

# Very Slowly Varying Functions

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## Abstract

A real-valued function  $f$  of a real variable is said to be  $\varphi$ -slowly varying ( $\varphi$ -s.v.) if  $\lim_{x \rightarrow \infty} \varphi(x) [f(x + \alpha) - f(x)] = 0$  for each  $\alpha$ . It is said to be uniformly  $\varphi$ -slowly varying (u.  $\varphi$ -s.v.) if  $\lim_{x \rightarrow \infty} \sup_{\alpha \in I} \varphi(x) |f(x + \alpha) - f(x)| = 0$  for every bounded interval  $I$ .

It is supposed throughout that  $\varphi$  is positive and increasing. It is proved that if  $\varphi$  increases rapidly enough, then every  $\varphi$ -s.v. function  $f$  must be u.  $\varphi$ -s.v. and must tend to a limit at  $\infty$ . Regardless of the rate of increase of  $\varphi$ , a measurable function  $f$  must be u.  $\varphi$ -s.v. if it is  $\varphi$ -s.v. Examples of pairs  $(\varphi, f)$  are given that illustrate the necessity for the requirements on  $\varphi$  and  $f$  in these results.

## Introduction

The theory of slowly varying functions plays a role in analysis and number theory and has recently come to the fore in probability theory [3]. We consider here some simple, but basic questions about slowly varying functions. We prove four theorems and a lemma.

### I. Statement of Results

Let  $\varphi$  be a positive non-decreasing real-valued function defined on  $[0, \infty)$  and let  $f$  be any real-valued (not necessarily measurable) function defined on  $[0, \infty)$ . The object of this paper is to study the condition

$$\text{for every } \alpha, \quad \varphi(x) [f(x + \alpha) - f(x)] \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (1.1)$$

Whenever (1.1) holds, we will say that  $f$  is  $\varphi$ -slowly varying, and abbreviate this by  $\varphi$ -s.v. If (1.1) holds uniformly for  $\alpha$  in each bounded interval, then we say that  $f$  is uniformly  $\varphi$ -slowly varying (u.  $\varphi$ -s.v.). In other words,  $f$  is u.  $\varphi$ -s.v. if

$$\lim_{x \rightarrow \infty} \sup_{\alpha \in I} \varphi(x) |f(x + \alpha) - f(x)| = 0 \quad \text{for each bounded interval } I.$$

Throughout this paper, the words 'measurable' and 'measure' refer to Lebesgue measure.

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Of course, if  $f$  is u. $\varphi$ -s.v. then it is  $\varphi$ -s.v. The converse is 'almost' true.

**THEOREM 1.** *If  $f$  is  $\varphi$ -slowly varying and measurable, then  $f$  is uniformly  $\varphi$ -slowly varying.*

**THEOREM 2.** *If  $f$  is  $\varphi$ -slowly varying and if  $\varphi$  satisfies*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty, \quad (1.2)$$

*then  $f$  tends to a finite limit at  $\infty$ . Conversely, if*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} = \infty, \quad (1.3)$$

*then there is a continuous function  $f$  (whose choice depends on  $\varphi$ ) that is  $\varphi$ -slowly varying (and, hence, uniformly  $\varphi$ -slowly varying by Theorem 1), but that does not tend to a limit, finite or infinite, at  $\infty$ .*

**THEOREM 3.** (a) *If  $f$  is  $\varphi$ -slowly varying and if  $\varphi$  satisfies*

$$\varphi(x) \sum_{j=0}^{\infty} \frac{1}{\varphi(x+j)} \leq B < \infty \quad \text{for all } x \geq 0, \quad (1.4)$$

*or, equivalently,*

$$\varphi(x) \int_x^{\infty} \frac{dt}{\varphi(t)} \leq C < \infty \quad \text{for all } x \geq 0, \quad (1.4)'$$

*then  $f$  is uniformly  $\varphi$ -slowly varying.*

(b) *Conversely, if  $\varphi$  does not satisfy (1.4), then there is a function  $f=f(\varphi)$  which is  $\varphi$ -s.v. but not uniformly  $\varphi$ -s.v.<sup>3)</sup>*

The proof of the first part of Theorem 3 may be easily modified to prove the next result.

