Approximate and $L^p$ Peano derivatives of non-integral order

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Abstract. Let $n$ be a nonnegative integer and let $u \in (n, n+1]$. We say that $f$ is $u$-times Peano bounded in the approximate (resp. $L^p$, $1 \leq p \leq \infty$) sense at $x \in \mathbb{R}^m$ if there are numbers $\{f_\alpha(x)\}$, $|\alpha| \leq n$ such that $f(x+h) - \sum_{|\alpha| \leq n} f_\alpha(x) h^\alpha / \alpha!$ is $O(h^u)$ in the approximate (resp. $L^p$) sense as $h \to 0$. Suppose $f$ is $u$-times Peano bounded in either the approximate or $L^p$ sense at each point of a bounded measurable set $E$. Then for every $\epsilon > 0$ there is a perfect set $\Pi \subset E$ and a smooth function $g$ such that the Lebesgue measure of $E \setminus \Pi$ is less than $\epsilon$ and $f = g$ on $\Pi$. The function $g$ may be chosen to be in $C^n$ when $u$ is integral, and, in any case, to have for every $j$ of order $\leq n$ a bounded $j$th partial derivative that is Lipschitz of order $u - |j|$.

Pointwise boundedness of order $u$ in the $L^p$ sense does not imply pointwise boundedness of the same order in the approximate sense. A classical extension theorem of Calderón and Zygmund is confirmed.

1. Introduction

Throughout this paper $n$ denotes a fixed nonnegative integer, and $u$ a real number in $(n, n+1]$. All functions will be defined on subsets of $m$ dimensional Euclidean space and will be real valued.

Definition 1. We say that $f$ is $u$-times approximately Peano bounded at $x$ if $f$ is Lebesgue measurable and for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$
all $\alpha_i$ being nonnegative integers and order of $\alpha$, $|\alpha| = \sum_{i=1}^{n} \alpha_i \leq n$ there is a number $f_{\alpha}(x)$ such that

$$f(x + h) = \sum_{|\alpha| \leq n} \frac{h^\alpha}{\alpha!} f_{\alpha}(x) + M_x(x + h)||h||^u$$

where $||h||$ denote Euclidean distance in $\mathbb{R}^m$, $h^\alpha = h_1^{\alpha_1} \cdots h_m^{\alpha_m}$, $\alpha! = \alpha_1! \cdots \alpha_m!$, $f_0(x) = f(x)$ and $M_x(x + h)$ remains bounded as $h \to 0$ through a set of density 1 at $h = 0$. The set $A$ has density 1 at $x$ (equivalently, $x$ is a point of density of $A$) if $A$ is Lebesgue measurable and $\lim_{r \to 0} \frac{\lambda(A \cap B(x, r))}{\lambda(B(x, r))} = 1$, where $B(x, r)$ denotes the closed ball of radius $r$ centered at $x$ and $\lambda$ denotes Lebesgue measure.

**Definition 2.** A function, $g : \mathbb{R}^m \to \mathbb{R}$ is in the class $B_u$, if for every multi-index $j$ with $0 \leq |j| \leq n$, we have $g^j(x)$ a bounded function of Lipschitz class $u - |j|$. The functions $g^j$ are the ordinary partial derivatives of $g$, i.e.,

$$g^j(x) = \frac{\partial^j}{\partial x_1^{j_1} \partial x_2^{j_2} \cdots \partial x_m^{j_m}} g(x).$$

The main result is this.

**Theorem 1.** Suppose $f$ is $u$-times approximately Peano bounded on a bounded measurable set $E$. Then for every $\epsilon > 0$ there is a perfect set $\Pi \subset E$ and a $B_u$ function $g$ such that $\lambda(E \setminus \Pi) < \epsilon$ and $f = g$ on $\Pi$. Furthermore if $u = n + 1$, then $g$ can be chosen to belong to $C^{n+1}$.

A weaker version of this theorem specialized to dimension $m = 1$ and $u = n + 1 \in \mathbb{Z}$ was established by Marcinkiewicz [3], [7, vol. II, p. 73]. Marcinkiewicz’s result has had many applications. In the hope that our theorems will also prove useful, we try to increase visibility by giving an equivalent statement of Theorem 1 using the language of decomposition: Suppose $f$ is $u$-times approximately Peano bounded at all $x \in E$, where $E$ is a bounded measurable set. Then for every $\epsilon > 0$ there are functions $g$ and $h$ such that

$$f(x) = g(x) + h(x),$$

$$g(x) \in B_u,$$

and

the measure of (support of $h(x)) \cap E$ is $< \epsilon$.

The same result holds in $L^p$ norm. Explicitly, for each $p \in [1, \infty]$, we have the following definition.
DEFINITION 3. We say that $f$ is $u$-times Peano bounded in the $L^p$ sense at $x$ if $f$ is $L^p$ in a neighborhood of $x$ and there are numbers $f_s(x)$ such that

$$
\left( \frac{1}{h^m} \int_{|t| \leq h} \left| f(x + t) - \sum_{|s| \leq n} \frac{t^s}{s!} f_s(x) \right|^p dt \right)^{1/p} = L_x(h)\|h\|^u ;
$$

where $L_x(h)$ remains bounded as $h$ tends to zero. When $p = \infty$, the left side of (1.1) means, as usual, $	ext{esssup}_{|t| \leq h} \left| f(x + t) - \sum_{|s| \leq n} \frac{t^s}{s!} f_s(x) \right|$.

If, further, $f \in L^p(\mathbb{R}^m)$ and $L_x(h)$ is uniformly bounded for all $h$, in reference [1] the function $f$ is then said to belong to $T^p_u(x)$.

THEOREM 2. Suppose $f$ is $u$-times Peano bounded in the $L^p$ sense on a bounded measurable set $E$. Then for every $\epsilon > 0$, there is a perfect set $\Pi \subset E$ and a $B_u$ function $g$ such that $\lambda(E \setminus \Pi) < \epsilon$ and $f = g$ on $\Pi$. Furthermore if $u = n + 1$, then $g$ can be chosen to belong to $C^{m+1}$.

One might think that this Theorem is an immediate consequence of the “folklore fact” that for $p \in [1, \infty)$, if $\limsup_{h \to 0} \left( h^{-m} \int_{|t| \leq h} |g(t)|^p dt \right)^{1/p} h^{-u}$ is $\leq M$, as $h \to 0$, then the approximate lim sup of $|g(h)| \|h\|^{-u}$ is also less than or equal to $M$. Actually, this is not true, as we will point out in the first part of the next section wherein the relation between $L^p$ and approximate differential behavior is discussed. The failure of this “fact” requires us to adjoin an additional final section for the $L^p$ case.

In [1], Theorem 9 Part I, the authors prove that if $f \in T^p_u(x)$ uniformly for all $x$ in a closed set, then it is a restriction of a $B_u$ function. In this paper we will show that if $f$ is $L^p$ $u$-times Peano bounded not necessarily uniformly on a compact set $E$, then $E$ is a union of a sequence of nested closed sets $A_k$ so that on each $A_k$, $f$ is a restriction of a $B_u$ function. The result from [1] is a special case of our results because under the hypotheses of the corresponding Theorem 9 in [1], we have $E = A_k$ for some integer $k$.

