Comparing algebraic and finite group cohomology

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- G - simple, simply-connected algebraic group over $\overline{\mathbb{F}}_p$
- $G(\mathbb{F}_q)$ - finite subgroup of $\mathbb{F}_q$-rational points in G, $q = p^r$
- B - Borel subgroup of G
- U - unipotent radical of B
- G_r - Frobenius kernel of G
- $L(\lambda)$ - irreducible G-module of highest weight λ
- $V(\lambda)$ - Weyl module of highest weight λ
- $H^0(\lambda) = \mathrm{ind}_B^G(\lambda)$ - induced module

e.g.,

- $G = SL_n(\overline{\mathbb{F}}_p)$
- $G(\mathbb{F}_q) = SL_n(\mathbb{F}_q)$
- B - lower triangular invertible matrices
- U - lower triangular unipotent matrices

Problem

Compute

$$\mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) \quad \text{and} \quad \mathrm{H}^2(G(\mathbb{F}_q), L(\lambda))$$

for λ small, say, less than or equal to a fundamental dominant weight.

Cline, Parshall, Scott (1975, 1977), Jones (1975)

Let λ be a minimal nonzero dominant weight. Then

$$\dim \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) \leq \begin{cases} 2 & \text{if } p = 2 \\ 1 & \text{if } p \neq 2 \end{cases}$$

Our goal: Compare $\mathrm{H}^i(G(\mathbb{F}_q), L(\lambda))$ to $\mathrm{H}^i(G, L(\lambda))$.

Commutative square of restriction maps:

$$\begin{array}{ccc} \mathrm{H}^i(G,V) & \xrightarrow{\sim} & \mathrm{H}^i(B,V) \\ \downarrow & & \downarrow \\ \mathrm{H}^i(G(\mathbb{F}_q),V) & \hookrightarrow & \mathrm{H}^i(B(\mathbb{F}_q),V). \end{array}$$

Cline, Parshall, Scott, van der Kallen (1977)

Let V be a finite-dimensional rational G-module, and let $i \in \mathbb{N}$. Then for all sufficiently large e and q, the restriction map is an isomorphism

$$\mathrm{H}^i(G,V^{(e)}) \xrightarrow{\sim} \mathrm{H}^i(G(\mathbb{F}_q),V^{(e)}).$$

Avoid twists and $q \gg 0$ by more direct appeal to the left column.

Consider the functor $\mathrm{ind}_{G(\mathbb{F}_q)}^{G}(-)$. There exists a short exact sequence

$$0 \to k \to \mathrm{ind}_{G(\mathbb{F}_q)}^{G}(k) \to N \to 0.$$

Let M be a rational G-module. Then there exists a short exact sequence

$$0 \to M \to \mathrm{ind}_{G(\mathbb{F}_q)}^{G}(M) \to M \otimes N \to 0.$$

Using $\mathrm{Ext}_G^n(k, \mathrm{ind}_{G(\mathbb{F}_q)}^{G}(M)) \cong \mathrm{Ext}_{G(\mathbb{F}_q)}^n(k, M)$, we get:

Long exact sequence for restriction

$$\begin{array}{cccccccc} 0 & \to & \mathrm{Hom}_G(k, M) & \overset{\text{res}}{\to} & \mathrm{Hom}_{G(\mathbb{F}_q)}(k, M) & \to & \mathrm{Hom}_G(k, M \otimes N) & \\ & \to & \mathrm{Ext}_G^1(k, M) & \overset{\text{res}}{\to} & \mathrm{Ext}_{G(\mathbb{F}_q)}^1(k, M) & \to & \mathrm{Ext}_G^1(k, M \otimes N) & \\ & \to & \mathrm{Ext}_G^2(k, M) & \overset{\text{res}}{\to} & \mathrm{Ext}_{G(\mathbb{F}_q)}^2(k, M) & \to & \mathrm{Ext}_G^2(k, M \otimes N) & \to \end{array}$$

Restriction Isomorphism Theorem

Let λ be less than or equal to a fundamental dominant weight. Suppose p and q are as below. Then $\text{Ext}^i_G(k, L(\lambda)) \cong \text{Ext}^i_{G(\mathbb{F}_q)}(k, L(\lambda))$ for $i \leq 2$.