**THEOREM 3'.** *If  $f$  is  $\varphi$ -slowly varying and if  $\psi$  is a positive increasing function on  $[0, \infty)$  such that  $\varphi/\psi$  is increasing, then  $f$  is uniformly  $\varphi/\psi$ -slowly varying provided*

<sup>3)</sup> The completion of this half of the theorem, together with Theorem 4, was inspired by a note communicated to us by Tord Ganelius [5].

that

$$\varphi(x) \sum_{j=0}^{\infty} \frac{1}{\varphi(x+j)} \leq B\psi(x) \quad (1.5)$$

for all  $x \geq 0$  and some finite constant  $B$ .

The following result shows that the more strongly (1.4) fails, the more disjoint become the conditions of slowly varying and of uniformly slowly varying.

**THEOREM 4.** *If*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} = \infty,$$

*then there is a function  $f=f(\varphi)$  which is  $\varphi$ -slowly varying, but not even uniformly 1-slowly varying.*

The changes of variables  $h=e^{-x}$ ,  $a=e^{-\alpha}$ ,  $f(x)=g(e^{-x})$ ,  $\eta(h)=1/\varphi(\log 1/h)$  convert condition (1.1) to

$$\text{for every } a > 0, \quad \frac{g(ah) - g(h)}{\eta(h)} \rightarrow 0 \quad \text{as } h \rightarrow 0+ \quad (1.6)$$

and conditions (1.2) and (1.4) respectively, to

$$\sum_{n=1}^{\infty} \eta(e^{-n}) < \infty$$

and

$$\frac{1}{\eta(e^{-x})} \sum_{k=0}^{\infty} \eta(e^{-x-k}) \leq B < \infty.$$

From (1.6), we see that for studying differentiation theory, the function  $\varphi(x)=e^x$ , which corresponds to  $\eta(h)=h$ , is of special import. In fact, Theorem 2 with  $\varphi(x)=e^x$  provides a negative answer to question (c) on page 501 of [1]. Another change of variables converts our study to that of multiplicatively slowly oscillating functions – we omit the details (see [6], p. 79). The next lemma supplies an affirmative answer to question (b) on page 501 of [1].

**LEMMA 1.** *The function  $f$  is  $\varphi$ -slowly varying if it satisfies the apparently weaker condition*

$$\left. \begin{aligned} &\text{for each } \lambda \text{ belonging to a set } E \text{ of positive measure,} \\ &\varphi(x) [f(x+\lambda) - f(x)] \rightarrow 0 \quad \text{as } x \rightarrow \infty. \end{aligned} \right\} \quad (1.7)$$

## II. Proofs of Results

*Proof of Theorem 1.* We give a slight variation on the proof given in [6; pp. 81–82] for the case  $\varphi(x) \equiv 1$ . We assume that  $f$  is measurable and  $\varphi$ -s.v. For simplicity, we will prove that

$$\lim_{x \rightarrow \infty} \sup_{\alpha \in [0, 1]} \varphi(x) |f(x + \alpha) - f(x)| = 0. \quad (2.1)$$

Supposing, by way of contradiction, that (2.1) fails, there is a  $\delta > 0$ , and there exist sequences  $\{x_n\}$  and  $\{\alpha_n\}$  such that  $x_n \rightarrow \infty$  and  $\alpha_n \in [0, 1]$  such that for each positive integer  $n$ ,

$$\varphi(x_n) |f(x_n + \alpha_n) - f(x_n)| > \delta. \quad (2.2)$$

Let

$$V_n = \{\alpha \in [0, 2]: |f(\alpha + x_k) - f(x_k)| \varphi(x_k) < \delta/2 \text{ for all } k \geq n\},$$

$$W_n = \{\beta \in [0, 1]: |f(\beta + \alpha_k + x_k) - f(\alpha_k + x_k)| \varphi(x_k + \alpha_k) < \delta/2 \text{ for all } k \geq n\}$$

and let

$$W'_n = \alpha_n + W_n = \{\eta: \eta = \alpha_n + \beta \text{ for some } \beta \in W_n\}.$$