The second part of Theorem 9 of [1] asserts that under certain additional assumptions $B_u$ can be replaced by $b_u$ in the conclusion. (See Subsection 2.2, Definition 6 below for the definition of $b_u$.) Actually this is not true as we will point out in Subsection 2.2. Our results below show that this was not a very serious defect in the overall program developed in the paper [1]. For example, both [1, Theorem 13] and its given proof are fine if, in the proof, one uses our Theorems 5 and 1 in place of [1, Theorem 9, second part].

Let $h \in [0, 1]$. The condition $\limsup_{h \rightarrow 0} |f(h)| < \infty$ is equivalent to the condition that $\lim_{h \rightarrow 0} f(h)\epsilon(h) = 0$ for every nondecreasing function
\( \epsilon(h) \) satisfying \( \lim_{h \to 0} \epsilon(h) = 0 \). In subsection 2.1 we show that this equivalence fails for approximate limits and that this failure is responsible for the breakdown of the “folklore fact” mentioned above.

2. Two “big oh” and “little oh” comparisons

2.1. Connections between \( L^p \) and approximate behavior. There has been an idea in the folklore of analysis that approximate behavior is always more general than \( L^1 \) behavior. An example on which this notion is based is the fact that if a function is differentiable at a point in the \( L^1 \) sense, then it is differentiable in the approximate sense at that point. This section contains three theorems: the first supports the folklore, the second contradicts it, while the third supports it again. The first says that if a function’s rate of growth near a point is \( \ll \|h\|^u \) in the \( L^p \) sense, then its rate of growth must also be \( \ll \|h\|^u \) in the approximate sense; the second says that if a function’s rate of growth near every point of a set is \( O(\|h\|^u) \) in the \( L^p \) sense, then at almost every point of that set its rate of growth must also be \( O(\|h\|^u) \) in the approximate sense.

Abbreviate \( f \mid_{R^m : P(x)} \to f \mid_{P(x)} \).

Definition 4. We say that \( \lim_{\|x\| \to 0} f(x) = M \) if there is a set \( E \subset R^m \) so that 0 is a point of density of \( E \) and \( \lim_{\|x\| \to 0, x \in E} f(x) = M \). Zero is a point of dispersion of a set \( F \) if \( \lim_{h \to 0} \frac{\lambda^*(E \cap B(0,h))}{\lambda(B(0,h))} = 0 \), where \( \lambda^* \) denotes outer Lebesgue measure. We say that \( \limsup_{\|x\| \to 0} f(x) = M \) if for every \( N > M \), 0 is a point of dispersion of \( \{f(x) > N\} \) and \( M \) is the infimum of all \( N \) with this property. We say that \( \liminf_{\|x\| \to 0} f(x) = M \) if for every \( N < M \), 0 is a point of dispersion of \( \{f(x) < N\} \) and \( M \) is the supremum of all \( N \) with this property.

The definitions of \( \limsup_{\|x\| \to 0} \) and \( \liminf_{\|x\| \to 0} \) can be found on page 218 of [4] and the definition of \( \lim_{\|x\| \to 0} \) can be found on page 323 of [7]. For measurable functions we have \( \liminf_{\|x\| \to 0} f(x) = \limsup_{\|x\| \to 0} f(x) = M \) if and only if \( \lim_{\|x\| \to 0} f(x) = M \).

Theorem 3. Let \( g \) have an \( n \)th \( L^p \) Peano derivative at \( x \in R^m \) so that \( f(t) = \left| g(x + t) - \sum_{|j| \leq n} g_j(x) t^j \right| \) satisfies

\[
\frac{1}{h^m} \int_{B(0,h)} f^p = o(h^{np})
\]

as \( h \to 0 \). Then \( g \) also has an \( n \)th approximate derivative at \( x \), in other words, \( \lim_{\|t\| \to 0} \frac{f(t)}{\|t\|^u} = 0 \).
Proof. We have \( \epsilon_N \to 0 \), where \( \epsilon_N \) is defined by
\[
\frac{1}{2^{-N_m}} \int_{\|x\| \leq 2^{-N}} f^p = \epsilon_N^2 2^{-npN}.
\]
Let \( I_N = B(0,2^{-N}) \setminus B(0,2^{-N-1}) \), and let \( E_N \) be defined by
\[
E_N = \{ x \in I_N : f^p(x) \geq \epsilon_N 2^{-npN} \}.
\]
From
\[
\epsilon_N^2 2^{-(np+m)N} \geq \int_{I_N} f^p \geq \int_{E_N} \epsilon_N 2^{-npN} = \epsilon_N 2^{-npN} \lambda(E_N),
\]
it follows that
\[
\epsilon_N \geq \frac{\lambda(E_N)}{2^{-N_m}} = c_m \frac{\lambda(E_N)}{\lambda(I_N)},
\]
so 0 is a point of density of \( \cup E_N^c = G \) and
\[
\lim_{\|t\| \to 0, t \in G} \frac{f(t)}{\|t\|^n} = 0.
\]
\[
\square
\]

The above proof is a routine adaptation of a \( p = 2 \) one dimensional argument given on page 324 of [7] and is only worth mentioning because of the following example. For the example we specialize to \( m = 1 \), \( p = 1 \), and \( f \) supported in \([0,1]\).

**Theorem 4.** There is a function \( f \) satisfying
\[
\frac{1}{h} \int_0^h f = O(h)
\]
as \( h \) goes to 0 such that for every finite number \( M \), \( \limsup_{x \to 0} \frac{f(x)}{x} > M \).

Proof. We give an example of a nonnegative function \( f \) satisfying
\[
(2.1) \quad \frac{1}{h} \int_0^h f = O(h)
\]
such that \( \limsup_{h \to 0} \frac{f(h)}{h} \) is infinite. It is sufficient to prove that for every positive integer \( j \), \( \{ f(x) \geq jx \} \) does not have 0 as a point of dispersion. Let \( e_j^k, j = 1,2,\ldots,k \) be disjoint subintervals of \([2^{-k-1},2^{-k}]\), such that \( \lambda(e_j^k) = 2^{-k-j-1} \). Let
\[
f(x) = \sum_{k=0}^\infty \sum_{j=1}^k j 2^{-k} \chi_{e_j^k}(x).
\]
Then
\[
\int_0^{2^{-N}} f(x)dx = \sum_{k=0}^\infty \sum_{j=1}^k j 2^{-k} \lambda(e_j^k) = \sum_{k=0}^\infty \sum_{j=1}^k j 2^{-k} 2^{-k-j-1} \leq
\]
\[
\sum_{k=0}^\infty \sum_{j=1}^k j 2^{-k-j-1} \leq
\]
\[
\sum_{k=0}^\infty \sum_{j=1}^k j 2^{-k} 2^{-k-j-1} \leq
\]
Thus, for any positive integer $H$.

