	6, 18, 20
40	6, 18, 22

Values of n and j for which $\mathrm{H}^2(Sp_{2n}, L(\omega_j))$ is 1-dimensional, $p = 5$.

n	j
10	10
11	10
12	10
13	10
14	
15	10
16	10
17	10
18	10
19	

n	j
20	10
21	10
22	10
23	10
24	
25	10
26	10
27	10
28	10
29	

n	j
30	10
31	10
32	10
33	10
34	
35	10, 12
36	10, 14
37	10, 16
38	10, 18
39	

n	j
40	10, 22
41	10, 24
42	10, 26
43	10, 28
44	
45	10, 32
46	10, 34
47	10, 36
48	10, 38
49	

n	j
50	10, 42
51	10, 44
52	10, 46
53	10, 48
54	50

- Are these cohomology groups always at most one-dimensional?
- Can the non-vanishing be described p-adically in terms of n and j?