Since  $V_n \subseteq V_{n+1}$  and since every  $\alpha \in [0, 2]$  lies in some  $V_n$ , we have  $|V_n| > \frac{3}{2}$  if  $n$  is sufficiently large, where  $|\cdot|$  denotes Lebesgue measure. Similarly,  $|W'_n| = |W_n| > \frac{1}{2}$  if  $n$  is sufficiently large. Since  $W'_n \subseteq [0, 2]$ , we see that  $W'_n \cap V_n$  is not empty for some large  $n$ . This leads to a contradiction, since if  $\gamma \in W'_n$ , we have

$$\begin{aligned} |f(\gamma + x_n) - f(x_n)| \varphi(x_n) &\geq |f(\alpha_n + x_n) - f(x_n)| \varphi(x_n) \\ &\quad - \left\{ |f(\gamma + x_n) - f(\alpha_n + x_n)| \varphi(x_n + \alpha_n) \frac{\varphi(x_n)}{\varphi(x_n + \alpha_n)} \right\} \\ &> \delta - \delta/2 = \delta/2 \end{aligned}$$

so that  $\gamma$  cannot belong to  $V_n$ .

*Proof of Theorem 2.* We begin with the proof of the first assertion, and suppose that  $\sum 1/\varphi(n) < \infty$ . If  $f$  satisfies (1.1), then  $f$  cannot tend to an infinite limit at  $\infty$ , since for every positive integer  $n$ ,

$$|f(n)| \leq |f(1)| + \sum_{k=1}^{n-1} |f(k+1) - f(k)| \leq |f(1)| + B \sum_{k=1}^{\infty} 1/\varphi(k) < \infty.$$

Therefore, if  $f$  does not have a finite limit at  $\infty$ , we may assume without loss of generality that

$$\limsup_{x \rightarrow \infty} f(x) > 1 \quad \text{and} \quad \liminf_{x \rightarrow \infty} f(x) < -1, \quad (2.3)$$

since we could otherwise replace  $f$  by  $cf + d$  for suitable constants  $c$  and  $d$ . Since  $f$  is

$\varphi$ -s.v., we see in particular that

$$\varphi(x) |f(x+1) - f(x)| < 1 \quad (2.4)$$

if  $x$  is sufficiently big, say  $x \geq M$ . Also, since  $\sum 1/\varphi(n)$  converges, we have

$$\sum_{n=[x]}^{\infty} \frac{1}{\varphi(n)} < \frac{1}{2} \quad (2.5)$$

if  $x \geq N$ , say. By (2.3) we may find two numbers  $x$  and  $y$  with  $x > y$  and

$$x > \max(M, N), \quad f(x) > 1$$

and

$$y > \max(M, N), \quad f(y) < -1.$$

This leads to a contradiction since on the one hand

$$|f(x+n) - f(y+n)| = \left| \frac{\varphi(y+n) [f(y+n+(x-y)) - f(y+n)]}{\varphi(y+n)} \right| \leq 1$$

for  $n$  a sufficiently large positive integer, while on the other hand, for any positive integer  $n$ ,

$$\begin{aligned} f(x+n) &= f(x) + \sum_{k=1}^n [f(x+k) - f(x+k-1)] > f(x) - \sum_{k=1}^n \frac{1}{\varphi(x+k-1)} \\ &\geq f(x) - \sum_{k=[x]}^{\infty} \frac{1}{\varphi(k)} > 1 - \frac{1}{2} = \frac{1}{2}, \end{aligned}$$

and similarly  $f(y+n) < -\frac{1}{2}$ , so that  $f(x+n) - f(y+n) > 1$ .