Hence

Thus,

$$e_j^k \subset \{ x : f(x) \geq jx \} \cap [0, 2^{-k}].$$

Hence

$$\lambda(\{ x : f(x) \geq jx \} \cap [0, 2^{-k}]) \geq \lambda(e_j^k) = 2^{-k-j-1}.$$ 

Thus, for any positive integer $j$ and for any $k \geq j$

$$\frac{\lambda(\{ x : f(x) \geq jx \} \cap [0, 2^{-k}])}{\lambda([0, 2^{-k}])} \geq 2^{-j-1}.$$ 

Hence, for any positive integer $j$, zero is not a point of dispersion for the set $\{ f(x) \geq jx \}$. 

Let $h \in [0, 1]$. The condition $\limsup_{h \searrow 0} |f(h)| < \infty$ is equivalent to the condition that $\lim_{h \searrow 0} f(h) \epsilon(h) = 0$ for every nondecreasing function $\epsilon(h)$ satisfying $\lim_{h \searrow 0} \epsilon(h) = 0$. We use the example from the previous theorem to show that this equivalence fails for approximate limits:

**Proposition 1.** Assume that for any nondecreasing function $\epsilon(x)$ on $(0,1]$ such that $\lim_{h \to 0} \epsilon(h) = 0$, we have

$$\lim_{x \to 0} \text{limsup} f(x) \epsilon(x) = 0.$$ 

Then it does not follow that there must exist a constant $M$ so that $0$ is a point of dispersion of $\{ x : f(x) \geq M \}$. Consequently $f$ need not have a finite limsupap at $x = 0$.

**Proof.** Let $f$ be the example function just above and let $g(x) := f(x)/x$. We’ve already shown that there does not exist a constant $M$ so that $\{ x : g(x) \geq M \}$ has 0 as a point of dispersion. Let $\epsilon(x)$ be a nondecreasing function on $(0,1]$ such that $\lim_{h \to 0} \epsilon(h) = 0$. Let $\zeta, \eta > 0$. Pick $k$ so large that $\sum_{i=1}^{k} 2^{-i} > 1 - \zeta$. Then pick $N$ so large that $\epsilon(2^{-N}) < \frac{\eta}{M}$. For every $M \geq N$, $\{ x \in [2^{-M-1}, 2^{-M}] : g(x) \epsilon(x) > \eta \} \subset \cup_{i=k+1}^{M} e_i M$ so that the relative density of $\{ g > \eta \}$ in $[2^{-M-1}, 2^{-M}]$ is less than $\zeta$. Hence the relative density of $\{ g \epsilon > \eta \}$ in $[0, 2^{-N}]$ is less than $\zeta$. Since $\zeta$ and $\eta$ were arbitrary, relation (2.2) holds for $g$ and $\epsilon(x)$. Since $\epsilon(x)$ was arbitrary, relation (2.2) holds for $g$ and every such $\epsilon(x)$. 

As Theorem 4 shows we cannot prove a pointwise analogue of Theorem 3 in the “big oh” case. The following corollary of Theorem 8 of the final section is a substitute.
Theorem 5. Suppose that \( \lambda(E) > 0 \) and at each \( x \in E \) we have

\[
(2.3) \quad \left( \frac{1}{|B(0, h)|} \int_{B(0, h)} \left| g(x + t) - \sum_{|j| \leq n} g_j(x) t^j \right|^p \, dt \right)^{1/p} = O(h^u).
\]

Then at a.e. \( x \in E \),

\[
\limsup_{\|h\| \to 0} \frac{|g(x + h) - \sum_{|j| \leq n} g_j(x) h^j|}{\|h\|^u} < \infty.
\]

2.2. A \( T_u^p \) extension theorem without a \( t_u^0 \) analogue. Roughly speaking, Theorem 9 of [1] asserts that a uniformly \( T_u^p \) function can be extended to a \( B_u \) function and that “similarly” a \( t_u^0 \) function can be extended to a \( b_u \) function. The proof of the former statement appearing in [1] is correct. The latter statement is false when \( u \) is an integer. To make this assertion precise, we first give Calderón and Zygmund’s definitions of \( t_u^p \) \((x_0)\), \( B_u \) \((Q)\) and \( b_u \) \((Q)\), where \( x_0 \) is a point of \( \mathbb{R}^m \) and \( Q \) is a closed set in \( \mathbb{R}^m \).

Definition 5. Let \( f \) be a function in \( T_u^p \) \((x_0)\). We shall say that \( f \in t_u^p \) \((x_0)\) if there exists a polynomial \( P(x - x_0) \) of degree \( \leq u \) such that

\[
\left( \rho^{-m} \int_{|x - x_0| \leq \rho} |f(x) - P(x - x_0)|^p \, dx \right)^{1/p} = o(\rho^u) \quad \text{as} \ \rho \to 0.
\]

Here \( 1 \leq p < \infty \).

Definition 6. Let \( Q \) be a closed set. We shall say that the bounded function \( f \in B_u \) \((Q)\), \( u > 0 \), if there exist bounded functions \( f_\alpha, |\alpha| < u \), such that

\[
f_\alpha(x + h) = \sum_{|\beta| < u - |\alpha|} f_{\alpha + \beta}(x) \frac{h^\beta}{\beta!} + R_\alpha(x, h)
\]

for all \( x \) and \( x + h \) in \( Q \), with \(|R_\alpha(x, h)| \leq C \|h\|^{u - |\alpha|} \). We say that \( f \in b_u \) \((Q)\), \( u \geq 0 \), if there exist functions \( f_\alpha, |\alpha| \leq u \), such that

\[
f_\alpha(x + h) = \sum_{|\beta| \leq u - |\alpha|} f_{\alpha + \beta}(x) \frac{h^\beta}{\beta!} + R_\alpha(x, h)
\]

for all \( x \) and \( x + h \) in \( Q \), with \(|R_\alpha(x, h)| \leq C \|h\|^{u - |\alpha|} \) and, in addition, \( R_\alpha(x, h) = o \left( \|h\|^{u - |\alpha|} \right) \) as \( h \to 0 \), uniformly in \( x \in Q \). The connection between \( B_u \) \((Q)\) and the previously defined \( B_u \) is that \( f \in B_u \) if \( f \in B_u \) \((\mathbb{R}^m)\) and, additionally for all \( j \) with \( |j| \leq n \) the Peano derivative \( f_j \) is equal to \( f^j \), the ordinary partial derivative of \( f \). Similarly we define \( f \in b_u \) to mean \( f \in b_u \) \((\mathbb{R}^m)\) and \( f^j = f_j \) whenever \( |j| \leq u \).
Now the second half of Theorem 9 of [1] asserts that if $f \in B_u(Q)$ and in addition $f \in t_u^b(x_0)$ for all $x_0$ in the closed set $Q$, then $f$ can be chosen to be in $b_u(\mathbb{R}^m)$ in such a way that $(\partial / \partial x)^{\beta} \tilde{f}(x_0) = f_{\beta}(x_0)$ for $|\beta| \leq u$, and all $x_0 \in Q$.

