Support varieties for irreducible modules of small quantum groups

Christopher M. Drupieski

Department of Mathematics
University of Georgia

April 21, 2012

Joint work with Daniel Nakano (UGA) and Brian Parshall (UVA).

Appearing in Adv. Math. **229** (2012) 2656–2668

- $\mathfrak{g}$ finite-dimensional simple complex Lie algebra
- Φ root system of $\mathfrak{g}$, with highest short root α_0
- $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ the Weyl weight
- $h = (\rho, \alpha_0^\vee) + 1$ the Coxeter number of Φ
- W the Weyl group of Φ

- $\ell \in \mathbb{N}$ odd integer with $\ell > h$ and $3 \nmid \ell$ if Φ is of type G_2
- $\zeta \in \mathbb{C}$ primitive ℓ-th root of unity
- $u_\zeta(\mathfrak{g})$ small quantum group associated to $\mathfrak{g}$, a finite-dimensional Hopf subalgebra of the Lusztig quantum group $U_\zeta(\mathfrak{g})$ with parameter ζ.

- $W_\ell = W \ltimes \ell\mathbb{Z}\Phi$ affine Weyl group
- $\mathcal{N}$ nullcone of $\mathfrak{g}$, consisting of the nilpotent elements in $\mathfrak{g}$

Let A be a Hopf algebra over an algebraically closed field k.
Suppose $R = \mathrm{H}^{2\bullet}(A, k)$ is finitely-generated as an algebra over k.

Cohomological spectrum

$V_A(k) = \mathit{MaxSpec}\,\mathrm{H}^{2\bullet}(A, k)$ (maximal ideal spectrum).

Let M be a finite-dimensional A-module. Set $I_A(M) = \mathrm{Ann}_R \operatorname{Ext}_A^\bullet(M, M)$.

Support variety of a module

$V_A(M) = \mathit{MaxSpec}(\mathrm{H}^{2\bullet}(A, \mathbb{C})/I_A(M))$, closed subvariety of $V_A(k)$

The cases $A = kG$, the group ring of a finite group G, and $A = u(\mathfrak{g})$, the restricted enveloping algebra of a p-restricted Lie algebra $\mathfrak{g}$, have been of interest since at least the early 1980s.

Ginzburg–Kumar (1993)

$\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$, hence $V_{u_\zeta(\mathfrak{g})}(\mathbb{C}) \cong \mathcal{N}$.

General problem that few explicit examples of support varieties of known.

Ginzburg–Kumar (1993)

$\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$, hence $V_{u_\zeta(\mathfrak{g})}(\mathbb{C}) \cong \mathcal{N}$.

General problem that few explicit examples of support varieties of known.

For $\lambda \in X^+$, have $H^0(\lambda)$ and $V(\lambda)$ (induced and Weyl modules for $U_\zeta(\mathfrak{g})$).
Set $\Phi_\lambda = \{\alpha \in \Phi : (\lambda + \rho, \alpha^\vee) \equiv 0 \bmod \ell\}$.
There exists $w \in W$ and a subset of simple roots J such that $w(\Phi_\lambda) = \Phi_J$.
Let $\mathfrak{u}_J$ be the nilradical of the standard parabolic subalgebra $\mathfrak{p}_J \subset \mathfrak{g}$.

Ginzburg–Kumar (1993)

$\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$, hence $V_{u_\zeta(\mathfrak{g})}(\mathbb{C}) \cong \mathcal{N}$.

General problem that few explicit examples of support varieties of known.

For $\lambda \in X^+$, have $H^0(\lambda)$ and $V(\lambda)$ (induced and Weyl modules for $U_\zeta(\mathfrak{g})$).
Set $\Phi_\lambda = \{\alpha \in \Phi : (\lambda + \rho, \alpha^\vee) \equiv 0 \bmod \ell\}$.
There exists $w \in W$ and a subset of simple roots J such that $w(\Phi_\lambda) = \Phi_J$.
Let $\mathfrak{u}_J$ be the nilradical of the standard parabolic subalgebra $\mathfrak{p}_J \subset \mathfrak{g}$.

Ostrik (1998), Bendel–Nakano–Parshall–Pillen (2011)

$V_{u_\zeta(\mathfrak{g})}(H^0(\lambda)) = V_{u_\zeta(\mathfrak{g})}(V(\lambda)) = G \cdot \mathfrak{u}_J$, irreducible of dimension $|\Phi| - |\Phi_J|$

Question

What is the support variety of each irreducible $u_\zeta(\mathfrak{g})$-module $L(\lambda)$?

No previous calculation of the support varieties for all irreducible modules of a finite-dimensional Hopf algebra (except in cases where all $V_A(L)$ equal the full cohomological spectrum, i.e., the variety of the trivial module).