To prove the second half of Theorem 1, let a non-decreasing positive function  $\varphi$  be given that satisfies (1.3), namely,  $\sum 1/\varphi(n) = \infty$ . We will construct a continuous function  $f$  that is  $\varphi$ -s.v. and that satisfies

$$\lim_{x \rightarrow \infty} \sup f(x) = +\infty, \quad \lim_{x \rightarrow \infty} \inf f(x) = -\infty. \quad (2.6)$$

Let  $A = A(\varphi)$  be the set of positive integers  $m$  satisfying  $\varphi(m+1) \leq 2\varphi(m)$ . By (1.3), we have

$$\sum_{n \in A} \frac{1}{\varphi(n)} = \sum_{n=1}^{\infty} \frac{1}{\varphi(n)} - \sum_{n \notin A} \frac{1}{\varphi(n)} = \infty \quad (2.7)$$

since

$$\sum_{n \notin A} \frac{1}{\varphi(n)} \leq \frac{1}{\varphi(1)} + \frac{1}{2} \frac{1}{\varphi(1)} + \frac{1}{2^2} \frac{1}{\varphi(1)} + \cdots = \frac{2}{\varphi(1)} < \infty.$$

In particular,  $A$  is infinite, and we write  $A = \{m_1, m_2, m_3, \dots\}$ . Now there are positive constants  $a_i$  with  $a_{i+1} < a_i$  for  $i = 1, 2, 3, \dots$  and  $a_i \varphi(m_i) \rightarrow 0$  as  $i \rightarrow \infty$  and  $\sum_{i=1}^{\infty} a_i = \infty$  (see [2], p. 47). We now define a sequence  $\{b_i\}$  by  $b_i = \pm a_i$ , where the signs are chosen in blocks so that  $\sum b_i$  has both  $+\infty$  and  $-\infty$  as limits of subsequences of its partial sums. We define  $f$  by  $f = 0$  on  $[0, m_1]$ ,  $f = b_1$  on  $[m_1 + 1, m_2]$ ,  $f = b_1 + b_2$  on  $[m_2 + 1, m_3]$ ,  $\dots$ ,  $f = b_1 + b_2 + \dots + b_k$  on  $[m_k + 1, m_{k+1}]$ ,  $\dots$ , and extend  $f$  to be linear and continuous on each interval  $[m_k, m_{k+1}]$ ,  $k = 1, 2, 3, \dots$ . It is clear that (2.6) holds. To verify that  $f$  is  $\varphi$ -s.v., we note that

$$\left. \begin{aligned} \varphi(x) |f(x + \alpha) - f(x)| &= \varphi(x) |f(x + \alpha) - f([x + \alpha] + 1) \\ &\quad + \sum_{i=0}^{[x + \alpha] - [x]} \{f([x] + i + 1) - f([x] + i)\} \\ &\quad + f([x]) - f(x)| \leq \\ &|f([x + \alpha]) - f([x + \alpha] + 1)| \varphi([x + \alpha] + 1) \\ &\quad + \sum_{i=0}^{[x + \alpha] - [x]} |f([x] + i + 1) \\ &\quad - f([x] + i)| \varphi([x] + i + 1) \\ &\quad + |f([x]) - f([x] + 1)| \varphi([x] + 1), \end{aligned} \right\} \quad (2.8)$$

since  $f$  is monotone between consecutive integers and  $\varphi$  is non-decreasing. For fixed  $\alpha$ , there are at most  $[\alpha] + 4$  terms on the right hand side of (2.8), and as  $x \rightarrow \infty$ , each term tends to 0 since

$$|f(m) - f(m + 1)| \varphi(m + 1) = \begin{cases} 0 & \text{if } m \notin A \\ a_k \varphi(m_k + 1) & \text{if } m = m_k \in A \end{cases}$$

and

$$a_k \varphi(m_k + 1) = \frac{\varphi(m_k + 1)}{\varphi(m_k)} a_k \varphi(m_k) \leq 2a_k \varphi(m_k),$$

which tends to 0 as  $k \rightarrow \infty$ .