To see what the problem is, let $u = m = p = 1$. Then let $Q = \mathbb{R}^1$ be the entire space, so that the original function $f$ and the extension function $\tilde{f}$ coincide. Suppose that $f$ is compactly supported and has a uniformly bounded derivative which is not continuous. Then $f \in t^1_1(x_0)$ for every real $x_0$, but $f \notin b_1(\mathbb{R}^1)$ since $f'$ is not continuous.

3. Proofs: the approximate case

Recall that $n$ is the fixed nonnegative integer $[u] - 1$ so that $n < u \leq n + 1$.

Some additional notation and simple facts about multi-indices will be needed. If $\alpha$ and $\beta$ are two multi-indices, then $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_m + \beta_m)$. Moreover $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for each $i = 1, 2, \ldots, m$. For $\beta \leq \alpha$ we set $\alpha - \beta = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots, \alpha_m - \beta_m)$ and 

$$
(\alpha \choose \beta) = \frac{\alpha!}{\beta!(\alpha - \beta)!}.
$$

Using this notation the following version of the Binomial Theorem holds. If $x$ and $y \in \mathbb{R}^m$, then $(x + y)^\alpha = \sum_{\beta \leq \alpha} (\alpha \choose \beta) x^\beta y^{\alpha - \beta}$. Recall that $g^j$ means the ordinary $j$th partial derivative.

The first part of Theorem 1 is immediate from the following result which may be worthwhile on its own.

**Theorem 6.** Suppose $f$ is $u$-times approximately Peano bounded on a compact set $E$. Then there is a decomposition of $E$ into a nested sequence of closed sets $\{A_k\}$, such that on $A_k$ the function $f$ is a restriction of a $B_u$ function.

In order to define the sets $A_k$ of Theorem 6 we need some additional notation and the following Lemma. For the rest of this paper we will set $S = \sum_{s=0}^{n} \binom{s + m - 1}{s} = \binom{n + m}{n}$. $S$ denotes the number of multi-indices less than or equal $n$.

**Lemma 1.** Let $P_i : \mathbb{R}^m \to \mathbb{R}, i = 1, 2, \ldots, S'$ be polynomials that are independent vectors over $\mathbb{R}$. For any $S'$ points of $\mathbb{R}^m, h_1, h_2, \ldots, h_{S'},$ set

$$
M = \begin{pmatrix}
P_1(h_1) & P_2(h_1) & \cdots & P_{S'}(h_1) \\
\vdots & \vdots & \ddots & \vdots \\
P_1(h_{S'}) & P_2(h_{S'}) & \cdots & P_{S'}(h_{S'})
\end{pmatrix}.
$$

Then one can choose the $h_i$ such that $\det M \neq 0$. 

PROOF. Independence means that if \( a_1, a_2, ..., a_{S'} \) are real numbers such that
\[
a_1P_1(h) + a_2P_2(h) + ... + a_{S'}P_{S'}(h) = 0 \quad \text{for all } h \in \mathbb{R}^m,
\]
then \( a_1 = a_2 = ... = a_{S'} = 0 \).

If \( S' = 1 \), \( M = [P_1(h)] \), \( \det M = 1 \cdot P_1(h) \), so there is an \( h \) for which \( P_1(h) \neq 0 \) by definition of independence.

Assume the lemma has been proved for \( S' - 1 \) and let \( M \) be as above. Expanding along the first row, we have
\[
\text{(3.1)} \quad \det M = P_1(h_1)C_{11}(h_2, ..., h_{S'}) + P_2(h_1)C_{12}(h_2, ..., h_{S'}) + ....
\]
By induction there are points \( h_2, ..., h_{S'} \) such that \( C_{11}(h_2, ..., h_{S'}) \neq 0 \). Now fixing such a choice \( h_2, ..., h_{S'} \) and thinking of \( h_1 \) as a variable, if \( \det M \) were identically 0, then equation (3.1) would contradict independence. \( \square \)

We will apply this lemma with \( S' = S \) and \( P_i(h) = h^i \) for \( 0 \leq |i| \leq n \) (the monomials \( h^i \) are independent over \( \mathbb{R} \)) to obtain \( S \) points \( \{h_i\} \) such that the determinant of the corresponding \( M \) is not zero. Since scaling a column scales the value of the determinant, there is no loss of generality in assuming that \( ||h_i|| < 1 \) for each multi-index \( 0 \leq |i| \leq n \). Continuity of \( \det M \) allows us to find a positive number \( \delta < 1 \) such that for every \( i \), \( B(h_i, \delta) \subset B(0, 1) \) and such that \( |\det M| \geq \delta \) for any \( S \) points \( q_i \in B(h_i, \delta) \).

For the rest of this section \( \delta \) and \( B(h_i, \delta) \) denote the number and the balls respectively that were just introduced.

Let \( M_x(y) \) be the real number defined by
\[
\text{(3.2)} \quad f(y) - \sum_{|\alpha|\leq n} \frac{(y-x)^{\alpha}}{\alpha!} f_{\alpha}(x) = M_x(y)||y-x||^u.
\]

For a positive integer \( k \) let \( N_k(x, r) = \{y \in B(x, r) : |M_x(y)| \leq k\} \). Since \( f \) is a measurable function \( N_k(x, r) \) is a measurable set. We define sets
\[
A_k = \left\{ x \in E : \lambda(N_k(x, r)) \geq (1 - \frac{1}{k}) \lambda(B(x, r)) \right\}
\]
Clearly the sets \( A_k \) are nested and since \( f \) is \( u \)-times approximately Peano bounded on \( E \), we have \( E = \bigcup_{k=1}^{\infty} A_k \). The proof of Theorem 6 follows from the Theorem below and the Extension Theorem 4 from [5, page 177].

THEOREM 7. The sets \( A_k \) are closed and there is a constant \( M \) such that for all \( x \) and \( y \) from \( A_k \) we have
\[
|f_s(y) - \sum_{i=s}^{n} \frac{(y-x)^{i-s}}{(i-s)!} f_i(x)| \leq M||y-x||^{u-|s|} \quad \text{for } 0 \leq |s| \leq n.
\]
Before we prove this theorem we will need several lemmas, the first of which is a several variables version of Lemma 5 in [2].
Lemma 2. Let \( x, y, h \in \mathbb{R}^m \). Suppose that \( f \) is \( u \)-times approximately Peano bounded at \( x \) and \( y \). Then

\[
\sum_{|\alpha| \leq n} h_{\alpha} \left( \frac{1}{\alpha!} \left( f_{\alpha}(y) - \sum_{|\alpha+\beta| \leq n} \frac{(y-x)^{\beta}}{\beta!} f_{\alpha+\beta}(x) \right) \right) = \|y - x + h\|^u M_x(y + h) - \|h\|^u M_y(y + h). 
\]

(3.3)

Proof. This identity is obtained by writing \( f(y+h) \) two ways as follows. First we have

\[
f(y+h) = \sum_{|\beta| \leq n} \frac{(y-x+h)^{\beta}}{\beta!} f_{\beta}(x) + M_x(y + h) \|y - x + h\|^u
\]

then write \( y+h \) as \( x + (y-x+h) \) and expand about \( x \) to get

\[
f(y+h) = \sum_{|\alpha| \leq n} \sum_{\beta \geq \alpha} \frac{(y-x)^{\beta-\alpha} h_{\alpha}}{\alpha!(\beta-\alpha)!} f_{\beta}(x) + \|y - x + h\|^u M_x(y + h)
\]

