On the other hand, if $c_M(x) = b_0 + b_1x + \cdots + b_{n-k-1}x^{n-k-1} + x^{n-k}$, then
\[
c_A(x) = b_0c_R(x) + b_1xc_R(x) + \cdots + b_{n-k-1}x^{n-k-1}c_R(x) + x^{n-k}c_R(x),
\]
which translates to the linear relation
\[
c = b_0c_0 + b_1c_1 + \cdots + b_{n-k-1}c_{n-k-1} + c_{n-k}.
\]
Thus the vector $c$ in Theorem 4 is in the kernel of $B_v$. This is the Cayley-Hamilton Theorem.

In an elementary treatment, determination of the rank of $B_v$ and the minimal polynomial of $A$ lend themselves to interesting computer experiments using the Gauss-Jordan algorithm. Students can be led to discover the vagaries of the types of matrices and decompositions that result.

For a more advanced treatment, the proof can be summarised as follows. The map $B_v$ is morally a linear one from the space $\mathcal{P}_n$ of polynomials of degree at most $n$ to $F^n$. Its image is the cyclic subspace spanned by $v, Av, A^2v, \ldots, A^{k-1}v$ and its kernel is a near-ideal that is generated by the polynomial $c_R(x)$. It suffices to show that $c_A(x)$ is a multiple of this generator.

---

Mathematical Sciences Dept., Chancellor College, University of Malawi, P.O. Box 280, Zomba, MALAWI
chisala@Unima.wn.apc.org

---

**The Probability of a Tie in an $n$-Game Match**

**J. Marshall Ash**

Deep Blue and Gary Kasparov recently played two 6-game matches. My son, Andrew, thought that it was a bad idea to have the number of games be even, because this would make the probability of a tie for the match too high. That seemed like pretty sound intuition to me. What follows is an analysis of a fairly realistic model of match play between approximately equal players. In this model Andrew’s intuition fails.

**The model.** Let $A$ and $B$ play a match of $n$ games. Our basic assumption is that for each game, $P(A$ wins $) = P(A$ and $B$ tie $) = P(B$ wins $) = \frac{1}{3}$. For each game, score 1 point when $A$ wins, 0 points for a tie, and $-1$ points when $B$ wins. For each $j \in \{-n, \ldots, -1, 0, 1, \ldots, n\}$, let $p_j^{(n)}$ be the probability that a match of $n$ games ends with a score of $j$. Thus $p_0^{(n)}$ is the probability that an $n$-game match ends in a draw.

If no game could end in a tie, there would be 0 probability of a drawn 5- or 7-game match, while the probability of a tie in a 6-game match would be strictly positive. However, with our assumptions, we have the following result.

**Theorem 1.1.** If, for each game, $P(A$ wins $) = P(A$ and $B$ tie $) = P(B$ wins $) = \frac{1}{3}$, then the probability of a drawn match becomes less as the match length increases. More precisely, $p_0^{(1)} = p_0^{(2)} = \frac{1}{3}$ and
\[
p_0^{(n)} > p_0^{(n+1)} \quad \text{for each } n \geq 2. \tag{1}
\]
In particular, \( p^{(5)}_0 = \frac{51}{243} = .21, p^{(6)}_0 = \frac{141}{729} = .19, \) and \( p^{(7)}_0 = \frac{393}{2187} = .18, \) so that
\[
p^{(5)}_0 > p^{(6)}_0 > p^{(7)}_0.
\]

Furthermore, we have
\[
p^{(n)}_0 = \frac{1}{3^n} \sum_{k=0}^{\lceil n/2 \rceil} \frac{n!}{k!(n-2k)!} = \frac{1}{3^n} \sum_{k=0}^{\lceil n/2 \rceil} \binom{n}{n-k} \binom{n-k}{n-2k}
\]
and
\[
p^{(n)}_0 = \sqrt{\frac{3}{4\pi n}} \quad \text{as} \quad n \to \infty.
\]

**Proof:** Let \( n \) be the number of games in a match. There are \( 3^n \) possible outcomes and only 1 of these involves player \( A \) winning every game, so \( p^{(n)}_n = 3^{-n} \). Symmetrically, \( p^{(n)}_n = 3^{-n} \). To simplify notation, let \( N^{(n)}_j = 3^n p^{(n)}_j \) be the number of ways that a score of \( j \) can arise. To get a score of \( n-1 \), player \( A \) must tie one game and win the rest. The tied game can be any one of the \( n \) games, so \( N^{(n)}_{n-1} = n \). Symmetrically, \( N^{(n)}_{n+1} = n \). Since a score of \( j \) after the \( n+1 \)st game can arise only from scores belonging to \( \{j-1, j, j+1\} \) after \( n \) games, we have the basic recurrence
\[
N^{(n+1)}_j = N^{(n)}_{j-1} + N^{(n)}_j + N^{(n)}_{j+1}, \quad j = 1, 2, \ldots, \lfloor j \rfloor < n.
\]

We can use our results to find \( N^{(n)}_j \) quickly for all \( n \leq 7 \) with the following Pascal triangle-like array.

\[
\begin{array}{cccccc}
 n = 1 & 1 & 1 & 1 & 1 & 1 \\
n = 2 & 1 & 2 & 3 & 2 & 1 \\
n = 3 & 1 & 3 & 6 & 7 & 6 & 3 & 1 \\
n = 4 & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\
n = 5 & 1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1 \\
n = 6 & 1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & 6 & 1 \\
n = 7 & 1 & 7 & 28 & 77 & 161 & 266 & 357 & 393 & 357 & 266 & 161 & 77 & 28 & 7 & 1
\end{array}
\]