*Proof of Theorem 3(a).* We prove a stronger result than asserted, using the same idea we used to prove Theorem 2. Namely, we prove that if  $f$  is  $\varphi$ -s.v. and if  $\varphi$  satisfies (1.4), then

$$\lim_{x \rightarrow \infty} \sup_{\alpha \geq 0} \varphi(x) |f(x + \alpha) - f(x)| = 0. \quad (2.9)$$

Since (1.4) implies (1.2), we know by Theorem 2 that  $f$  tends to a finite limit  $L$  at  $\infty$ . It follows from (2.9), on letting  $\alpha \rightarrow \infty$ , that

$$\lim_{x \rightarrow \infty} \varphi(x) |f(x) - L| = 0. \quad (2.10)$$

For the proof of (2.9), suppose it is false. Then we can find  $\delta > 0$  and arbitrarily large

$x$  such that for  $\alpha = \alpha(x) \geq 0$  we have

$$\left| f(x + \alpha + k) - f(x + k) \right| = \left| \sum_{j=0}^{k-1} \{f(x + \alpha + j + 1) - f(x + \alpha + j)\} - \sum_{j=0}^{k-1} \{f(x + j + 1) - f(x + j)\} + f(x + \alpha) - f(x) \right| \quad (2.11)$$

$$\geq \frac{\delta}{\varphi(x)} - \sum_{j=0}^{\infty} \frac{\varepsilon(x + \alpha + j) + \varepsilon(x + j)}{\varphi(x + j)},$$

where  $\varepsilon(y) = \varphi(y)|f(y+1) - f(y)|$ , which tends to 0 as  $y \rightarrow \infty$ . Now choose  $x$  so large in (2.11) that  $\varepsilon(y) < \delta/4B$  for  $y \geq x$ , to get  $\varphi(x+k)|f(x+\alpha+k) - f(x+k)| > \delta/2$ , which contradicts the hypothesis that  $f$  is  $\varphi$ -s.v., since  $(x+\alpha+k) - (x+k) = \alpha$ , which is independent of  $k$ .

*Proof of Theorem 3(b).* From the geometrically evident identity

$$\int_x^{\infty} \frac{dt}{\varphi(t)} \leq \sum_{j=0}^{\infty} \frac{1}{\varphi(x+j)} \leq \frac{1}{\varphi(x)} + \int_x^{\infty} \frac{dt}{\varphi(t)},$$

it follows that (1.4) and (1.4)' are equivalent. Assume now that (1.4)' fails. Let  $\{\beta_\lambda\}$  be a Hamel basis for the real numbers, i.e., every real number  $x$  has a unique representation  $x = \sum_{k=1}^n r_k \beta_{\lambda_k}$  with a finite number  $n = n(x)$  of non-zero rationals  $\{r_k\}$ . Evidently,  $|n(x+\alpha) - n(x)| \leq n(\alpha)$ . One may easily construct a function  $\psi \downarrow 0$  such that also

$$\limsup_{x \rightarrow \infty} \varphi(x) \int_x^{\infty} \frac{\psi(t)}{\varphi(t)} dt = \infty. \quad (2.12)$$

Let

$$f(x) = \int_0^{x+n(x)-1} \frac{\psi(t)}{\varphi(t)} dt.$$

If  $\alpha$  is fixed, then, since  $\psi/\varphi \downarrow$  and both limits of integration are greater than  $x$ ,

$$\begin{aligned} \varphi(x) |f(x+\alpha) - f(x)| &= \varphi(x) \left| \int_{x+n(x)-1}^{x+\alpha+n(x+\alpha)-1} \frac{\psi(t)}{\varphi(t)} dt \right| \\ &\leq \varphi(x) \cdot \frac{\psi(x)}{\varphi(x)} \cdot |\alpha + n(x+\alpha) - n(x)| \leq \psi(x) (\alpha + n(\alpha)), \end{aligned}$$

which tends to 0 as  $x$  tends to infinity so that  $f$  is  $\varphi$ -s.v. But  $f$  is not uniformly  $\varphi$ -s.v. In fact,

$$\limsup_{x \rightarrow \infty} \left( \sup_{\alpha \in [0, 1]} \varphi(x) |f(x + \alpha) - f(x)| \right) = \infty.$$