Change the order of summation to obtain

\[
f(y+h) = \sum_{|\alpha| \leq n} \sum_{\beta \geq \alpha} \frac{(y-x)^{\beta-\alpha} h_{\alpha}}{\alpha!(\beta-\alpha)!} f_{\beta}(x) + \|y - x + h\|^u M_x(y + h).
\]

Substitute for \( \beta - \alpha \) a new positive multi-index \( \beta \) to obtain

\[
f(y+h) = \sum_{|\alpha| \leq n} \sum_{|\alpha+\beta| \leq n} \frac{(y-x)^{\beta} h_{\alpha}}{\alpha!\beta!} f_{\alpha+\beta}(x) + \|y - x + h\|^u M_x(y + h)
\]

\[
= \sum_{|\alpha| \leq n} \frac{h_{\alpha}}{\alpha!} \left( \sum_{|\alpha+\beta| \leq n} \frac{(y-x)^{\beta}}{\beta!} f_{\alpha+\beta}(x) \right) + \|y - x + h\|^u M_x(y + h).
\]

Equating these two expansions gives the desired result. \( \square \)

Lemma 3. Let \( y, b, \) and \( c \in \mathbb{R}^m \). Suppose that \( f \) is \( u \)-times approximately Peano bounded at \( b \) and \( c \). Then

\[
f_{\alpha}(y) - \sum_{|\alpha+\beta| \leq n} \frac{(y-b)^{\beta}}{\beta!} f_{\alpha+\beta}(b)
\]

\[
= f_{\alpha}(y) - \sum_{|\alpha+\beta| \leq n} \frac{(y-c)^{\beta}}{\beta!} f_{\alpha+\beta}(c) + \sum_{|\alpha+\beta| \leq n} \frac{(y-c)^{\beta}}{\beta!} \left( f_{\alpha+\beta}(c) - \sum_{|\alpha+\beta+\eta| \leq n} \frac{(c-b)^{\eta}}{\eta!} f_{\alpha+\beta+\eta}(b) \right)
\]
PROOF. It is enough to check that
\[ \sum_{|\alpha| \leq n} \sum_{|\alpha + \beta| \leq n} \frac{(y - c)^\beta (c - b)^\eta}{\beta! \eta!} f_{\alpha + \beta + \eta}(b) = \sum_{|\alpha| \leq n} \frac{(y - b)^\beta}{\beta!} f_{\alpha + \beta}(b). \]
Indeed, if we introduce a new positive multi-index \( \kappa = \beta + \eta \), then the left side is
\[
\sum_{|\alpha + \beta| \leq n} \sum_{\beta \leq \kappa \atop \alpha + \beta \leq n} \frac{(y - c)^\beta (c - b)^{\kappa - \beta}}{\beta! (\kappa - \beta)!} f_{\alpha + \kappa}(b) \\
= \sum_{|\alpha + \kappa| \leq n} \left( \sum_{\beta \leq \kappa} \frac{(y - c)^\beta (c - b)^{\kappa - \beta}}{\beta! (\kappa - \beta)!} \right) f_{\alpha + \kappa}(b) \\
= \sum_{|\alpha + \kappa| \leq n} \frac{(y - b)^\kappa}{\kappa!} f_{\alpha + \k}(b). 
\]

\[ \square \]

Lemma 4. Let \( x \in A_k \) and \( r < \frac{1}{k} \). If \( I \) is a ball inside \( B(x, r) \) and such that \( \lambda(I) \geq \frac{\delta_m}{2^m} \lambda(B) \), then \( \lambda(N_k(x, r) \cap I) \geq \frac{2}{3} \lambda(I) \).

PROOF. Indeed \( \lambda(B(x, r)) - \lambda(I) + \lambda(N_k(x, r) \cap I) \geq \lambda(N_k(x, r) \cap B(x, r)) \geq (1 - \frac{\delta_m}{2m}) \lambda(B(x, r)) \). Hence \( \lambda(N_k(x, r) \cap I) \geq \lambda(I) - \frac{\delta_m}{2m} \lambda(B(x, r)) \geq \lambda(I) - \frac{1}{3} \lambda(I) = \frac{2}{3} \lambda(I) \).
\[ \square \]

Now we are ready to prove Theorem 7.

PROOF. Let \( x \in A_K \), and let \( K \geq k \) be such that \( x \in A_K \). Now let \( y \in A_k \) be such that \( ||y - x|| < \frac{1}{2K} \). For a multi-index \( i, 0 \leq |i| \leq n \), let \( I_i = y + ||y - x||B(h_i, \delta) \). Then \( I_i \) is inside \( B(y, ||y - x||) \subset B(x, 2||y - x||) \), and \( \lambda(I_i) = \delta^m \lambda(B(y, ||y - x||)) = \frac{\delta^m}{2^m} \lambda(B(x, 2||y - x||)) \). By Lemma 4 we have \( \lambda(N_k(y, ||y - x||) \cap I_i) \geq \frac{2}{3} \lambda(I_i) \) and \( \lambda(N_K(x, 2||y - x||) \cap I_i) \geq \frac{2}{3} \lambda(I_i) \). Therefore for each \( i \in \{ \alpha : |\alpha| \leq n \} \) there are points \( y_i \in N_k(y, ||y - x||) \cap N_K(x, 2||y - x||) \cap I_i \). Notice that \( q_i = \frac{y_i - y}{||y - x||} \in B(h_i, \delta) \) so that \( |\det M| \geq \delta \) where \( \det M \) was evaluated at \( \{ q_i \}_{0 \leq i \leq n} \).

By replacing \( h \) in (3.3) with \( (y_j - y) \) for \( j \in \{ \alpha : |\alpha| \leq n \} \), we obtain a system of S linear equations in the S unknowns \( X_{00\ldots0}, \ldots, X_{0\ldots0n} \),

\[
\sum_{0 \leq |s| \leq n} (y_j - y)^s X_s = b_j : j \in \{ \alpha : |\alpha| \leq n \}
\]
where
\[ X_s = \frac{1}{s!} \left( f_s(y) - \sum_{|s+i| \leq n} \frac{(y-x)^i}{i!} f_{s+i}(x) \right) \]
and
\[ b_j = ||y_j - x||^u M_x(y_j) - ||y_j - y||^u M_y(y_j). \]

