Math Bite: Once in a While, Differentiation is Multiplicative

Many students of calculus would be a lot happier if the Leibniz formulas
\[(fg)' = f'g + fg'\] and \[\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2},\]
where \(f'\) denotes the derivative of \(f\), could be replaced by the much simpler formulas
\[(fg)' = f'g' \tag{1}\]
and
\[\left(\frac{f}{g}\right)' = \frac{f'}{g'}. \tag{2}\]

Discovering exactly when the usually erroneous equations (1) and (2) are valid is a simple but neat exercise involving three separable differential equations. Paul Zorn called my attention to the article [1], in which the function \(f\) is fixed in (1), and the one-dimensional subspace of all corresponding \(g\) is determined. Fixing \(g\) in (2) and then determining \(f\) involves essentially the same calculation, while fixing \(f\) in (2) and finding \(g\) leads to the formula
\[g(x) = C \exp \left( \int \frac{f' \pm \sqrt{(f')^2 - 4ff'}}{2f} \, dx \right). \tag{3}\]

More concrete problems arise when these formulas are used to find companions for simple concrete choices of functions. For example, setting \(f(x) = x^r\) in (3) produces
\[
\left\{ \begin{array}{c}
x^r \\
C x^{r/2} \left( \frac{r - \sqrt{r^2 - 4r}}{r + \sqrt{r^2 - 4r}} \right)^{\pm r/2} \exp \left( \pm \sqrt{r^2 - 4r}x \right) \\
C x^{r/2} \left( \frac{r - \sqrt{r^2 - 4r}}{r + \sqrt{r^2 - 4r}} \right)^{\pm r/2} \exp \left( \pm \sqrt{r^2 - 4r}x \right)
\end{array} \right\}'.
\]

Several other examples can be found in [1].

Another question is whether we can find a pair of functions \(\{p, q\}\), neither identically zero, such that at least two of the relations
\[
\begin{align*}
(i) \quad (p')' &= \frac{p'}{q'} \\
(ii) \quad (q')' &= \frac{q'}{p'} \\
(iii) \quad (pq)' &= p'q'
\end{align*}
\]
hold simultaneously. Any such pair solving (iii) satisfies neither (i) nor (ii), and there are very few simultaneous solutions to (i) and (ii), namely the pairs
\[
\left\{ ce^{\frac{x}{2}+x}, \quad de^{\frac{x}{2}+x} \right\},
\]
where \( i = \sqrt{-1} \) and \( c \) and \( d \) are arbitrary non-zero constants.

**Remarks.** An easy related exercise for beginning calculus students is to find all pairs of polynomials \((f, g)\) such that \((fg)' = f'g'\). There are many similar questions. For example, given \( f \) and \( g \), it is easy to find all functions \( h \) such that \((fgh)' = f'g'h'\).

My interest in this question was motivated by Exercise 2 on page 545 of [2].

**References**


— J. Marshall Ash
DePaul University
Chicago, IL 60614