To see this, let  $M > 0$  be given. Pick  $y_0 > M$  such that

$$\varphi(y_0) \int_{y_0}^{\infty} \frac{\psi(t)}{\varphi(t)} dt > M.$$

Pick  $y_1 > y_0$  such that  $n(y_1) = 1$  and so close to  $y_0$  that

$$\varphi(y_1) \int_{y_1}^{\infty} \frac{\psi(t)}{\varphi(t)} dt > M$$

also. (This can be done since all the members of the dense set  $\{r\beta_{\lambda_1} : r \text{ is rational}\}$  satisfy  $n=1$ .) Finally, pick  $\alpha \in [0, 1]$  so that  $n(y_1 + \alpha)$  is so big that

$$\varphi(y_1) \int_{y_1 + n(y_1) - 1}^{y_1 + \alpha + n(y_1 + \alpha) - 1} \frac{\psi(t)}{\varphi(t)} dt = \varphi(y_1) |f(y_1 + \alpha) - f(y_1)|$$

is also greater than  $M$ . This shows the lim sup to be greater than (an arbitrarily chosen)  $M$  and hence infinite.

*Proof of Theorem 4.* The proof proceeds essentially as the proof of 3(b) above, so we will be brief. Equivalent to our assumption is the equality  $\int_x^{\infty} dt/\varphi(t) = \infty$ . Choose  $\psi \downarrow 0$  such that  $\int_x^{\infty} \psi(t)/\varphi(t) dt = \infty$ . Define

$$f(x) = \int_0^{x+n(x)} \frac{\psi(t)}{\varphi(t)} dt.$$

For fixed  $\alpha$  we have

$$\varphi(x) |f(x + \alpha) - f(x)| = \varphi(x) \left| \int_{x+n(x)}^{x+\alpha+n(x+\alpha)} \frac{\psi(t)}{\varphi(t)} dt \right| \leq \varphi(x) \cdot \frac{\psi(x)}{\varphi(x)} \cdot (\alpha + n(\alpha))$$

which tends to 0; while for each  $x$

$$\sup_{\alpha \in [0, 1]} |f(x + \alpha) - f(x)| = \sup_{\alpha \in [0, 1]} \left| \int_{x+n(x)}^{x+\alpha+n(x+\alpha)} \frac{\psi(t)}{\varphi(t)} dt \right| = \infty$$

since  $n(x + \alpha)$  may be arbitrarily large.



*Proof of Lemma 1.* Assume that (1.7) holds and that  $\lambda, \mu \in E$  with  $\lambda > \mu$ . We must prove that (1.1) holds. First we have the inequality

$$\begin{aligned} \varphi(x) |f(x + \lambda - \mu) - f(x)| &= \left| -\frac{\varphi(x)}{\varphi(x + \lambda - \mu)} \varphi(x + \lambda - \mu) \{f(x + \lambda) \right. \\ &\quad \left. - f(x + \lambda - \mu)\} + \varphi(x) \{f(x + \lambda) - f(x)\} \right| \\ &\leq \varphi(x + \lambda - \mu) |f((x + \lambda - \mu) + \mu) - f(x + \lambda - \mu)| + \varphi(x) |f(x + \lambda) - f(x)|. \end{aligned}$$

Then we apply Steinhaus' Theorem (see [4; p. 68] or [8; pp. 97-99]) that the difference set of a set of positive measure contains an open interval that contains 0, to deduce that (1.1) holds for all sufficiently small  $\alpha$ . Now repeated application of the inequality

$$\begin{aligned} \varphi(x) |f(x + 2\alpha) - f(x)| &\leq \varphi(x + \alpha) |f(x + 2\alpha) - f(x + \alpha)| \\ &\quad + \varphi(x) |f(x + \alpha) - f(x)| \end{aligned}$$

completes the proof. (See also [7; pp. 266-267], and [1; p. 493].)

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