Question

What is the support variety of each irreducible $u_\zeta(\mathfrak{g})$-module $L(\lambda)$?

No previous calculation of the support varieties for all irreducible modules of a finite-dimensional Hopf algebra (except in cases where all $V_A(L)$ equal the full cohomological spectrum, i.e., the variety of the trivial module).

$L(\lambda) = \mathrm{soc}_{U_\zeta(\mathfrak{g})} H^0(\lambda)$, follows via induction that $V_{u_\zeta(\mathfrak{g})}(L(\lambda)) \subseteq G \cdot \mathfrak{u}_J$.

Theorem (D–Nakano–Parshall)

Suppose $w(\Phi_\lambda) = \Phi_J$ for some $w \in W$. Then $V_{u_\zeta(\mathfrak{g})}(L(\lambda)) = G \cdot \mathfrak{u}_J$.

Let M be a finite-dimensional $U_\zeta(\mathfrak{g})$-module, with $M = \bigoplus_{\lambda \in X} M_\lambda$.

Generic dimension of a weight module

$\dim_t M = \sum_{\lambda \in X} (\dim M_\lambda) t^{-2\,\mathrm{wht}(\lambda)} \in \mathbb{Z}[t, t^{-1}]$

Here $\mathrm{wht}(\lambda) = \frac{1}{2} \sum_{\alpha \in \Phi^+} d_\alpha (\lambda, \alpha^\vee) \in \mathbb{Z}[\frac{1}{2}]$, where $d_\alpha = (\alpha, \alpha)/(\alpha_0, \alpha_0)$.

Let M be a finite-dimensional $U_\zeta(\mathfrak{g})$-module, with $M = \bigoplus_{\lambda \in X} M_\lambda$.

Generic dimension of a weight module

$\dim_t M = \sum_{\lambda \in X} (\dim M_\lambda) t^{-2\,\mathrm{wht}(\lambda)} \in \mathbb{Z}[t, t^{-1}]$

Here $\mathrm{wht}(\lambda) = \frac{1}{2} \sum_{\alpha \in \Phi^+} d_\alpha (\lambda, \alpha^\vee) \in \mathbb{Z}[\frac{1}{2}]$, where $d_\alpha = (\alpha, \alpha)/(\alpha_0, \alpha_0)$.

Nakano–Parshall–Vella (2002)

Suppose ζ is a root of multiplicity s in $\dim_t M$. Then

$$\dim V_{u_\zeta(\mathfrak{g})}(M) \geq |\Phi| - 2s.$$

Outline of the argument for the induced modules:

"Generic" Weyl Character Formula

$\dim_t H^0(\mu) = D_\lambda(t)/D_0(t)$, where

$$D_\lambda(t) = \prod_{\alpha \in \Phi^+} (t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)}).$$

Note that ζ is a root of $t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)}$ if and only if $\alpha \in \Phi_\lambda^+$.

Outline of the argument for the induced modules:

"Generic" Weyl Character Formula

$\dim_t H^0(\mu) = D_\lambda(t)/D_0(t)$, where

$$D_\lambda(t) = \prod_{\alpha \in \Phi^+} (t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)}).$$

Note that ζ is a root of $t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)}$ if and only if $\alpha \in \Phi_\lambda^+$.

Then ζ is a root of $\dim_t H^0(\lambda)$ with multiplicity $|\Phi_\lambda^+| = |\Phi_J^+|$, hence

$$\dim V_{u_\zeta(\mathfrak{g})}(H^0(\lambda)) \geq |\Phi| - 2|\Phi_J^+| = |\Phi| - |\Phi_J| = \dim G \cdot \mathfrak{u}_J.$$

But $V_{u_\zeta(\mathfrak{g})}(H^0(\lambda)) \subseteq G \cdot \mathfrak{u}_J$ from other techniques, so by dimension comparison and irreducibility of $G \cdot \mathfrak{u}_J$, the varieties must be equal.

Outline of the argument for the induced modules:

"Generic" Weyl Character Formula

$\dim_t H^0(\mu) = D_\lambda(t)/D_0(t)$, where

$$D_\lambda(t) = \prod_{\alpha\in\Phi^+}(t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)}).$$

Note that ζ is a root of $t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)}$ if and only if $\alpha \in \Phi_\lambda^+$.

Then ζ is a root of $\dim_t H^0(\lambda)$ with multiplicity $|\Phi_\lambda^+| = |\Phi_J^+|$, hence

$$\dim V_{u_\zeta(\mathfrak{g})}(H^0(\lambda)) \geq |\Phi| - 2|\Phi_J^+| = |\Phi| - |\Phi_J| = \dim G\cdot\mathfrak{u}_J.$$

But $V_{u_\zeta(\mathfrak{g})}(H^0(\lambda)) \subseteq G\cdot\mathfrak{u}_J$ from other techniques, so by dimension comparison and irreducibility of $G\cdot\mathfrak{u}_J$, the varieties must be equal.

