We have proved that a Frobenius transfer triple over an elementary abelian group $S$ induces a $p$-local finite group over $S$.

Conversely we will now show that for a $p$-local finite group $(S, \mathcal{F}, \mathcal{L})$ over any finite $p$-group $S$, the natural inclusion $\theta: BS \rightarrow |\mathcal{L}|_p^\wedge$ has a Frobenius transfer $t$, which makes $(\theta, t, |\mathcal{L}|_p^\wedge)$ a Frobenius transfer triple over $S$.

In the process of doing so we develop tools which allows us to prove properties of classifying spectra for saturated fusion systems, as defined in [BLO2], which are probably of greater interest than our original goal.
Given a saturated fusion system $\mathcal{F}$ over a finite $p$-group $S$, it is shown in [BLO2] how to associate a summand $\mathbb{B}\mathcal{F}$ of $\Sigma^\infty_+ BS$ to $\mathcal{F}$. Furthermore, it is shown that if $\mathcal{F}$ has an associated linking system $\mathcal{L}$, then

$$\mathbb{B}\mathcal{F} \simeq \Sigma^\infty_+ |\mathcal{L}|^\wedge_p.$$ 

Therefore Broto-Levi-Oliver refer to $\mathbb{B}\mathcal{F}$ as the classifying spectrum of $\mathcal{F}$.

We will enrich the classifying spectrum with a structure map

$$\sigma_\mathcal{F} : \Sigma^\infty_+ BS \longrightarrow \mathbb{B}\mathcal{F}$$

and a transfer

$$t_\mathcal{F} : \mathbb{B}\mathcal{F} \longrightarrow \Sigma^\infty_+ BS$$

such that

$$\sigma_\mathcal{F} \circ t_\mathcal{F} \simeq id_{\mathbb{B}\mathcal{F}}.$$
Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems over finite $p$-groups $S_1$ and $S_2$, respectively.

**Definition.** A group homomorphism

$$\gamma: S_1 \rightarrow S_2$$

is $(\mathcal{F}_1, \mathcal{F}_2)$-fusion-preserving if there exists a functor

$$F_\gamma: \mathcal{F}_1 \rightarrow \mathcal{F}_2$$

such that

$$F_\gamma(P) = \gamma(P)$$

for all $P \leq S$ and

$$\gamma|_Q \circ \varphi = F_\gamma(\varphi) \circ \gamma|_P$$

for all $\varphi \in \text{Hom}_{\mathcal{F}_1}(P, Q)$. 
We will prove that classifying spectra of saturated fusion systems have the following properties:

(A) $\varphi \in \text{Hom}_F(P, Q)$ if and only if
\[ \sigma_F \circ \sum^\infty B\iota_Q \circ \sum^\infty B\varphi \simeq \sigma_F \circ \sum^\infty B\iota_P. \]

(B) A $(F_1, F_2)$–fusion-preserving homomorphism $\gamma: S_1 \to S_2$ induces a map $B\gamma_{F_1}$ making the following diagram commute up to homotopy:

\[ \begin{array}{ccc} BS_1 & \xrightarrow{B\gamma} & BS_2 \\ \downarrow^{\sigma_{F_1}} & & \downarrow^{\sigma_{F_2}} \\ BF_1 & \xrightarrow{B\gamma} & BF_2. \end{array} \]

This assignment is functorial and extends the group-theoretic case.

(C) Fusion-preserving monomorphisms admit transfers, which behave like transfers between groups in cohomology.
We reach our original goal by the following:

\((D)\) When \(\mathcal{F}\) has an associated centric linking system \(\mathcal{L}\), the natural inclusion

\[ \theta: BS \rightarrow |\mathcal{L}|^\wedge_p \]

suspends to the structure map

\[ \sigma_{\mathcal{F}}: \Sigma^\infty_+ BS \rightarrow \mathbb{B}\mathcal{F} \]

and \(t_{\mathcal{F}}\) is a Frobenius transfer for \(\theta\).

