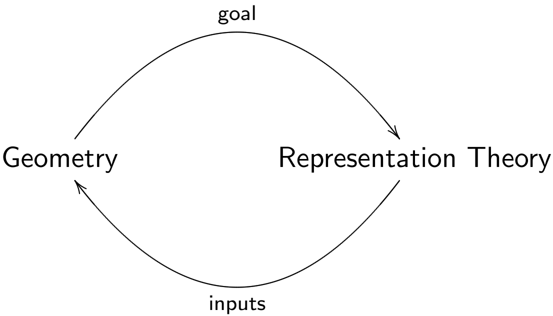
On injective modules and support varieties for the small quantum group

Christopher Drupieski

Southeastern Lie Theory Conference on
Homological Methods in Representation Theory

May 23, 2010


goal
Geometry
Representation Theory
inputs

Let A be a finite-dimensional Hopf algebra over the field k.
Assume that the cohomology ring $\mathrm{H}^\bullet(A,k)$ is finitely-generated.

$\mathrm{H}(A,k) := \mathrm{H}^{2\bullet}(A,k)$ is a finitely-generated commutative ring.
Set $J_A(M,N) = \mathrm{Ann}_{\mathrm{H}(A,k)} \mathrm{Ext}_A^\bullet(M,N)$ (homogeneous ideal).

Definition

1. $V_A(k) = \operatorname{MaxSpec} \mathrm{H}(A,k)$ (cohomological spectrum)
2. $V_A(M,N) = \operatorname{MaxSpec} \mathrm{H}(A,k)/J_A(M,N)$ (relative support variety)
3. $V_A(M) = \operatorname{MaxSpec} \mathrm{H}(A,k)/J_A(M,M)$ (ordinary support variety)

Most important example for us:

Theorem (Friedlander–Parshall, 1986)

Let $\mathfrak{g}$ be a p-restricted Lie algebra with p-map $X \mapsto X^{[p]}$. Let M be a finite-dimensional restricted $\mathfrak{g}$-module. Then

$$V_{u(\mathfrak{g})}(M) = \left\{ X \in \mathfrak{g} : X^{[p]} = 0 \text{ and } M|_{\langle X \rangle} \text{ is not projective} \right\} \cup \{0\}.$$

We'd like to study support varieties for quantum groups at roots of unity, which have much in common with restricted Lie algebras.

We'd like to study support varieties for quantum groups at roots of unity, which have much in common with restricted Lie algebras.

- $\mathfrak{g}$ simple Lie algebra over $\mathbb{C}$ with root system Φ
- $\zeta \in \mathbb{C}$ primitive ℓ-th root of unity (ℓ odd, coprime to 3 for type G_2)
- $U_\zeta(\mathfrak{g})$ quantized enveloping algebra (Lusztig form) with parameter ζ
- $u_\zeta(\mathfrak{g}) \subset U_\zeta(\mathfrak{g})$ the small quantum group
- $u_\zeta(\mathfrak{u}) \subset u_\zeta(\mathfrak{b}) \subset u_\zeta(\mathfrak{g})$ Borel and nilpotent subalgebras
- $u_\zeta^0 \subset u_\zeta(\mathfrak{b})$ small quantum torus
- $u_\zeta(\mathfrak{b}) \cong u_\zeta(\mathfrak{u}) \otimes u_\zeta^0$ as a vector space

Let $\mathcal{N} \subset \mathfrak{g}$ be the variety of nilpotent elements.

Theorem (Ginzburg–Kumar, 1993)

Suppose $\ell > h$, the Coxeter number of Φ. Then

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \cong S(\mathfrak{u}^*) \qquad \text{and}$$
$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}].$$

Let $\mathcal{N} \subset \mathfrak{g}$ be the variety of nilpotent elements.

