Projective modules for Frobenius kernels and finite groups of Lie type

Christopher M. Drupieski – University of Georgia

Notation:

G simple, simply-connected algebraic group scheme over $\overline{\mathbb{F}}_p$.

$B = U \rtimes T$ Borel subgroup, its unipotent radical, maximal torus

$F : G \to G$ standard Frobenius morphism

$G_r = \ker(F^r)$ r-th Frobenius kernel

$G(\mathbb{F}_q) = G^{F^r}$ finite subgroup of $\mathbb{F}_q$-rational points ($q = p^r$)

Question (Parshall, 1987): Let M be a finite-dimensional rational G-module. Suppose $M|_{G_1}$ is projective. Is $M|_{G(\mathbb{F}_p)}$ also projective?

1. True for the Steinberg module $St = L((p-1)\rho) = H^0((p-1)\rho) = V((p-1)\rho)$.

2. True if $p > 2(h-1)$, h the Coxeter number of G, since then the projective indecomposables for G_1 are all G-summands of $St \otimes V$ for appropriate rational G-modules V.

Theorem (Lin–Nakano, 1999): The answer to Parshall's question is "yes" provided $p \neq 2$ if G is of type B_n, C_n, or F_4, and $p \neq 2,3$ if G is of type G_2.

Call p "excellent" for G if these conditions are satisfied.

Lin–Nakano result follows from their study of complexity.

$cx_{G_1} M$: complexity of M as a G_1-module
= rate of growth of a minimal projective resolution for $M|_{G_1}$
= dimension of the associated cohomological support variety $V_{G_1}(M)$.
$cx_{G_1} M = 0$ if and only if $M|_{G_1}$ is projective.

Theorem (Lin – Nakano): Let M be a finite-dimensional rational G-module. Then

$$cx_{G(\mathbb{F}_p)} M \leq cx_{B(\mathbb{F}_p)} M = cx_{U(\mathbb{F}_p)} M \leq cx_{U_1} M = cx_{B_1} M \leq \frac{1}{2} \cdot cx_{G_1} M$$

Thus, if $M|_{G_1}$ is projective, then $cx_{G_1} M = 0$, hence $cx_{G(\mathbb{F}_p)} M = 0$, which implies that $M|_{G(\mathbb{F}_p)}$ is projective.

Key step: $cx_{U(\mathbb{F}_p)} M \leq cx_{U_1} M$

- $cx_{U_1} M$ = rate of growth of $\dim H^{\bullet}(U_1, M)$
- $cx_{U(\mathbb{F}_p)} M$ = rate of growth of $\dim H^{\bullet}(U(\mathbb{F}_p), M)$

The group ring $kU(\mathbb{F}_p)$ is filtered by powers of the augmentation ideal.

Theorem (Quillen + Lin–Nakano): If p is excellent, then $\mathrm{gr}\, kU(\mathbb{F}_p) \cong u(\mathrm{Lie}(U))$, the restricted enveloping algebra of $\mathrm{Lie}(U)$.

consequence: Since $u(\mathrm{Lie}(U))$-mod $\equiv U_1$-mod, one can construct a spectral sequence $E_1^{i,j} = H^{i+j}(U_1, M)_{(i)} \Rightarrow H^{i+j}(U(\mathbb{F}_p), M)$.

construction depends on M admitting a weight space decomposition.

$\boxed{\text{Question:}}$ Let M be a finite-dimensional rational G-module. Let $q = p^r$. Suppose $M|_{G_r}$ is projective. Is $M|_{G(\mathbb{F}_q)}$ projective?

Previous approach does not generalize to $r > 1$:
$\mathrm{gr}\, kU(\mathbb{F}_q)$ is a poor approximation for U_r. (more on this later)

$\boxed{\text{Observe:}}$

- $M|_{G_r}$ projective $\Rightarrow$ $M|_{U_r}$ projective
- $M|_{U(\mathbb{F}_q)}$ projective $\Rightarrow$ $M|_{G(\mathbb{F}_q)}$ projective
  (b/c $U(\mathbb{F}_q)$ is a Sylow p-subgroup)

$\boxed{\text{Theorem (D):}}$ Let M be a finite-dimensional rational B-module (p arbitrary). If $M|_{U_r}$ is projective, then $M|_{U(\mathbb{F}_q)}$ is projective.