Finally, we will also give an explicit basis for the \(\mathbb{Z}_p^\wedge\)-modules \([\mathbb{B}\mathcal{F}_1, \mathbb{B}\mathcal{F}_2]\).
Preliminaries on the Segal conjecture

We need to have a good understanding of the group \( \{BS_1^+, BS_2^+\} \) of homotopy classes of stable maps \( \Sigma^\infty_+ BS_1 \to \Sigma^\infty_+ BS_2 \) for finite \( p \)-groups.

The Segal conjecture relates stable maps \( \Sigma^\infty_+ BG_1 \to \Sigma^\infty_+ BG_2 \) to \((G_1, G_2)\)-bisets for finite groups.

\( \text{Mor}(G_1, G_2) := \) Set of isomorphism classes of finite sets with right \( G_1 \)-action and free left \( G_2 \)-action such that the actions commute.

\( \text{Mor}(G_1, G_2) \) is a monoid under disjoint union.

\( A(G_1, G_2) := \) Grothendieck group completion of \( \text{Mor}(G_1, G_2) \).
We describe a “natural” homomorphism
\[ \alpha: A(G_1, G_2) \longrightarrow \{BG_1+, BG_2+\}. \]

Take a representative \( \Omega \) of an element in \( \text{Mor}(G_1, G_2) \) and put
\[ \Lambda := \Omega / G_2. \]

The projection map
\[ EG_1 \times_{G_1} \Lambda \rightarrow BG_1 \]

is a finite covering (the fibre is \( \Lambda \)) and we get a transfer
\[ \tau: \Sigma^\infty+BG_1 \rightarrow \Sigma^\infty+BG_2. \]

We also have a principal \( G_2 \)-fibre sequence
\[ G_2 \rightarrow EG_1 \times_{G_1} \Omega \rightarrow EG_1 \times_{G_1} \Lambda \]

since the \( G_2 \)-action on \( \Omega \) was free. Let
\[ \xi: EG_1 \times_{G_1} \Lambda \rightarrow BG_2 \]

be the classifying map.

Finally,
\[ \alpha(\Omega) := \Sigma^\infty+\xi \circ \tau. \]
There is a pairing

\[ A(G_2, G_3) \times A(G_1, G_2) \rightarrow A(G_1, G_3) \]

\[ (\Lambda, \Omega) \mapsto \Lambda \times_{G_2} \Omega. \]

This makes \( A(G, G) \) a ring called the double Burnside ring of \( G \).

The morphism \( \alpha \) sends this pairing to the composition pairing of stable maps:

\[ \alpha(\Lambda \times_{G_2} \Omega) = \alpha(\Lambda) \circ \alpha(\Omega). \]

We could regard \( \alpha \) as a functor from the category whose objects are finite groups and whose morphism sets are \( A(G_1, G_2) \) to the category whose objects are finite groups and whose morphism sets are \( \{BG_{1+}, BG_{2+}\} \).
Put $A(G) := A(G, 1)$.

There is a pairing

$$A(G) \times A(G, G') \to A(G, G'),$$

$$(X, \Omega) \to X \times \Omega,$$

where the actions are given by

$$(x, y).g = (x.g, y.g), \quad g'.(x, y) = (x, g'.y).$$

In particular, $A(G)$ is a ring and $A(G, G')$ is a module over $A(G)$. $A(G)$ is called the Burnside ring of $G$.

$A(G)$ has an augmentation

$$A(G) \to \mathbb{Z}, \quad \Omega \mapsto |\Omega|.$$ 

We let $I(G)$ denote the augmentation ideal (i.e. the kernel of the augmentation).
Lewis-May-McClure have shown that the following theorem is a consequence of the Segal conjecture, which was proved by Carls-
son.

