On injective modules and support varieties for the small quantum group

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Background and motivation

Main idea

Geometry

goal

Representation Theory

inputs
Let $A$ be a finite-dimensional Hopf algebra over the field $k$. Assume that the cohomology ring $H^\bullet(A, k)$ is finitely-generated.

$H(A, k) := H^2\bullet(A, k)$ is a finitely-generated commutative ring. Set $J_A(M, N) = \text{Ann}_{H(A, k)} \text{Ext}_A^\bullet(M, N)$ (homogeneous ideal).

**Definition**

1. $V_A(k) = \text{MaxSpec } H(A, k)$ (cohomological spectrum)
2. $V_A(M, N) = \text{MaxSpec } H(A, k)/J_A(M, N)$ (relative support variety)
3. $V_A(M) = \text{MaxSpec } H(A, k)/J_A(M, M)$ (ordinary support variety)
Most important example for us:

**Theorem (Friedlander–Parshall, 1986)**

Let $\mathfrak{g}$ be a $p$-restricted Lie algebra with $p$-map $X \mapsto X^{[p]}$. Let $M$ be a finite-dimensional restricted $\mathfrak{g}$-module. Then

$$V_{u(\mathfrak{g})}(M) = \left\{ X \in \mathfrak{g} : X^{[p]} = 0 \text{ and } M|_{\langle X \rangle} \text{ is not projective} \right\} \cup \{0\}.$$
We’d like to study support varieties for quantum groups at roots of unity, which have much in common with restricted Lie algebras.
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- $\mathfrak{g}$ simple Lie algebra over $\mathbb{C}$ with root system $\Phi$
- $\zeta \in \mathbb{C}$ primitive $\ell$-th root of unity ($\ell$ odd, coprime to 3 for type $G_2$)
- $U_\zeta(\mathfrak{g})$ quantized enveloping algebra (Lusztig form) with parameter $\zeta$
- $u_\zeta(\mathfrak{g}) \subset U_\zeta(\mathfrak{g})$ the small quantum group
- $u_\zeta(\mathfrak{u}) \subset u_\zeta(\mathfrak{b}) \subset u_\zeta(\mathfrak{g})$ Borel and nilpotent subalgebras
- $u_\zeta^0 \subset u_\zeta(\mathfrak{b})$ small quantum torus
- $u_\zeta(\mathfrak{b}) \cong u_\zeta(\mathfrak{u}) \otimes u_\zeta^0$ as a vector space
Let $\mathcal{N} \subset \mathfrak{g}$ be the variety of nilpotent elements.

**Theorem (Ginzburg–Kumar, 1993)**

Suppose $\ell > h$, the Coxeter number of $\Phi$. Then

\[
H^\bullet(u_\zeta(b), \mathbb{C}) \cong S(u^*) \quad \text{and} \quad H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}].
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Let $M$ be a finite-dimensional $u_\zeta(\mathfrak{g})$-module. Then:

- $V_{u_\zeta(\mathfrak{g})}(M)$ is a closed subvariety of $\mathcal{N}$
- $V_{u_\zeta(\mathfrak{b})}(M)$ is a closed subvariety of $u$
Some known calculations:

**Theorem (Ostrik, BNPP, DNP)**

Suppose $\ell > h$. Let $\lambda$ be a dominant weight. Then there exists a subset of simple roots $J \subset \Phi$, depending on $\ell$ and $\lambda$, such that

$$V_{u_\zeta(g)}(L_\zeta(\lambda)) = V_{u_\zeta(g)}(H^0_\zeta(\lambda)) = G \cdot u_J \subset \mathcal{N}$$
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Open questions:

1. Naturality: Is $V_{u_{\zeta}(\mathfrak{g})}(M) \cap u = V_{u_{\zeta}(\mathfrak{b})}(M)$?
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4. Supports of quantized baby Verma modules?
Question

Can we provide rank variety interpretations for $V_{u_\zeta(g)}(M)$ or $V_{u_\zeta(b)}(M)$ similar to those of Friedlander and Parshall for restricted Lie algebras?
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To what subalgebra in $u\zeta(g)$ does an arbitrary $X \in g$ correspond?

Questions is less ambiguous if $X$ is a root vector:

$u\zeta(u)$ is spanned by monomials of root vectors $E_{\gamma_1}^{a_1} \cdots E_{\gamma_N}^{a_N}$, $0 \leq a_i < \ell$. 
Main Theorem (D, 2009)

Let $M$ be a finite-dimensional $u_\zeta(b)$-module. Then

$$E_\gamma \in V_{u_\zeta(b)}(M) \iff M|_{\langle E_\gamma \rangle} \text{ is not projective.}$$
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Let $M$ be a finite-dimensional $u_\zeta(\mathfrak{b})$-module. Then

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One direction of proof of Main Theorem

Let $M$ be a $u_\zeta(\mathfrak{b})$-module. Set $V = M \otimes M^*$. 