Theorem (Ginzburg–Kumar, 1993)

Suppose $\ell > h$, the Coxeter number of Φ. Then

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \cong S(\mathfrak{u}^*) \qquad \textit{and}$$
$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}].$$

Let M be a finite-dimensional $u_\zeta(\mathfrak{g})$-module. Then:

- $V_{u_\zeta(\mathfrak{g})}(M)$ is a closed subvariety of $\mathcal{N}$
- $V_{u_\zeta(\mathfrak{b})}(M)$ is a closed subvariety of $\mathfrak{u}$

Some known calculations:

Theorem (Ostrik, BNPP, DNP)

Suppose $\ell > h$. Let λ be a dominant weight. Then there exists a subset of simple roots $J \subset \Phi$, depending on ℓ and λ, such that

$$V_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) = V_{u_\zeta(\mathfrak{g})}(\mathrm{H}^0_\zeta(\lambda)) = G \cdot \mathfrak{u}_J \subset \mathcal{N}$$

Some known calculations:

Theorem (Ostrik, BNPP, DNP)

Suppose $\ell > h$. Let λ be a dominant weight. Then there exists a subset of simple roots $J \subset \Phi$, depending on ℓ and λ, such that

$$V_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) = V_{u_\zeta(\mathfrak{g})}(\mathrm{H}^0_\zeta(\lambda)) = G \cdot \mathfrak{u}_J \subset \mathcal{N}$$

Open questions:

1. Naturality: Is $V_{u_\zeta(\mathfrak{g})}(M) \cap \mathfrak{u} = V_{u_\zeta(\mathfrak{b})}(M)$?

Some known calculations:

Theorem (Ostrik, BNPP, DNP)

Suppose $\ell > h$. Let λ be a dominant weight. Then there exists a subset of simple roots $J \subset \Phi$, depending on ℓ and λ, such that

$$V_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) = V_{u_\zeta(\mathfrak{g})}(\mathrm{H}^0_\zeta(\lambda)) = G \cdot \mathfrak{u}_J \subset \mathcal{N}$$

Open questions:

1. Naturality: Is $V_{u_\zeta(\mathfrak{g})}(M) \cap \mathfrak{u} = V_{u_\zeta(\mathfrak{b})}(M)$?
2. Is $V_{u_\zeta(\mathfrak{g})}(M \otimes N) = V_{u_\zeta(\mathfrak{g})}(M) \cap V_{u_\zeta(\mathfrak{g})}(N)$?

Some known calculations:

Theorem (Ostrik, BNPP, DNP)

Suppose $\ell > h$. Let λ be a dominant weight. Then there exists a subset of simple roots $J \subset \Phi$, depending on ℓ and λ, such that

$$V_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) = V_{u_\zeta(\mathfrak{g})}(\mathrm{H}^0_\zeta(\lambda)) = G \cdot \mathfrak{u}_J \subset \mathcal{N}$$

Open questions:

1. Naturality: Is $V_{u_\zeta(\mathfrak{g})}(M) \cap \mathfrak{u} = V_{u_\zeta(\mathfrak{b})}(M)$?
2. Is $V_{u_\zeta(\mathfrak{g})}(M \otimes N) = V_{u_\zeta(\mathfrak{g})}(M) \cap V_{u_\zeta(\mathfrak{g})}(N)$?
3. Supports of tilting modules? (Relative supports by Bezrukavnikov.)

Some known calculations:

Theorem (Ostrik, BNPP, DNP)

Suppose $\ell > h$. Let λ be a dominant weight. Then there exists a subset of simple roots $J \subset \Phi$, depending on ℓ and λ, such that

$$V_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) = V_{u_\zeta(\mathfrak{g})}(\mathrm{H}^0_\zeta(\lambda)) = G \cdot \mathfrak{u}_J \subset \mathcal{N}$$

Open questions:

1. Naturality: Is $V_{u_\zeta(\mathfrak{g})}(M) \cap \mathfrak{u} = V_{u_\zeta(\mathfrak{b})}(M)$?
2. Is $V_{u_\zeta(\mathfrak{g})}(M \otimes N) = V_{u_\zeta(\mathfrak{g})}(M) \cap V_{u_\zeta(\mathfrak{g})}(N)$?
3. Supports of tilting modules? (Relative supports by Bezrukavnikov.)
4. Supports of quantized baby Verma modules?