(In order to apply standard matrix methods such as Cramer’s rule, we assume that the set \( \{\alpha : |\alpha| \leq n\} \) of \( S \) elements is linearly ordered.) The main determinant \( \det \Delta \) of the system (3.4) is \( ||y - x||^T \det M \), where \( T = \sum_{|\alpha| \leq n} |\alpha| \). Hence \( |\det \Delta| \geq ||y - x||^T \delta \). On the other hand if \( \det \Delta_s \) is the determinant obtained by replacing the \( s \)th column of \( \Delta \) with the values \( b_1, \ldots, b_S \), then in the expansion of \( \Delta_s \) about the \( s \)th column, each minor is the sum of \( (S-1)! \) terms of the form \( \pm \prod_{j \neq j_0, |\alpha| \leq n} \alpha_{x\beta}(y_j - y)^\alpha \). Since \( ||y_j - y|| \leq ||y - x||, \ |||y_j - x|| \leq 2||y - x|| \) and \( |M_x(y_j)| \leq K, |M_y(y_j)| \leq k \) we have \( |b_j| \leq ||y - x||^u (k + 2uK) \). Hence
\[ |\Delta_s| \leq \sum_{j \in \{k : |k| \leq n\}} |b_j| (S - 1)! ||y - x||^{T - |s|} \leq S! ||y - x||^{T + u - |s|} (k + 2uK). \]

By Cramer’s Rule
\[ |X_s| = \left| \frac{\det \Delta_s}{\det \Delta} \right| \leq \frac{S! ||y - x||^{u - |s|} (k + 2uK)}{\delta}. \]

(3.5)

In particular for \( 0 \leq |s| \leq n \lim_{y \to x, y \in A_k} f_s(y) = f_s(x) \). Next let \( x_j \in A_k \) be a sequence converging to \( x \). Fix \( r < \frac{1}{k} \). Let \( y \in \bigcap_{j=1}^{\infty} N_k(x_j, r) \). Then \( y \) is in \( N_k(x_j, r) \) for infinitely many \( j \) and thus continuity of \( f_s \) establishes
\[ \left| f(y) - \sum_{|i| \leq n} \frac{(y-x)^i}{i!} f_i(x) \right| \leq k||y - x||^u. \]

Therefore \( y \in N_k(x, r) \) and thus \( \bigcap_{j=1}^{\infty} N_k(x_j, r) \subset N_k(x, r) \). Since
\[ \lambda\left( \bigcup_{s=j}^{\infty} N_k(x_s, r) \right) \geq (1 - \frac{1}{3^{2m}}) \lambda(B(x, r)) \]
we have
\[ \lambda(N_k(x, r)) \geq (1 - \frac{1}{3^{2m}}) \lambda(B(x, r)). \]
Thus \( x \in A_k \) and \( A_k \) is closed. Hence in the proof of the theorem we could take \( K = k \) to obtain

\[
(3.6) \quad \left| f_s(y) - \sum_{|i+s| \leq n} \frac{(y-x)^i}{i!} f_{i+s}(x) \right| \leq M \|y - x\|^{u-s}
\]

for \( 0 \leq |s| \leq n \) where \( M = \max\{\alpha!\} \frac{1 + 2^m}{\delta} k \), (in 3.5 replace \( S! \) by \( \max\{\alpha!\} \) and \( K \) by \( k \)) whenever \( x, y \) are in \( A_k \) such that \( \|y - x\| < \frac{1}{2k} \). We would like to get these inequalities for any \( x \) and \( y \) from \( A_k \). To that end let \( \{I_j\} \) be a finite open cover of \( A_k \) with centers from \( A_k \) and radii equal to \( \frac{1}{2k} \).

Since \( J \), the set of all centers of balls \( \{I_j\} \) is finite, there is a constant \( W \) such that

\[
(3.7) \quad \max_{b,c \in J} \left| f_s(c) - \sum_{|r+s| \leq n} \frac{(c-b)^r}{r!} f_{r+s}(b) \right| \leq W\|c - b\|^{u-s}, 0 \leq |s| \leq n.
\]

We may assume that \( x \) and \( y \) are in two different balls centered at \( b \) and \( c \) respectively, and that \( \|y - x\| > \frac{1}{2k} \). We first show that there is a constant \( M' \) such that

\[
(3.8) \quad \left| f_s(y) - \sum_{|i+s| \leq n} \frac{(y-b)^i}{i!} f_{i+s}(b) \right| \leq M'\|y - b\|^{u-s}, 0 \leq |s| \leq n.
\]

By Lemma 3

\[
\left| f_s(y) - \sum_{|i+s| \leq n} \frac{(y-b)^i}{i!} f_{i+s}(b) \right| = \sum_{|i+s| \leq n} \left[ \frac{(y-c)^i}{i!} f_{i+s}(c) + \sum_{|i+s+r| \leq n} \frac{(c-b)^r}{r!} f_{i+s+r}(b) \right] \leq M\|y - c\|^{u-s} + \sum_{|i+s| \leq n} \frac{||y - c||^{|i|}}{i!} W\|c - b\|^{u-s+i+s}
\]

Since \( ||y - c|| \leq ||y - b|| \), and \( ||c - b|| \leq 2||y - b|| \) the last inequality is

\[
\leq M\|y - b\|^{u-s} + \sum_{|i+s| \leq n} \frac{||y - b||^{|i|}}{i!} 2^{u-s} W\|y - b\|^{u-s} = M_s\|y - b\|^{u-s}.
\]

Setting \( M' = \max\{M_s\} \) establishes (3.8).
We use Lemma 3 again but this time applied to \( y, x, \) and \( b \).

\[
\left| f_s(y) - \sum_{|i+s| \leq n} \frac{(y-x)^i}{i!} f_{i+s}(x) \right| = \\
\left| f_s(y) - \sum_{|i+s| \leq n} \frac{(y-b)^i}{i!} f_{i+s}(b) + \\
\sum_{|i+s| \leq n} \frac{(y-b)^i}{i!} \left[ f_{i+s}(b) - \sum_{|i+s+r| \leq n} \frac{(b-x)^r}{r!} f_{i+s+r}(x) \right] \right|
\]

The inequalities 3.8 and 3.6 applied to the right hand side yield

\[
\leq M'||y-b||^{u-s}| + \sum_{|i+s| \leq n} \frac{|y-b|^{|i|}}{i!} M||b-x||^{u-s}|.
\]

Finally since \( ||y-b|| \leq 2||y-x||, \) and \( ||b-x|| \leq ||y-x|| \) the last inequality is

\[
\leq 2^{u-s}M'||y-x||^{u-s}| + \sum_{|i+s| \leq n} \frac{2^{|i|}||y-x||^{|i|}}{i!} M||y-x||^{u-s}| = \\
\left[ 2^{u-s}M' + \sum_{|i+s| \leq n} \frac{2^{|i|}}{i!} M \right] ||y-x||^{u-s}| \leq \\
\left[ 2^u M' + \sum_{|i| \leq n} \frac{2^{|i|}}{i!} M \right] ||y-x||^{u-s}|.
\]

\[\square\]

In the special case \( u = n + 1 \), we can improve Theorem 6, as in the following corollary, which is also the second part of Theorem 1.

**Corollary 1.** Suppose \( f \) is \( n+1\)-times approximately Peano bounded on a bounded measurable set \( E \). Then for every \( \epsilon > 0 \) there is a closed set \( \Pi \) with \( \lambda(E-\Pi) < \epsilon \) and a \( C^{n+1} \) function \( h \) such that on \( \Pi \) the function \( f \) and its partial derivatives agree with \( h \) and the corresponding partial derivatives of \( h \).