$\boxed{\text{Main tool of proof:}}$ Algebra of distributions on U, $\mathrm{Dist}(U)$
(aka, the hyperalgebra of U).

For each root subgroup $U_\alpha \subset U$, $\mathrm{Dist}(U_\alpha)$ has basis $\{ X_\alpha^{(n)} \mid n \geq 0 \}$

$$X_\alpha^{(n)} X_\alpha^{(m)} = \binom{n+m}{n} X_\alpha^{(n+m)}$$

- $\mathrm{Dist}(U)$ has a basis of PBW-monomials $X_{\alpha_1}^{(n_1)} \cdots X_{\alpha_N}^{(n_N)}$.
- $\mathrm{Dist}(U_r)$ is spanned by those monomials with $0 \leq n_i < p^r = q$

- $U_r\text{-mod} \equiv \mathrm{Dist}(U_r)\text{-mod}$.
- Action of $\mathrm{Dist}(U_\alpha)$ on a U_α-module M determines the action of U_α:
If $x_\alpha : \mathbb{G}_a \to U_\alpha$ is a fixed isomorphism, $a \in \mathbb{G}_a$ and $m \in M$, then

$$\boxed{x_\alpha(a).m = \sum_{n \geq 0} (a^n X_\alpha^{(n)}).m.}$$

- $x_\alpha(a)$ acts on a rational module via $\sum_{n\geq 0} a^n X_\alpha^{(n)}$.
- If $a \in \mathbb{F}_q$, then $a^q = a$, so then $x_\alpha(a) \in U_\alpha(\mathbb{F}_q)$ acts as $\sum_{i=0}^{q-1} a^i \left(\sum_{n\geq 0} X_\alpha^{(i+n(q-1))} \right)$
- Set $y_{\alpha,0} = 1$, and for $1 \leq i \leq q-1$, set $y_{\alpha,i} = \sum_{n \geq 0} X_\alpha^{(i+n(q-1))}$.

$\boxed{\text{Lemma:}}$ Let M be a rational U_α-module. Then the span in $\mathrm{End}_k(M)$ of $y_{\alpha,0}, y_{\alpha,1}, \cdots, y_{\alpha,q-1}$ is the same as the k-span of the operators $x_\alpha(a) \in U_\alpha(\mathbb{F}_q)$ $(a \in \mathbb{F}_q)$.

$\boxed{\text{Idea of proof:}}$ Linear transformation sending one set of operators to the other is given by an invertible Vandermonde matrix.

$\boxed{\text{Proof of the theorem:}}$

Let M be a finite-dimensional rational B-module. Suppose $M|_{U_r}$ is projective. Then M admits a $\mathrm{Dist}(U_r)$-basis $\{m_1, m_2, \cdots, m_s\}$ consisting of weight vectors. So $\dim M = s \cdot \dim \mathrm{Dist}(U_r) = s \cdot p^{r \cdot |\Phi^+|}$.

Since $\dim \mathrm{Dist}(U_r) = \dim kU(\mathbb{F}_q)$, M is free (hence projective) over $kU(\mathbb{F}_q)$ provided $\{m_1, m_2, \cdots, m_s\}$ generates M as a $kU(\mathbb{F}_q)$-module.

- $kU(\mathbb{F}_q)$ is spanned by $x_{\alpha_1}(a_1)\, x_{\alpha_2}(a_2) \cdots x_{\alpha_N}(a_N)$ $\quad (a_i \in \mathbb{F}_q)$
- can replace by $(y_{\alpha_1,n_1})(y_{\alpha_2,n_2}) \cdots (y_{\alpha_N,n_N})$ $\quad (0 \leq n_i \leq q-1)$

- lowest weight vector $X_{\alpha_1}^{(q-1)} X_{\alpha_2}^{(q-1)} \cdots X_{\alpha_N}^{(q-1)} m_j = (y_{\alpha_1,q-1})(y_{\alpha_2,q-1}) \cdots (y_{\alpha_N,q-1}). m_j$
- other weight vectors $X_{\alpha_1}^{(n_1)} X_{\alpha_2}^{(n_2)} \cdots X_{\alpha_N}^{(n_N)} . m_j \equiv (y_{\alpha_1,n_1})(y_{\alpha_2,n_2}) \cdots (y_{\alpha_N,n_N}) . m_j$ modulo lower weight vectors

Theorem: M a finite-dimensional rational G-module.
Then $M|_{G_r}$ projective $\Rightarrow$ $M|_{G(\mathbb{F}_q)}$ projective.