Question

Can we provide rank variety interpretations for $V_{u_\zeta(\mathfrak{g})}(M)$ or $V_{u_\zeta(\mathfrak{b})}(M)$ similar to those of Friedlander and Parshall for restricted Lie algebras?

Question

Can we provide rank variety interpretations for $V_{u_\zeta(\mathfrak{g})}(M)$ or $V_{u_\zeta(\mathfrak{b})}(M)$ similar to those of Friedlander and Parshall for restricted Lie algebras?

To what subalgebra in $u_\zeta(\mathfrak{g})$ does an arbitrary $X \in \mathfrak{g}$ correspond?

Questions is less ambiguous if X is a root vector:
$u_\zeta(\mathfrak{u})$ is spanned by monomials of root vectors $E_{\gamma_1}^{a_1} \cdots E_{\gamma_N}^{a_N}$, $0 \leq a_i < \ell$.

Main Theorem (D, 2009)

Let M be a finite-dimensional $u_\zeta(\mathfrak{b})$-module. Then

$$E_\gamma \in V_{u_\zeta(\mathfrak{b})}(M) \iff M|_{\langle E_\gamma \rangle} \text{ is not projective.}$$

Main Theorem (D, 2009)

Let M be a finite-dimensional $u_\zeta(\mathfrak{b})$-module. Then

$$E_\gamma \in V_{u_\zeta(\mathfrak{b})}(M) \iff M|_{\langle E_\gamma \rangle} \text{ is not projective.}$$

One direction of proof of Main Theorem

Let M be a $u_\zeta(\mathfrak{b})$-module. Set $V = M \otimes M^*$.

If $V|_{\langle E_\gamma \rangle}$ is projective, then $x_\gamma \in S(\mathfrak{u}^*) = \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ acts nilpotently on $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V)$.

We'll outline some main ideas for the special case $\gamma = \gamma_1$ (simple root).

Product we want to investigate:

$$\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V)$$

Product we want to investigate:

$$\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V)$$

1
- $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V) = \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V)^{u_\zeta^0}$
- $\Delta(u_\zeta(\mathfrak{u})) \subset u_\zeta(\mathfrak{b}) \otimes u_\zeta(\mathfrak{u})$ (Caution!)
- $\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V)$ makes sense.

Product we want to investigate:

$$\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V)$$

1.
   - $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V) = \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V)^{u_\zeta^0}$
   - $\Delta(u_\zeta(\mathfrak{u})) \subset u_\zeta(\mathfrak{b}) \otimes u_\zeta(\mathfrak{u})$ (Caution!)
   - $\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V)$ makes sense.
2. Let $A_m = \langle E_{\gamma_1}, \dots, E_{\gamma_m} \rangle \subset u_\zeta(\mathfrak{u})$, $1 \leq m \leq N$.
   - $\Delta(A_m) \subseteq u_\zeta(\mathfrak{b}) \otimes A_m$

Product we want to investigate:

$$\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V)$$

1.
   - $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V) = \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V)^{u_\zeta^0}$
   - $\Delta(u_\zeta(\mathfrak{u})) \subset u_\zeta(\mathfrak{b}) \otimes u_\zeta(\mathfrak{u})$ (Caution!)
   - $\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V)$ makes sense.
2. Let $A_m = \langle E_{\gamma_1}, \dots, E_{\gamma_m} \rangle \subset u_\zeta(\mathfrak{u})$, $1 \leq m \leq N$.
   - $\Delta(A_m) \subseteq u_\zeta(\mathfrak{b}) \otimes A_m$
   - Specifically, $\Delta(E_{\gamma_j}) \in \langle E_{\gamma_j}, \dots, E_{\gamma_N} \rangle u_\zeta^0 \otimes \langle E_{\gamma_1}, \dots, E_{\gamma_j} \rangle$.