**Proof.** Let \( A_k \) and \( g_k \) be from Theorem 6 such that \( \lambda(E- A_k) < \frac{\epsilon}{3} \).

By Theorem 6, for each multi-index \( j \) with \( |j| = n \), the function \( g_k^j \) is Lipschitz. Hence by a theorem of H. Rademacher \( g_k^j \) is totally differentiable at almost every \( x \in \mathbb{R}^m \). Now let \( P \supset A_k \) be a bounded open set such that \( \lambda(P-A_k) < \frac{\epsilon}{3} \). The function \( g_k \) on \( P \) satisfies the conditions of Theorem 4 from [6], so by that theorem, there is a closed set \( Q \subset P \) such that \( \lambda(P-Q) < \frac{\epsilon}{3} \) and a \( C^{n+1} \) function \( h \) that agrees with \( g_k \) on \( Q \). Notice that \( \lambda(E- A_k \cap Q) < \epsilon \) and that \( h = f \) on the closed set \( \Pi = A_k \cap Q \). \[\square\]
Corollary 1 in the case $n = 0$ was proved by H. Whitney. (See Theorem 1 [6].) The proof of this result from [6] uses the fact that approximate partial derivatives of a measurable function are measurable. (See Theorem 11.2 page 299 from [4].) The proof of Corollary 1 doesn’t require measurability of the approximate partial derivatives $f_s$ for $|s| \geq 1$. However measurability of the $f_s$ is an immediate consequence of this corollary and Lusin’s theorem.

**Corollary 2.** Let $f : \mathbb{R}^m \to \mathbb{R}$ be a measurable function. Suppose $f$ is $n + 1$—times approximately Peano differentiable on a measurable set $E$. Then for every multi-index $|s| = n + 1$, the partials $f_s$ are measurable.

**Proof.** By the term “$f$ is $n$—times approximately Peano differentiable at a point $x$” we mean that the left hand side of expression (3.2) is $o(||y - x||^n)$ as $y \to x$ through a set of density 1 at $x$. If $f$ is $n + 1$—times approximately Peano differentiable on a measurable set $E$, then clearly $f$ is $n + 1$—times approximately Peano bounded with $|M_x(y)| \leq \sum_{|s|=n+1} \frac{1}{s!} \max_{|s|=n+1} |f_s(x)| + 1$. For an integer $i$ let $E_i = E \cap B(0, i)$. Then $E$ is a countable union of bounded measurable sets $E_i$. By corollary 1, for each $\epsilon > 0$, $f_s$ agrees with a continuous function on a set $F_i$ with $\lambda(E_i - F_i) < \epsilon$. Hence by Lusin’s theorem $f_s$ is measurable on each $E_i$. Since $f_s = \lim_{i \to \infty} f_s \chi_{E_i}$ where $\chi_{E_i}$ denotes the characteristic function of $E_i$ we see that $f_s$ is measurable. 

**4. Proofs: The $L^p$ case**

**Definition 7.** Let $f : \mathbb{R}^m \to \mathbb{R}$ be a measurable function. We say that $f$ is locally $L^p$ $u$—times Peano bounded at $x$ if for each multi-index $\alpha$, with $|\alpha| \leq n$ there is a number $f_\alpha(x)$, such that

\[
\left( \frac{1}{\rho^n} \int_{||y-x|| \leq \rho} \left| f(y) - \sum_{0 \leq |\alpha| \leq n} \frac{(y-x)^\alpha}{\alpha!} f_\alpha(x) \right|^p dy \right)^{\frac{1}{p}} = L_x(\rho)^u
\]

where $L_x(\rho)$ remains bounded as $\rho \to 0$. In this definition we will assume that $f_{(0,0,\ldots,0)}(x) = f(x)$.

Recall $S$ denotes the number of multi-indices less than or equal $n = [u] - 1$.

The main result of this section is this.

**Theorem 8.** Suppose $f$ is locally $L^p$ $u$—times Peano bounded on a compact set $E$. Then there is a decomposition of $E$ into a nested sequence of closed sets $\{A_k\}$, such that on $A_k$ the function $f$ is a restriction of a $B_u$ function.
Proof. Let $M_x(y) = f(y) - \sum_{0 \leq |\alpha| \leq n} \frac{(y-x)^{\alpha}}{\alpha!} f_\alpha(x)$. Then $f$ is locally $L^p$ $u$-times Peano bounded at $x$ means that there is $\delta > 0$ and a constant $M$ such that
\[
\left( \int_{||y-x|| \leq \rho} |M_x(y)|^p dy \right)^{1/p} \leq M \rho^{u+m/p} \text{ for all } 0 < \rho < \delta.
\]
For a positive integer $k$ let
\[
A_k = \{ x \in E : \int_{||y-x|| \leq \rho} |M_x(y)|^p dy \leq k \rho^{u+m/p} \text{ for all } 0 < \rho < \frac{1}{k} \}.
\]
Clearly the sets $A_k$ are nested and since $f$ is locally $L^p$ $u$-times Peano bounded on $E$, we have $E = \bigcup_{k=1}^{\infty} A_k$. The proof of Theorem 8 follows from Theorem 9 below and the extension Theorem 4 from [5, page 177].

**Theorem 9.** The sets $A_k$ are closed and there is a constant $M$ such that for all $x$ and $y$ from $A_k$ we have
\[
|f_s(y) - \sum_{|s+i| \leq n} \frac{(y-x)^i}{i!} f_{s+i}(x)| \leq M |y - x|^{|s|} \text{ for } 0 \leq |s| \leq n.
\]

In the proof of this theorem we will use the following two lemmas.

**Lemma 5.** Let $x, y, h \in \mathbb{R}^m$. Then
\[
\sum_{|\alpha| \leq n} \frac{h^\alpha}{\alpha!} \left( f_\alpha(y) - \sum_{|\alpha+\beta| \leq n} \frac{(y-x)^\beta}{\beta!} f_{\alpha+\beta}(x) \right) = M_x(y+h) - M_y(y+h).
\]

The proof of this lemma is implicit in the proof of Lemma 2 of the last section. We will also need the following version of Lemma 2.6 from page 182 of reference [1].

**Lemma 6.** Let $C$ denote the vector space of continuous functions defined on the closed ball $\overline{B}(0, 1) \subset \mathbb{R}^m$. Then the linear map $T : C \to \mathbb{R}^S$ defined by $T(\varphi) = \left( \int_{||h|| \leq 1} \varphi(h) h^\alpha dh : |\alpha| \leq n \right)$ is onto.

**Proof.** Indeed if $T$ were not onto then there are $S$ numbers $\{c_s\}$ not all of which are zero and such that for every $\varphi \in C$ we have
\[
\sum_{|s| \leq n} c_s \int_{||h|| \leq 1} \varphi(h) h^s dh = 0.
\]
In particular this would be true for \( \varphi(h) = \sum_{|s| \leq n} c_s h^s \). Substitution in (4.3) yields
\[
\int_{||h|| \leq 1} \left( \sum_{|s| \leq n} c_s h^s \right)^2 dh = 0; \text{ thus } c_s = 0 \text{ for all } s. \text{ This is a contradiction.}
\]

Now we are ready to prove Theorem 9.