The leftmost diagonal of 1's corresponds to \( N^{(n)}_0 = 1, n = 1, \ldots, 7 \); the next diagonal, \([1, 2, 3, 4, 5, 6, 7]\), corresponds to \( N^{(n)}_{n+1} = n, n = 1, \ldots, 7 \). The rightmost and next-to-rightmost diagonals are symmetrical. The remaining elements can be filled in easily from top to bottom by using the recursion relation (4). This amounts to letting each item be the sum of the number above it and the numbers to the left and right of that number. For example, \( 393 = 126 + 141 + 126 \), since \( N^{(7)}_0 = N^{(6)}_{-1} + N^{(6)}_0 + N^{(6)}_1 \). It follows that \( p^{(7)}_0 = N^{(7)}_0 / 3^7 = 393 / 2187 \).

For a score of \( n-2 \) we must have \( n \geq 2 \) and player \( A \) must have either all wins except for 1 loss, which can happen in \( \binom{n}{1} \) ways, or all wins except for 2 ties, which can happen in \( \binom{n}{2} \) ways, so that \( N^{(n)}_{n-2} = \binom{n}{1} + \binom{n}{2} = n(n+1)/2 \). Symmetrically, \( N^{(n)}_{-n+2} = n(n+1)/2 \). The recurrence (4) leads to the generating function relation
\[
\frac{(x^{-1} + 1 + x)^n}{3^n} = \sum_{j=-n}^{n} p^{(n)}_j x^j,
\]
and (by the trinomial theorem) to (2). See [1, Section VI.9], [2, Section 5.4], or [3, Section 1.5].
A straightforward induction argument, using only the exact formulas for \( p_j^{(n)} \) when \( j \in \{ n - 2, n - 1, n \} \) and the recurrence relations, shows the mound-shaped behavior of the \( \{ p_j^{(n)} \} \), i.e., for every \( n \geq 2 \),

\[
 p_n^{(n)} < p_{n-1}^{(n)} < \cdots < p_0^{(n)} \quad \text{and} \quad p_j^{(n)} = p_j^{(n)}, \ |j| \leq n.
\]

From this and the recurrence relation, it follows that for each integer \( n \geq 2 \)

\[
 p_0^{(n+1)} = \frac{1}{3} p_1^{(n)} + \frac{1}{3} p_0^{(n)} + \frac{1}{3} p_0^{(n)} < \frac{1}{3} p_0^{(n)} + \frac{1}{3} p_0^{(n)} + \frac{1}{3} p_0^{(n)} = p_0^{(n)},
\]

which establishes (1).

To prove (3), notice that our game may be thought of as a trinomial distribution where each individual trial has mean 0 and variance \( \frac{2}{3} \), i.e., as \( n \) “flips” of a three-sided fair “coin” with sides marked \(-1, 0, \) and \(+1\). Making the standard approximation via the central limit theorem (see [2, p. 215] or [3, p. 330]),

\[
 p_0^{(n)} = P\left( \sum_{i=1}^{n} X_i = 0 \right) = \frac{1}{\sqrt{2\pi(2/3)n}} \int_{-1/2}^{1/2} e^{-x^2/(2(2/3)n)} \, dx = \sqrt{\frac{3}{4\pi n}} \quad \text{as} \ n \to \infty.
\]

\[ \blacksquare \]

**Remark.** There is a reason why the Deep Blue-Kasparov match length was set at 6 games: playing the white pieces gives an important advantage, so having an even number of games allows each player to enjoy this advantage the same number of times. Perhaps it will come as a relief to chess tournament designers to know that when they choose an even match length, they are not necessarily decreasing the likelihood of a decisive outcome.

**Extensions.** We assumed \( P(A \text{ wins}) = P(A \text{ and } B \text{ tie}) = P(B \text{ wins}) = \frac{1}{3} \) and found, in particular, that \( p_0^{(5)} > p_0^{(6)} \). If we had assumed \( P(A \text{ wins}) = P(B \text{ wins}) = \frac{1}{2} \) and \( P(A \text{ and } B \text{ tie}) = 0 \), we would have found \( p_0^{(5)} = 0 < p_0^{(6)} \). More generally, assume that \( P(A \text{ wins}) = P(B \text{ wins}) = p \), so that \( P(A \text{ and } B \text{ tie}) = 1 - 2p \). A continuity argument makes it clear that there is a \( p \in (\frac{1}{3}, \frac{1}{2}) \) for which \( p_0^{(5)} = p_0^{(6)} \). The associated generating function would be \( (px^{-1} + (1 - 2p) + px)^n \). Similarly, as \( p \) moves up from \( \frac{1}{3} \), there must be a point at which the \( \{ p_j^{(n)} \} \) stop being mound-shaped. It might also be interesting to study the cases where players \( A \) and \( B \) are not evenly matched. If \( P(A \text{ wins}) = p, P(B \text{ wins}) = q \neq p \), so that \( P(A \text{ and } B \text{ tie}) = 1 - p - q \), then the generating function becomes \( (qx^{-1} + (1 - p - q) + px)^n \). Here the probability of a tie decreases exponentially as \( n \) increases.

**ACKNOWLEDGMENTS.** This paper was improved by discussions with Andrew Ash, Michael Ash, Alan Berele, and Herman Rubin.

**REFERENCES**