Product we want to investigate:

$$\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V)$$

1.
   - $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V) = \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V)^{u_\zeta^0}$
   - $\Delta(u_\zeta(\mathfrak{u})) \subset u_\zeta(\mathfrak{b}) \otimes u_\zeta(\mathfrak{u})$ (Caution!)
   - $\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), V)$ makes sense.
2. Let $A_m = \langle E_{\gamma_1}, \dots, E_{\gamma_m} \rangle \subset u_\zeta(\mathfrak{u})$, $1 \leq m \leq N$.
   - $\Delta(A_m) \subseteq u_\zeta(\mathfrak{b}) \otimes A_m$
   - Specifically, $\Delta(E_{\gamma_j}) \in \langle E_{\gamma_j}, \dots, E_{\gamma_N} \rangle u_\zeta^0 \otimes \langle E_{\gamma_1}, \dots, E_{\gamma_j} \rangle$.
   - $\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(A_m, V) \to \mathrm{H}^\bullet(A_m, V)$ makes sense.

Cobar resolution $C^\bullet(A, V) = \mathrm{Hom}_k(A_+^{\otimes \bullet}, V)$ computes $\mathrm{H}^\bullet(A, V)$.

Cobar resolution $C^\bullet(A, V) = \mathrm{Hom}_k(A_+^{\otimes\bullet}, V)$ computes $\mathrm{H}^\bullet(A, V)$.

3

- Choose cocycle representative $x_\gamma \in C^2(u_\zeta(\mathfrak{b}), \mathbb{C})$ for x_γ.
- x_γ has weight $-\ell\gamma = -\ell\gamma_1$ (γ_1 simple).
- Then $x_\gamma([u_1, u_2]) = 0$ unless $u_1, u_2 \in \langle E_{\gamma_1}, u_\zeta^0 \rangle$.

Cobar resolution $C^\bullet(A,V) = \mathrm{Hom}_k(A_+^{\otimes\bullet}, V)$ computes $\mathrm{H}^\bullet(A,V)$.

3
- Choose cocycle representative $x_\gamma \in C^2(u_\zeta(\mathfrak{b}), \mathbb{C})$ for x_γ.
- x_γ has weight $-\ell\gamma = -\ell\gamma_1$ (γ_1 simple).
- Then $x_\gamma([u_1, u_2]) = 0$ unless $u_1, u_2 \in \langle E_{\gamma_1}, u_\zeta^0 \rangle$.

4
- Cup product at level of cochains: For $g \in C^n(A_m, V)$,

$$(x_\gamma \cup g)([u_1, u_2, a_1, \ldots, a_n]) = \sum x_\gamma([u_1^{(1)}, u_2^{(1)}]) \otimes u_1^{(2)} u_2^{(2)}.g([a_1, \ldots, a_n]).$$

Cobar resolution $C^\bullet(A, V) = \mathrm{Hom}_k(A_+^{\otimes \bullet}, V)$ computes $\mathrm{H}^\bullet(A, V)$.

3
- Choose cocycle representative $x_\gamma \in C^2(u_\zeta(\mathfrak{b}), \mathbb{C})$ for x_γ.
- x_γ has weight $-\ell\gamma = -\ell\gamma_1$ (γ_1 simple).
- Then $x_\gamma([u_1, u_2]) = 0$ unless $u_1, u_2 \in \langle E_{\gamma_1}, u_\zeta^0 \rangle$.

4
- Cup product at level of cochains: For $g \in C^n(A_m, V)$,

$$(x_\gamma \cup g)([u_1, u_2, a_1, \dots, a_n]) = \sum x_\gamma([u_1^{(1)}, u_2^{(1)}]) \otimes u_1^{(2)} u_2^{(2)}.g([a_1, \dots, a_n]).$$

- Since $\Delta(E_{\gamma_j}) \in \langle E_{\gamma_j}, \dots, E_{\gamma_N} \rangle u_\zeta^0 \otimes \langle E_{\gamma_1}, \dots, E_{\gamma_j} \rangle$ for all $1 \leq j \leq N$, $x_\gamma \cup g = 0$ unless $u_1, u_2 \in \langle E_{\gamma_1}, u_\zeta^0 \rangle$

5

- LHS Spectral Sequence

$$E_1^{i,j} \cong \mathrm{Hom}_k((A_m//A_{m-1})_+^{\otimes i}, \mathrm{H}^j(A_{m-1}, V)) \Rightarrow \mathrm{H}^{i+j}(A_m, V)$$

- Arises from the decreasing filtration

$$F^p C^n(A_m, V) = \{g \in C^n(A_m, V) : g([a_1, \dots, a_n]) = 0 \text{ if any of } a_{n-p+1}, \dots, a_n \in K\},$$

  where $K \subset A_m$ is the ideal generated by $(A_{m-1})_+$.