Type	Conditions on p and q
A_n	p odd, $q > 3$
B_n	$p > 3$ ($q > 5$ if $n \leq 3$)
C_n	$p > 3$, $q > 5$
D_n	p odd, $q > 3$
E_6	$p > 3$
E_7	$p > 3$, $q > 5$
E_8	$p > 5$
F_4	$p > 3$, $q > 5$
G_2	$p > 5$

Bendel, Nakano, Pillen (2010)

$\mathrm{ind}_{G(\mathbb{F}_q)}^{G}(k)$ admits a filtration by G-submodules with sections of the form

$$H^0(\mu) \otimes H^0(\mu^*)^{(r)} \quad \mu \in X(T)_+.$$

Corollary: $N = \mathrm{coker}(k \to \mathrm{ind}_{G(\mathbb{F}_q)}^{G}(k))$ admits such a filtration with $\mu \neq 0$.

Then $\mathrm{Ext}_G^i(k, L(\lambda) \otimes N) = 0$ if it is zero for each section, i.e., if for $\mu \neq 0$,

$$\begin{aligned} &\mathrm{Ext}_G^i(k, L(\lambda) \otimes H^0(\mu) \otimes H^0(\mu^*)^{(r)}) \\ &\cong \mathrm{Ext}_G^i(V(\mu)^{(r)}, L(\lambda) \otimes H^0(\mu)) = 0. \end{aligned}$$

Analyze the spectral sequences

$$E_2^{i,j} = \mathrm{Ext}^i_{G/G_r}(V(\mu)^{(r)}, \mathrm{Ext}^j_{G_r}(k, L(\lambda) \otimes H^0(\mu))) \Rightarrow \mathrm{Ext}^{i+j}_G(V(\mu)^{(r)}, L(\lambda) \otimes H^0(\mu))$$

and $E_2^{i,j} = R^i \operatorname{ind}_{B/B_r}^{G/G_r} \mathrm{Ext}^j_{B_r}(k, L(\lambda) \otimes \mu) \Rightarrow \mathrm{Ext}^{i+j}_{G_r}(k, L(\lambda) \otimes H^0(\mu))$.

Critical calculation

Let $\lambda \in X(T)_+$ be less than or equal to a fundamental dominant weight, and let p and q be as above. Then there exists $I \subseteq \Delta$ such that

$$\mathrm{Ext}^1_{U_r}(k, L(\lambda)) \cong \bigoplus_{\alpha \in I} -s_\alpha \cdot \lambda^* \oplus \bigoplus_{\sigma \uparrow \lambda} (-\sigma)^{\oplus m_\sigma}$$

where $m_\sigma = \dim \mathrm{Ext}^1_G(L(\lambda^*), H^0(\sigma))$.

First Cohomology Main Theorem

Let $\lambda \in X(T)_+$ be a fundamental dominant weight. Assume $q > 3$ and

$$\begin{array}{ll} p > 2 & \text{if } \Phi \text{ has type } A_n,\ D_n; \\ p > 3 & \text{if } \Phi \text{ has type } B_n,\ C_n,\ E_6,\ E_7,\ F_4,\ G_2; \\ p > 5 & \text{if } \Phi \text{ has type } E_8. \end{array}$$

Then $\dim \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) = \dim \mathrm{H}^1(G, L(\lambda)) \leq 1$.

Space is one-dimensional in the following cases:

- Φ has type E_7, $p = 7$, and $\lambda = \omega_6$; and
- Φ has type C_n, $n \geq 3$, and $\lambda = \omega_j$ with $\frac{j}{2}$ a nonzero term in the p-adic expansion of $n+1$, but not the last term in the expansion.