Theorem is false if G is replaced by U!

Example: $G = SL_2$, $U \cong \mathbb{G}_a$.
Define $f: U \to U$ by $f(t) = t - t^q$. Then $\ker(f) = U(\mathbb{F}_q)$.
Take $M = f^*(St_r)$, pullback of St_r via f.
$f|_{U_r} = \mathrm{id}$ (b/c $U_r = \{a \in \mathbb{G}_a \mid a^{p^r} = a^q = 0\}$).
$\therefore$ $M \cong St_r$ as a U_r-module, but M is trivial for $U(\mathbb{F}_q)$.

(Note: M does not lift to a rational B-module b/c $t - t^q$ is non-homogeneous.)

Weil restriction

Set $\mathfrak{n} = \mathrm{Lie}(U)$. $\mathfrak{n} = \mathfrak{n}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} k$, $\mathfrak{n}_{\mathbb{F}_q} = \mathfrak{n}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_q$.

$\mathfrak{n}_{\mathbb{F}_p}$ is a p-restricted Lie algebra over $\mathbb{F}_p$.
$\mathfrak{n}_{\mathbb{F}_q}$ is also a p-restricted Lie algebra over $\mathbb{F}_p$ (forget $\mathbb{F}_q$-vector space structure)

Theorem (Quillen + Lin, Nakano): $\mathrm{gr}\, kU(\mathbb{F}_q) \cong u(\mathfrak{n}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k)$.

$$\mathfrak{n}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k = (\mathfrak{n}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_q) \otimes_{\mathbb{F}_p} k \cong \mathfrak{n}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} (\mathbb{F}_q \otimes_{\mathbb{F}_p} k)$$

$$\boxed{\mathbb{F}_q \otimes_{\mathbb{F}_p} \mathbb{F}_q \cong \mathbb{F}_q \times \cdots \times \mathbb{F}_q \quad (r \text{ times})}$$

$\therefore$ $\mathfrak{n}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k \cong (\mathfrak{n}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} k)^{\oplus r} = \mathfrak{n}^{\oplus r}$ as p-restricted Lie algebras $/ k$.

If M is a rational G-module, the action of $\mathfrak{n}_{\mathbb{F}_q}$ on M extends linearly to $\mathfrak{n}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k$. Identifying $\mathfrak{n}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k = \mathfrak{n}^{\oplus r}$, the action is the composite of the r-fold addition map $\mathfrak{n}^{\oplus r} \to \mathfrak{n}$, $(x_1, x_2, \cdots, x_r) \mapsto x_1 + x_2 + \cdots + x_r$, with the ordinary action of $\mathfrak{n}$ on M.

In particular, $(x, -x, 0, \cdots, 0) \in \mathfrak{n}^{\oplus r}$ acts as zero on M.

Theorem (Friedlander): Let M be a rational G-module. Then

$$cx_{G(\mathbb{F}_q)} M \leq \tfrac{1}{2} \cdot cx_{u(\mathfrak{g}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k)} M.$$

In particular, if M is projective over $u(\mathfrak{g}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k) \cong u(\mathfrak{g}^{\oplus r})$, then M is projective over $G(\mathbb{F}_q)$.

The second statement is vacuously true! (for $r > 1$)

Claim: If M is a rational G-module, then M is not projective for $u(\mathfrak{g}^{\oplus r})$.

Look at the support variety $V_{u(\mathfrak{g}^{\oplus r})}(M) \cong \left\{ z \in \mathfrak{g}^{\oplus r} \;\middle|\; z^{[p]} = 0 \text{ and } M|_{\langle z \rangle} \text{ is not projective} \right\}$

Choose any $0 \neq x \in \mathcal{N}_p(\mathfrak{g})$. Then $z := (x, -x, 0, \cdots, 0) \in V_{u(\mathfrak{g}^{\oplus r})}(M)$.