- Be careful about products on the LHS spectral sequence because A_m, A_{m-1} are not Hopf algebras.

Goal: Show $x_\gamma \in S(\mathfrak{u}^*) = \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ acts nilpotently on $\mathrm{H}^\bullet(A_m, V)$.

6

- Let $g \in C^n(A_m, V)$ be a cocycle.

Goal: Show $x_\gamma \in S(\mathfrak{u}^*) = \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ acts nilpotently on $\mathrm{H}^\bullet(A_m, V)$.

6
- Let $g \in C^n(A_m, V)$ be a cocycle.
- If $(x_\gamma^{\cup r}) \cup g \in F^{n+1}C^{n+2r}(A_m, V)$, then $(x_\gamma^{\cup r}) \cup g \equiv 0$.
  (What is a_{2r} if $((x_\gamma^{\cup r}) \cup g)([a_1, \ldots, a_{n+2r}]) \neq 0$?)

Goal: Show $x_\gamma \in S(\mathfrak{u}^*) = \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ acts nilpotently on $\mathrm{H}^\bullet(A_m, V)$.

6

- Let $g \in C^n(A_m, V)$ be a cocycle.
- If $(x_\gamma^{\cup r}) \cup g \in F^{n+1}C^{n+2r}(A_m, V)$, then $(x_\gamma^{\cup r}) \cup g \equiv 0$.
  (What is a_{2r} if $((x_\gamma^{\cup r}) \cup g)([a_1, \ldots, a_{n+2r}]) \neq 0$?)
- Iterated multiplication by x_γ does not push a fixed homogeneous element into arbitrarily high filtered degree. (This is the step that I think could be problematic if working over $u_\zeta(\mathfrak{g})$ instead of $u_\zeta(\mathfrak{b})$.)

Goal: Show $x_\gamma \in S(\mathfrak{u}^*) = \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ acts nilpotently on $\mathrm{H}^\bullet(A_m, V)$.

6. - Let $g \in C^n(A_m, V)$ be a cocycle.
   - If $(x_\gamma^{\cup r}) \cup g \in F^{n+1}C^{n+2r}(A_m, V)$, then $(x_\gamma^{\cup r}) \cup g \equiv 0$. (What is a_{2r} if $((x_\gamma^{\cup r}) \cup g)([a_1, \ldots, a_{n+2r}]) \neq 0$?)
   - Iterated multiplication by x_γ does not push a fixed homogeneous element into arbitrarily high filtered degree. (This is the step that I think could be problematic if working over $u_\zeta(\mathfrak{g})$ instead of $u_\zeta(\mathfrak{b})$.)
7. Inspect isomorphism for E_1-term and use induction. □

Quantum version of a classical result.

Theorem

Let M be a finite-dimensional $U_\zeta(G_1 T) = u_\zeta(\mathfrak{g})U_\zeta^0$-module. Then M is projective if and only if $M|_{\langle E_\gamma \rangle}$ is projective for all $\gamma \in \Phi$.

Quantum version of a classical result.

Theorem

Let M be a finite-dimensional $U_\zeta(G_1T) = u_\zeta(\mathfrak{g})U_\zeta^0$-module. Then M is projective if and only if $M|_{\langle E_\gamma \rangle}$ is projective for all $\gamma \in \Phi$.

Hard direction of proof ($\Leftarrow$).

- First reduce to case of Borel subalgebra.
- If M is not projective for $u_\zeta(\mathfrak{b})$, then $V_{u_\zeta(\mathfrak{b})}(M) \neq \{0\}$.
- $V_{u_\zeta(\mathfrak{b})}(M) \subset \mathfrak{u}$ is closed, T-stable, so contains a root vector E_γ.
- Then $M|_{\langle E_\gamma \rangle}$ is not projective. □