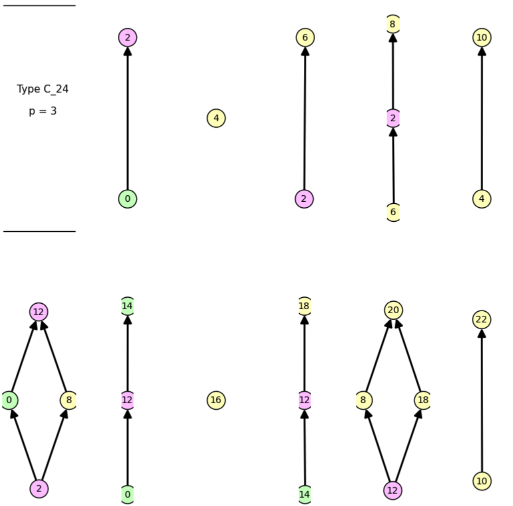
Reasons for vanishing: Linkage principle for G, $\mathrm{Ext}^1_G(V(0), H^0(\lambda)) = 0$.

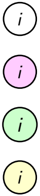
Second Cohomology Main Theorem

Let $\lambda \in X(T)_+$ be less than or equal to a fundamental dominant weight. Let $p > 7$. Then $\operatorname{Ext}^2_{G(\mathbb{F}_q)}(k, L(\lambda)) \cong \operatorname{Ext}^2_G(k, L(\lambda)) = 0$, except possibly in the cases

- $\Phi = E_8$, $p = 31$, and $\lambda \in \{\omega_6 + \omega_8, \omega_7 + \omega_8\}$
- $\Phi = C_n$, $n \geq 3$, and $\lambda = \omega_j$ with j even

Adamovich described combinatorially the submodule structure of Weyl modules in Type C having fundamental highest weight. We use this and $\text{Ext}^2_{C_n}(k, L(\omega_j)) \cong \text{Ext}^1_{C_n}(\text{rad}_G\, V(\omega_j), k)$ to make computations.





$i \longleftrightarrow L(\omega_i)$

$\text{Ext}^1_{C_n}(k, L(\omega_i)) \cong k.$

$[V(\omega_i) : k] = 1$

neither

Values of n and j for which $\mathrm{H}^2(Sp_{2n}, L(\omega_j)) \neq 0$, $p = 3$.

In each case, H^2 is 1-dimensional.

n	j	n	j	n	j	n	j
6	6	15	6, 8	24	6, 8, 18	33	6, 8, 18
7	6	16	6, 10	25	6, 10, 18	34	6, 10, 18
8		17		26		35	
9	6	18	6, 14	27	6, 14	36	6, 14
10	6	19	6, 16	28	6, 16	37	6, 16
11		20	18	29	18	38	18
12	6	21	6, 18	30	6, 18	39	6, 18, 20
13	6	22	6, 18	31	6, 18	40	6, 18, 22
14		23	18	32	18		

For $n = 12$, we have also $\mathrm{H}^1(Sp_{2n}, L(\omega_6)) \neq 0$ (parity vanishing violated).

Values of n and j for which $\mathrm{H}^2(Sp_{2n}, L(\omega_j)) \neq 0$: $p = 5$.

In each case, H^2 is 1-dimensional.

n	j	n	j	n	j	n	j	n	j
10	10	20	10	30	10	40	10, 22	50	10, 42
11	10	21	10	31	10	41	10, 24	51	10, 44
12	10	22	10	32	10	42	10, 26	52	10, 46
13	10	23	10	33	10	43	10, 28	53	10, 48
14		24		34		44		54	50
15	10	25	10	35	10, 12	45	10, 32		
16	10	26	10	36	10, 14	46	10, 34		
17	10	27	10	37	10, 16	47	10, 36		
18	10	28	10	38	10, 18	48	10, 38		
19		29		39		49			

For $n = 30$, we also have $\mathrm{H}^1(Sp_{2n}, L(\omega_{10})) \neq 0$ (parity vanishing violated).