To imitate this approach for the $L(\lambda)$, we need to know their characters.

"Generic" Lusztig Character Formula

Let $\lambda \in X^+$. Choose $\lambda^- \in \overline{C}_{\mathbb{Z}}^-$ (alcove opposite to the lowest ℓ-alcove) and $w \in W_\ell$ of minimal length such that $\lambda = w \cdot \lambda^-$. Then

$$\dim_t L(\lambda) = \sum_{y \in W_\ell} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \cdot \dim_t H^0(y \cdot \lambda^-).$$

Let $W_{\ell,I}$ be the standard parabolic subgroup stabilizing λ^-, and let W_ℓ^I be the set of minimal length right coset representatives for $W_{\ell,I}$. Then

$$\dim_t L(\lambda) = \sum_{y \in W_\ell^I} (-1)^{\ell(w)-\ell(y)} P_{y,w}^{I,-1}(1) \cdot \dim_t H^0(y \cdot \lambda^-).$$

Recall $D_\lambda(t) = \prod_{\alpha \in \Phi^+}(t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)})$. Then

$$f(t) = D_0(t) \cdot \dim_t L(\lambda) = \sum_{\substack{y \in W_\ell^I \\ y \cdot \lambda^- \in X^+}} (-1)^{\ell(w)-\ell(y)} P_{y,w}^{I,-1}(1) \cdot D_{y \cdot \lambda^-}(t).$$

Set $s = |\Phi_J^+|$. Now ζ is a root with multiplicity s in $f(t)$ if $f^{(s)}(\zeta) \neq 0$.

Recall $D_\lambda(t) = \prod_{\alpha\in\Phi^+}(t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)})$. Then

$$f(t) = D_0(t)\cdot \dim_t L(\lambda) = \sum_{\substack{y\in W_\ell^I \\ y\cdot\lambda^-\in X^+}} (-1)^{\ell(w)-\ell(y)} P_{y,w}^{I,-1}(1)\cdot D_{y\cdot\lambda^-}(t).$$

Set $s = |\Phi_J^+|$. Now ζ is a root with multiplicity s in $f(t)$ if $f^{(s)}(\zeta) \neq 0$.

The derivative

$$f^{(s)}(\zeta) = z\cdot\left(\sum_{y\in W_\ell^I, y\cdot\lambda^-\in X^+} P_{y,w}^{I,-1}(1)\left(\prod_{\alpha\in\Phi^+_{y\cdot\lambda^-}} 2d_\alpha(y\cdot\lambda^- + \rho,\alpha^\vee)\right)\right)$$

for some explicitly describable nonzero element $z\in\mathbb{C}$.

Recall $D_\lambda(t) = \prod_{\alpha\in\Phi^+}(t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)})$. Then

$$f(t) = D_0(t)\cdot \dim_t L(\lambda) = \sum_{\substack{y\in W_\ell^I \\ y\cdot\lambda^-\in X^+}} (-1)^{\ell(w)-\ell(y)} P_{y,w}^{I,-1}(1)\cdot D_{y\cdot\lambda^-}(t).$$

Set $s = |\Phi_J^+|$. Now ζ is a root with multiplicity s in $f(t)$ if $f^{(s)}(\zeta)\neq 0$.

The derivative

$$f^{(s)}(\zeta) = z\cdot\left(\sum_{y\in W_\ell^I, y\cdot\lambda^-\in X^+} P_{y,w}^{I,-1}(1)\left(\prod_{\alpha\in\Phi^+_{y\cdot\lambda^-}} 2d_\alpha(y\cdot\lambda^- +\rho,\alpha^\vee)\right)\right)$$

for some explicitly describable nonzero element $z\in\mathbb{C}$.

$P_{y,w}^{I,-1}(1)\in\mathbb{N}\cup\{0\}$, and $P_{w,w}^{I,-1}(1) = 1$. Follows that $f^{(s)}(\zeta)\neq 0$.

Summary:

- $V_{u_\zeta(\mathfrak{g})}(L(\lambda)) \subseteq G \cdot \mathfrak{u}_J$
- $\dim V_{u_\zeta(\mathfrak{g})}(L(\lambda)) \geq \dim G \cdot \mathfrak{u}_J$ from differentiating the generic LCF
- By irreducibility of $G \cdot \mathfrak{u}_J$, must have $V_{u_\zeta(\mathfrak{g})}(L(\lambda)) = G \cdot \mathfrak{u}_J$.

