Definition 1 A map $f: Y \to X$ between connected spaces is called a homotopy monomorphism at $p$ if its homotopy fibre $F$ is $B\mathbb{Z}/p$-local for every choice of basepoint.

In the special case where $Y = BP$ is the classifying space of a finite $p$-group, we say that $f$ is a $p$-subgroup inclusion.

Proposition 1 (Folklore) Let $f: Y \to X$ be a map between two $p$-complete spaces, with Noetherian cohomology rings. Then $f$ is a homotopy monomorphism if and only if the induced map in cohomology makes $H^*(Y; \mathbb{F}_p)$ a finitely generated $H^*(X; \mathbb{F}_p)$-module.
**Definition 2** Let $f : Y \to X$ be a map of spaces. A Frobenius transfer of $f$ is a stable map $t : \Sigma_{\infty}^+ X \to \Sigma_{\infty}^+ Y$ such that

$$\Sigma_{\infty}^+ f \circ t \simeq id_{\Sigma_{\infty}^+ X}$$

and the following diagram commutes up to homotopy

$$\begin{array}{ccc}
\Sigma_{\infty}^+ X & \xrightarrow{\Delta} & \Sigma_{\infty}^+ X \wedge \Sigma_{\infty}^+ X \\
\downarrow t & & \downarrow 1 \wedge t \\
\Sigma_{\infty}^+ Y & \xrightarrow{\Delta} & \Sigma_{\infty}^+ Y \wedge \Sigma_{\infty}^+ Y \\
& & \xrightarrow{f \wedge 1} \Sigma_{\infty}^+ X \wedge \Sigma_{\infty}^+ Y.
\end{array}$$

(1)
Definition 3 A Frobenius transfer triple over a finite $p$-group $S$ is a triple $(f, t, X)$, where

$X$ is a connected, $p$-complete space with finite fundamental group

$f$ is a subgroup inclusion $BS \rightarrow X$

$t$ is a Frobenius transfer for $f$. 
In general, given a map \( f : BS \to X \), we get a fusion system \( \mathcal{F}_{S,f}(X) \) over \( S \) by putting

\[
\text{Hom}_{\mathcal{F}_{S,f}}(P,Q) = \{ \varphi \in \text{Inj}(P,Q) \mid f|_{BP} \simeq f|_{BQ} \circ B\varphi \}
\]
for each \( P, Q \leq S \).

For such a fusion system \( \mathcal{F}_{S,f} \) we pose the following questions:

- Is \( \mathcal{F}_{S,f} \) saturated?

- Does it have an associated centric linking system \( \mathcal{L} \)? If so, is it unique?

- What is the relationship between \( BS \xrightarrow{f} X \) and \( BS \xrightarrow{\theta} |\mathcal{L}|_p^\wedge \)? Are they equivalent as objects under \( BS \).
In these lectures, these questions will be answered in a special case.

**Theorem 1** Let $S$ be a finite elementary abelian $p$-group. Let $(f, t, X)$ be a Frobenius transfer triple over $S$ and put $W := \text{Aut}_{F_{S,f}}(X)(S)$. Then the following hold

- $F_{S,f}(X)$ is equal to the saturated fusion system $F_{S}(W \ltimes S)$

- $F_{S,f}(X)$ has a unique associated centric linking system with classifying space $B(W \ltimes S)^\wedge_p$.

- There is a natural equivalence $B(W \ltimes S)^\wedge_p \xrightarrow{\simeq} X$ of objects under $BS$. 
Conversely, we will show the following for $p$-local finite groups over any finite $p$-group $S$:

**Theorem 2** Let $(S, F, \mathcal{L})$ be a $p$-local finite group. Then the natural inclusion
\[
\theta: BS \rightarrow |\mathcal{L}|_p^\wedge.
\]
has a Frobenius transfer
\[
t: \Sigma^\infty_+|\mathcal{L}|_p^\wedge \rightarrow \Sigma^\infty_+BS
\]
making $(\theta, t, X)$ a Frobenius transfer triple over $S$. 
In the course of the proof of Theorem 2, we also obtain the following interesting results:

**Theorem 3 (The next best thing)** There is a functorial assignment

\[ \Upsilon: (S, \mathcal{F}) \mapsto \Sigma^\infty BS \xrightarrow{\sigma_{\mathcal{F}}} B\mathcal{F} \]

of a classifying spectrum to every saturated fusion system, such that:

- \((S, \mathcal{F})\) can be recovered from \((\sigma_{\mathcal{F}}, B\mathcal{F})\).
- \(\sigma_{\mathcal{F}}\) admits a “transfer” \(t_{\mathcal{F}}: B\mathcal{F} \xrightarrow{\sigma_{\mathcal{F}}} \Sigma^\infty BS\), such that \(\sigma_{\mathcal{F}} \circ t_{\mathcal{F}} \simeq id_{B\mathcal{F}}\).
- \((\sigma_{\mathcal{F}}, B\mathcal{F})\) agrees with \((\Sigma^\infty \theta, \Sigma^\infty |\mathcal{L}|_p^\wedge)\) in the case of \(p\)-local finite groups.