**Theorem (Segal conjecture).** $\alpha$ induces an isomorphism

$$A(G_1, G_2) \overset{\wedge}{\to} \{BG_{1+}, BG_{2+}\},$$

where

$$A(G_1, G_2) \overset{\wedge}{=} \lim_{n} (A(G_1, G_2)/I^n)$$

is the completion with respect to the ideal $I = I(G_1)$.

When $G_1$ is a $p$-group this takes a simple form

**Theorem.** Let $S$ be a finite $p$-group and $G$ be a finite group. Let $\tilde{A}(S, G)$ be the kernel of the morphism

$$A(S, G) \to A(S), \ \Omega \to \Omega/G.$$ 

Then $\alpha$ induces an isomorphism

$$\tilde{A}(S, G) \overset{\wedge}{=} \mathbb{Z}_p \otimes \tilde{A}(S, G) \overset{\sim}{\to} \{BS, BG\}.$$
Historical note: Let $G$ be a finite group.

Atiyah (ca. 1960): There is an isomorphism

$$R(G)_I^\wedge \longrightarrow KU(BG),$$

where $R(G)$ is the complex representation ring and $I$ is the kernel of the augmentation

$$R(G) \rightarrow \mathbb{Z}, \ V \mapsto \dim(V).$$

Segal conjectured that the analogous result holds for stable cohomotopy.

**Segal conjecture (weak form):** The map

$$A(G)_I^\wedge \longrightarrow \pi^0_S(BG_+) := \{BG_+, S^0\}$$

is an isomorphism.

Lin: Proved conjecture for $G = \mathbb{Z}/2$.

Gunawardena: $G = \mathbb{Z}/p$, $p$ odd prime.

Ravenel: General finite cyclic groups.

Carlsson: Elementary abelian 2-groups.

Adams-Gunawardena-Miller:

Odd elementary abelian groups.

May-McClure: Reduce question to finite $p$-groups.

The proofs are actually of a stronger form of the conjecture. This was in fact necessary, since a statement involving only $\pi^0_S(BG_+)$ (and not higher cohomotopy groups) does not lend itself to induction.

Segal introduced equivariant stable cohomotopy groups

$$\pi^*_{G}(S^0) = \bigotimes_K \pi^S_*(BW(K)_+),$$

where the sum is taken over conjugacy classes of subgroups of $G$ and $W(K) = N_G(K)/K$.

Note that $\pi^*_G(S^0)$ is a an $A(G)$-module and $\pi^0_G(S^0) \cong A(G)$.

**Segal conjecture (strong form):** The map

$$\pi^*_G(S^0)^{\wedge}_{I(G)} \rightarrow \pi^*_S(BG_+)$$

is an isomorphism.
We will study $A(S_1, S_2)$ in detail for finite $p$-groups. Therefore we apply an unusual form of the Segal conjecture, which maintains more of the structure of $A(S_1, S_2)$.

**Theorem (Segal conjecture).** If $S$ is a finite $p$-group and $G$ any finite group, then $\alpha$ induces an isomorphism

$$A(S, G)^\wedge_p \xrightarrow{\cong} \{BS_+, BG_+\}_p^\wedge,$$

where $\{BS_+, BG_+\}_p^\wedge$ denotes the group of homotopy classes of stable maps $\Sigma_+^\infty BS^\wedge_p \to \Sigma_+^\infty BG^\wedge_p$.

Note that $\Sigma_+^\infty BG \simeq \Sigma_+^\infty BG \vee S^0$. We will only be interested in the case where $G$ is a $p$-group, in which case $\Sigma_+^\infty BG$ is $p$-complete and the effect of $p$-completing $\Sigma_+^\infty BG$ is exactly to $p$-complete the $S^0$-term.

It has been shown (Lewis-May-McClure) that in this case the difference between $\{BS_+, BG_+\}$ and $\{BS_+, BG_+\}_p^\wedge$ is that the $\mathbb{Z}$-term corresponding to $\{S^0_+, S^0_+\}$ in the former becomes a $\mathbb{Z}_p^\wedge$-term in the latter.