Theorem (D–Nakano–Parshall)

Let G be a simple simply-connected algebraic group over an algebraically closed field k of characteristic $p > h$. Assume that the Lusztig character formula holds for G for all restricted dominant weights. Let $\lambda \in X^+$, and suppose $\Phi_\lambda \sim \Phi_J$ for some subset of simple roots J. Then

$$V_{u(\mathfrak{g})}(L(\lambda)) = G \cdot \mathfrak{u}_J.$$

Equivalently, $V_{G_1}(L(\lambda)) = G \cdot \mathfrak{u}_J$.

Holds for groups of type A_1 if $p \geq 2$, A_2 if $p \geq 3$, B_2 if $p \geq 5$, G_2 if $p \geq 11$, A_3 if $p \geq 5$, A_4 if $p \in \{5, 7\}$, and $p \gg 0$ in general.

Suslin, Friedlander, Bendel (1997)

Suppose G admits an embedding of exponential type $G \hookrightarrow GL_n$. Then

$$V_{G_r}(k) \cong C_r(\mathcal{N}) = \{(x_0, \ldots, x_{r-1}) \in \mathcal{N}^r : [x_i, x_j] = 0 \text{ for all } i, j\}.$$

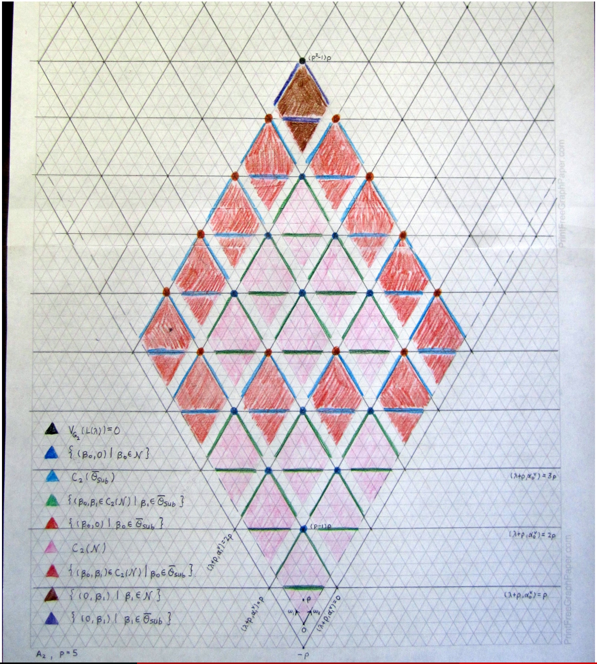
Using SFB's rank variety characterization of $V_{G_r}(M)$, Sobaje has proved:

Sobaje (2011)

Suppose G is a classical group, and that $p > hc$, where c is as given below. Let $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^s\lambda_s$ with $\lambda_i \in X_1(T)$. Then

$$V_{G_r}(L(\lambda)) = \{(x_0, \ldots, x_{r-1}) \in C_r(\mathcal{N}) : x_i \in V_{G_1}(L(\lambda_i))\}.$$

$c = \left(\frac{n+1}{2}\right)^2$ for A_n, $\frac{n(n+1)}{2}$ for B_n, $\frac{n^2}{2}$ for C_n, and $\frac{n(n-1)}{2}$ for D_n.


$(p^2-1)p$
$V_{G_2}(L(\lambda))=0$
$\{(\beta_0,0) \mid \beta_0 \in \mathcal{N}\}$
$C_2(\overline{\Theta}_{sub})$
$\{(\beta_0,\beta_1 \in C_2(\mathcal{N}) \mid \beta_1 \in \overline{\Theta}_{sub}\}$
$\{(\beta_0,0) \mid \beta_0 \in \overline{\Theta}_{sub}\}$
$C_2(\mathcal{N})$
$\{(\beta_0,\beta_1) \in C_2(\mathcal{N}) \mid \beta_0 \in \overline{\Theta}_{sub}\}$
$\{(0,\beta_1) \mid \beta_1 \in \mathcal{N}\}$
$\{(0,\beta_1) \mid \beta_1 \in \overline{\Theta}_{sub}\}$
$(\lambda+\rho,\alpha_0^\vee)=3p$
$(p-1)p$
$(\lambda+\rho,\alpha_0^\vee)=2p$
$(\lambda+\rho,\alpha_1^\vee)=2p$
$(\lambda+\rho,\alpha_0^\vee)=p$
ρ
ω_1
ω_2
$(\lambda+\rho,\alpha_1^\vee)=p$
$(\lambda+\rho,\alpha_2^\vee)=0$
0
$-\rho$
A_2, $p=5$