There is a theory of transfers for “injective” morphisms between saturated fusion systems.
The functor $\Upsilon$ also agrees with the $p$-completed stable classifying spaces of finite groups. That is, the following diagram of functors commutes:

$$
\begin{array}{ccc}
\text{Groups} & \xrightarrow{B(-)^\wedge_p} & \text{Spaces} \\
\downarrow \mathcal{F}(-) & & \downarrow \Sigma^\infty_+(-) \\
\text{Fusion systems} & \xrightarrow{\Upsilon} & \text{Spectra}.
\end{array}
$$
In view of Oliver’s solution of the Martino-Priddy conjecture, we get the following Corollary.

**Corollary 1** Let $G$ and $G'$ be finite groups with Sylow subgroups $S$ and $S'$, respectively. Then the following are equivalent

(i) $BG_p^{\wedge} \simeq BG'_p^{\wedge}$

(ii) There is an isomorphism $\varphi : S \to S'$ and a stable equivalence $h: \Sigma_+ \infty BS \rightarrow \Sigma_+ \infty BG_p^{\wedge}$ making the following diagram commute

$$
\begin{array}{ccc}
\Sigma_+ \infty BS & \rightarrow & \Sigma_+ \infty BG_p^{\wedge} \\
\downarrow B\varphi & & \downarrow h \\
\Sigma_+ \infty BS' & \rightarrow & \Sigma_+ \infty BG'_p^{\wedge}.
\end{array}
$$
Using the transfer theory, we also obtain the following corollary

**Corollary 2** Let $\mathcal{F}$ and $\mathcal{F}'$ be saturated fusion systems over a finite $p$-group $S$. Then $\mathcal{F} \subset \mathcal{F}'$ if and only if $B\mathcal{F}'$ is a stable summand of $B\mathcal{F}$ (as objects under $\Sigma_\infty^+ BS$).
Let $S$ be a finite abelian $p$-group and $\mathcal{F}$ be a fusion system over $S$. Then we get the following simplifications:

- Every $P \leq S$ is both fully centralized and fully normalized, since $C_S(P) = N_S(P) = S$.

- $Aut_S(P) = \{id\}$ for all $P \leq S$.

- For $P \leq S$ and $\varphi \in Hom_{\mathcal{F}}(P, S)$, we have
  \[
  N_{\varphi} = \{ g \in N_S(P) \mid \varphi \circ c_g \circ \varphi^{-1} \in Aut_S(\varphi P) \}
  = \{ g \in S \mid \varphi \circ id \circ \varphi^{-1} \in \{id\} \}
  = S.
  \]

- Since $C_S(P) = S$ for every $P \leq S$, the only $\mathcal{F}$-centric subgroup is $S$ itself.
Lemma 1 Let $\mathcal{F}$ be a fusion system over a finite abelian $p$-group $S$. Then $\mathcal{F}$ is saturated if and only if the following two conditions are satisfied:

(I\text{ab}) \quad |\text{Aut}_{\mathcal{F}}(S)| \text{ is prime to } p.

(II\text{ab}) \quad \text{Every } \varphi \in \text{Hom}_{\mathcal{F}}(P, Q) \text{ is the restriction of some } \tilde{\varphi} \in \text{Aut}_{\mathcal{F}}(S).
Therefore, the only saturated fusion systems over $S$ are the ones coming from semi-direct products $W \ltimes S$, where $W \leq Aut(S)$ has order prime to $p$.

These have a canonical classifying space $B(W \ltimes S)^\wedge_p$. The obstructions to existence and uniqueness of classifying spaces reduces to group cohomology

$$H^*(W; S),$$

which vanishes by a transfer argument. Hence the classifying space is unique.
Proposition 2  If $S$ is an abelian finite $p$-group, then the assignment
\[ W \mapsto (S, \mathcal{F}(W \rtimes S), \mathcal{L}_S^c(W \rtimes S)) \]
gives a bijective correspondence between subgroups $W \leq Aut(S)$ of order prime to $p$ and $p$-local finite groups over $S$. In particular, there are no exotic $p$-local finite groups over $S$. 
Outline of proof of Theorem 1:
1. Use a theorem of Adams and Wilkerson to show that
\[ H^*(X) = H^*(BS)^W = H^*(B(W \rtimes S)_p^\wedge), \]
for some \( W \leq Aut(S) \) of order prime to \( p \).

2. Use Lannes’s theorem to deduce that
\[ F_{S,f}(X) = F_S(W \rtimes S) \]

3. By classification of \( p \)-local finite groups, we have a unique classifying space \( B(W \rtimes S)_p^\wedge \) for \( F_{S,f}(X) \). We use Wojtkowiak’s obstruction theory to produce an equivalence
\[ B(W \rtimes S)_p^\wedge \xrightarrow{\simeq} X \]
of objects under \( BS \).