**Proof.** Let \( x \in \overline{A}_k \), and let \( K \geq k \) be such that \( x \in A_K \). Now let \( y \in A_k \) be such that \( ||y - x|| < \frac{1}{2K} \). By Lemma 6 for every multi-index \( |\alpha| \leq n \) there is \( \varphi_\alpha \in C \) such that \( T(\varphi_\alpha) = (0, 0, ..., \alpha!, 0) \) where in the vector \( (0, 0, ..., \alpha!, 0) \in \mathbb{R}^s \), the only nonzero entry is one that corresponds to \( \alpha \). By Lemma 5

(4.4)
\[
\int_{||h|| \leq ||y - x||} \varphi_\alpha \left( \frac{h}{||y - x||} \right) \sum_{|\alpha| \leq n} h^\alpha \frac{\partial^{\alpha-\beta}}{\partial^\beta} f_{\alpha+\beta}(x) dh =
\]
\[
\int_{||h|| \leq ||y - x||} \varphi_\alpha \left( \frac{h}{||y - x||} \right) (M_x(y + h) - M_y(y + h)) dh.
\]

The left hand side of (4.4) reduces to
\[
\left( f_{\alpha}(y) - \sum_{|\alpha+\beta| \leq n} \frac{(y - x)^\beta}{\beta!} f_{\alpha+\beta}(x) \right) \int_{||h|| \leq ||y - x||} \varphi_\alpha \left( \frac{h}{||y - x||} \right) \frac{h^\alpha}{\alpha!} dh =
\]

The change of variable \( k = \frac{h}{||y - x||} \) yields
\[
\left( f_{\alpha}(y) - \sum_{|\alpha+\beta| \leq n} \frac{(y - x)^\beta}{\beta!} f_{\alpha+\beta}(x) \right) ||y - x||^{||\alpha|+m} \int_{||k|| \leq 1} \varphi_\alpha(k) \frac{k^\alpha}{\alpha!} dk =
\]

(4.5)
\[
\left( f_{\alpha}(y) - \sum_{|\alpha+\beta| \leq n} \frac{(y - x)^\beta}{\beta!} f_{\alpha+\beta}(x) \right) ||y - x||^{||\alpha|+m} \text{ by the choice of } \varphi_\alpha.
\]

On the other hand if \( N \) is a bound for \( \{ |\varphi_\alpha| : |\alpha| \leq n \} \), the right-hand side of (4.4) is bounded by

\[
N \left( \int_{||h|| \leq ||y - x||} |M_x(y + h)| dh + \int_{||h|| \leq ||y - x||} |M_y(y + h)| dh \right) \leq
\]
Therefore

\[
N \left( \int_{|h| \leq 2||y-x||} |M_x(x + h)| dh + \int_{|h| \leq ||y-x||} |M_y(y + h)| dh \right) \leq
\]

By Holder’s inequality we have

\[
N \left( \int_{|h| \leq 2||y-x||} |M_x(x + h)|^p dh \right)^{\frac{1}{p}} (\lambda(B(0, 2||y-x||))^\frac{1}{q} +
\]

\[
N \left( \int_{|h| \leq ||y-x||} |M_y(y + h)|^p dh \right)^{\frac{1}{p}} (\lambda(B(0, ||y-x||))^\frac{1}{q} +
\]

\[
\leq N \left( K(2||y-x||)^{u+m/p} 2^{\frac{m}{p}} \lambda(B(0, ||y-x||))^\frac{1}{q} +
\right)
\]

\[
k||y-x||^{u+m/p} \lambda(B(0, ||y-x||))^\frac{1}{q} =
\]

\[
N \left( K||y-x||^{u+m/p} 2^{u+m} \lambda(B(0, ||y-x||))^\frac{1}{q} +
\right)
\]

\[
k||y-x||^{u+m/p} \lambda(B(0, ||y-x||))^\frac{1}{q})
\]

\[
= N'||y-x||^{u+m}(K2^{u+m} + k)
\]

where \(N'\) is independent of \(K\) and \(k\).

Combining this with (4.5) we obtain that whenever \(||y-x|| \leq \frac{1}{2K}\),

\[
\left| f_\alpha(y) - \sum_{|\alpha + \beta| \leq n} \frac{(y-x)^\beta}{\beta!} f_\alpha(x) \right| \leq M||y-x||^{u-|\alpha|} \text{ for all } |\alpha| \leq n.
\]

In particular for \(0 \leq |s| \leq n \lim_{y \to x, y \in A_k} f_s(y) = f_s(x)\). Next let \(x_j \in A_k\) be a sequence converging to \(x\). Fix \(\rho < \frac{1}{k}\). Then for infinitely many \(j\)'s we can find \(\rho \leq \rho_j < \frac{1}{k}\) such that \(B(x, \rho) \subset B(x_j, \rho_j)\) and \(\rho_j \to \rho\) as \(j \to \infty\). Then

\[
(4.6) \int_{||y-x|| \leq \rho} \left| f(y) - \sum_{|i| \leq n} \frac{(y-x)^i}{i!} f_i(x_j) \right|^p \, dy \leq
\]

\[
\int_{||y-x|| \leq \rho_j} \left| f(y) - \sum_{|i| \leq n} \frac{(y-x)^i}{i!} f_i(x_j) \right|^p \, dy \leq k\rho_j^{u+p+m}.
\]

Letting \(j \to \infty\) in (4.6) gets

\[
\int_{||y-x|| \leq \rho} \left| f(y) - \sum_{|i| \leq n} \frac{(y-x)^i}{i!} f_i(x) \right|^p \, dy \leq k\rho^{u+p+m}.
\]

Therefore \(x \in A_k\). Hence \(A_k\) is closed and in the proof of the theorem we
may take $K = k$ to obtain

\begin{equation} 
(4.7) \quad \left| f_s(y) - \sum_{|i+s| \leq n} \frac{(y-x)^i}{i!} f_{i+s}(x) \right| \leq M |y - x|^{u-|s|}
\end{equation}

for $0 \leq |s| \leq n$ where $M = N'(k2^{u+m} + k)$ and whenever $x, y$ are in $A_k$ such that $\|y - x\| < \frac{1}{2k}$. To complete the proof of Theorem 9, these inequalities must be shown to also hold for any $x$ and $y$ in $A_k$. The argument for this appears above in the last part of the proof of Theorem 7.

It is now easy to see that Theorem 5 is a corollary of Theorem 8. In fact by Theorem 8 there are sets $A_k$ and a $B_n$ function that agrees with $f$ on $A_k$. Thus if $x$ is a density point of $A_k$ we have

\[
\limsup_{y \to x} \left| f_\beta(y) - \sum_{\beta \leq \alpha \leq n} \frac{(y-x)^\alpha}{\alpha!} f_\alpha(x) \right| / \|y - x\|^{u-|\beta|} < \infty.
\]

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References


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