If $V|_{\langle E_\gamma \rangle}$ is projective, then $x_\gamma \in S(u^*) = H^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ acts nilpotently on $H^\bullet(u_\zeta(\mathfrak{b}), V)$.

We’ll outline some main ideas for the special case $\gamma = \gamma_1$ (simple root).
Product we want to investigate:

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2. \( \Delta(u_\zeta(u)) \subset u_\zeta(b) \otimes u_\zeta(u) \) (Caution!)
3. \( \cup : H^\bullet(u_\zeta(b), \mathbb{C}) \otimes H^\bullet(u_\zeta(u), V) \to H^\bullet(u_\zeta(u), V) \) makes sense.
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4. \( \cup : H^\bullet(u_\zeta(b), \mathbb{C}) \otimes H^\bullet(A_m, V) \to H^\bullet(A_m, V) \) makes sense.
Cobar resolution $C^\bullet(A, V) = \text{Hom}_k(A_+^\bullet, V)$ computes $H^\bullet(A, V)$.
Cobar resolution $C^\bullet(A, V) = \text{Hom}_k(A \otimes^\bullet, V)$ computes $H^\bullet(A, V)$.

- Choose cocycle representative $x_\gamma \in C^2(u_\zeta(b), \mathbb{C})$ for $x_\gamma$.
- $x_\gamma$ has weight $-\ell \gamma = -\ell \gamma_1$ ($\gamma_1$ simple).
- Then $x_\gamma([u_1, u_2]) = 0$ unless $u_1, u_2 \in \langle E_{\gamma_1}, u_\zeta^0 \rangle$. 

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4. Cup product at level of cochains: For $g \in C^n(A_m, V),$
   \[
   (x_\gamma \cup g)([u_1, u_2, a_1, \ldots, a_n])
   = \sum x_\gamma([u_1^{(1)}, u_2^{(1)}]) \otimes u_1^{(2)} u_2^{(2)}. g([a_1, \ldots, a_n]).
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- Since $\Delta(E_{\gamma_j}) \in \langle E_{\gamma_j}, \ldots, E_{\gamma_N} \rangle u_0^\gamma \otimes \langle E_{\gamma_1}, \ldots, E_{\gamma_j} \rangle$ for all $1 \leq j \leq N$, $x_\gamma \cup g = 0$ unless $u_1, u_2 \in \langle E_{\gamma_1}, u_0^\gamma \rangle$. 
LHS Spectral Sequence

\[ E_1^{i,j} \cong \text{Hom}_k((A_m/A_{m-1})_+^i, H^j(A_{m-1}, V)) \Rightarrow H^{i+j}(A_m, V) \]

- Arises from the decreasing filtration

\[ F^p C^n(A_m, V) = \{ g \in C^n(A_m, V) : g([a_1, \ldots, a_n]) = 0 \]  
if any of \( a_{n-p+1}, \ldots, a_n \in K \}, \]

where \( K \subset A_m \) is the ideal generated by \( (A_{m-1})_+ \).

- Be careful about products on the LHS spectral sequence because \( A_m, A_{m-1} \) are not Hopf algebras.
Goal: Show $x_{\gamma} \in S(u^*) = H^\bullet(u_\zeta(b), \mathbb{C})$ acts nilpotently on $H^\bullet(A_m, V)$.

6. Let $g \in C^n(A_m, V)$ be a cocycle.
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- Let $g \in C^n(A_m, V)$ be a cocycle.
- If $(x_\gamma^Ur) \cup g \in F^{n+1}C^{n+2r}(A_m, V)$, then $(x_\gamma^Ur) \cup g \equiv 0$.

(What is $a_{2r}$ if $((x_\gamma^Ur) \cup g)([a_1, \ldots, a_{n+2r}]) \neq 0$?)
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- Iterated multiplication by $x_\gamma$ does not push a fixed homogeneous element into arbitrarily high filtered degree. (This is the step that I think could be problematic if working over $u_\zeta(g)$ instead of $u_\zeta(b)$.)
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7. Inspect isomorphism for $E_1$-term and use induction.
Quantum version of a classical result.

**Theorem**

Let $M$ be a finite-dimensional $U_\zeta(G_1 T) = u_\zeta(g)U^0_\zeta$-module. Then $M$ is projective if and only if $M|_{\langle E_\gamma \rangle}$ is projective for all $\gamma \in \Phi$. 
Quantum version of a classical result.

**Theorem**

Let $M$ be a finite-dimensional $U_\zeta(G_1 T) = u_\zeta(g) U_\zeta^0$-module. Then $M$ is projective if and only if $M|_\langle E_\gamma \rangle$ is projective for all $\gamma \in \Phi$.

**Hard direction of proof ($\Leftarrow$).**

- First reduce to case of Borel subalgebra.
- If $M$ is not projective for $u_\zeta(b)$, then $V_{u_\zeta(b)}(M) \neq \{0\}$.
- $V_{u_\zeta(b)}(M) \subset u$ is closed, $T$-stable, so contains a root vector $E_\gamma$.
- Then $M|_\langle E_\gamma \rangle$ is not projective.