On the $n$th quantum derivative

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Abstract. The $n$th quantum derivative $D_n f(x)$ of the real-valued function $f$ is defined for each real non-zero $x$ as

$$
\lim_{q \to 1} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(k-1)k/2} f(q^{n-k}x) \frac{q(q-1)^n}{q^{(n-1)m/2}(q-1)^n} x^n,
$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the $q$-binomial coefficient. If the $n$th Peano derivative exists at $x$, which is to say that if $f$ can be approximated by an $n$th degree polynomial at the point $x$, then it is not hard to see that $D_n f(x)$ must also exist at that point. Consideration of the function $|1-x|$ at $x = 1$ shows that the second quantum derivative is more general than the second Peano derivative. However, we can show that the existence of the $n$th quantum derivative at each point of a set necessarily implies the existence of the $n$th Peano derivative at almost every point of that set.

1. Introduction

Let $f$ be a real valued function of a real variable. The first quantum difference is defined as

$$D_1(q,x) = \frac{\Delta_1(q,x)}{q^{-1}-x} = \frac{f(qx) - f(x)}{q^{-1}-x},$$

and for $n = 2, 3, \ldots$, the higher quantum differences are defined inductively by

$$D_n(q,x) = \frac{D_{n-1}(q,qx) - D_{n-1}(q,x)}{q^{n-1}x^n}.$$

For $n = 1, 2, \ldots$, the $n$th difference quotient can be written

$$D_n(q,x) = \frac{\Delta_n(q,x)}{q^{(n-1)m/2}(q-1)^n x^n}$$

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with
\[
\Delta_n (q, x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{(k-1)k/2} f \left( q^{n-k} x \right),
\]
where the $q$-binomial coefficients are defined by setting \([n]_q = \frac{q^n - 1}{q-1}\) if $n = 1, 2, \ldots$,
\[
[n]_q! = \begin{cases} 1 & \text{if } n = 0 \\ [1]_q [2]_q \cdots [n]_q & \text{if } n = 1, 2, \ldots \\ \end{cases},
\]
and
\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} \text{ for } n \geq k \geq 0.
\]
For each positive integer $n$, and each non-zero real number $x$, the $n$th quantum derivative of $f$ is defined to be
\[
\mathcal{D}_n f (x) = \lim_{q \to 1} D_n (q, x).
\]
From equation (1.2) it is clear that the $n$th quantum derivative is an analogue of the $n$th forward Riemann derivative
\[
\lim_{h \to 0} \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{(k-1)k/2} f \left( q^{n-k} x + (n-k) h \right)}{q^n}.
\]
There is also a symmetric Riemann derivative based on $n+1$ uniformly spaced points distributed symmetrically about $x$. The analogue of this is the $n$th symmetric quantum derivative,
\[
\mathcal{D}_n^S f (x) = \lim_{q \to 1} \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{(k-1)k/2} f \left( q^{n-k} x \right)}{q^{(n-1)n/2} (q-1)^n x^n}.
\]
The simple substitution $qx = x + h$ shows that $\mathcal{D}_1 f (x)$ is identical to the ordinary first derivative. The second quantum derivative $\mathcal{D}_2 f (x)$ is given by
\[
\mathcal{D}_2 f (x) = \lim_{q \to 1} \frac{f(q^2 x) - f(qx) - f(qx) + f(x)}{q^2 x - x} = \lim_{q \to 1} \frac{f(q^2 x) - (1 + q) f(qx) + q f(x)}{q (q-1)^2 x^2}.
\]
Perhaps the most important notion of higher order differentiation is that of the $n$th Peano derivative. The function $f (x)$ has an $n$th Peano derivative if $f$ can be approximated to $n$th order by an $n$th degree polynomial at the point $x$. More explicitly, there must exist constants $f_0 (x), f_1 (x), \ldots, f_n (x)$ such that
\[
f (x + h) = f_0 (x) + f_1 (x) h + \ldots + f_n (x) \frac{h^n}{n!} + o (h^n)
\]
as $h$ tends to 0. That the existence of the ordinary $n$th derivative at a point $x$ implies the existence of $f_n (x)$ is the content of a strong version of Taylor’s Theorem that was proved by Peano in 1884. (See page 246 of [AGV] for details.) If $n = 0$, $f_0 (x)$ exists and coincides with $f (x)$ exactly when $f$ is continuous at $x$, and if $n = 1$, the definitions of $f_1 (x)$ and the ordinary first derivative are the same. But if $n \geq 2$, the existence of the $n$th Peano derivative at each point of a set of positive Lebesgue measure does not imply the existence of the ordinary $n$th derivative at
almost every point of that set. [O] Thus if one accepts the centrality of the notion of nth degree polynomial approximation (and there are many physical reasons for doing so), when \( n \geq 2 \), it is the nth Peano derivative and not the ordinary nth derivative that is the benchmark for testing generalized differentiation schemes against. For example, each of the nth forward Riemann derivative and the nth symmetric Riemann derivative is equivalent almost everywhere to the nth Peano derivative. [MZ], [A]

If the nth Peano derivative \( f_n \) exists at \( x \), then it is easy to see that \( D_n f(x) \) must also exist and that then \( D_n f(x) = f_n(x) \). Consideration of the function \( |x - 1| \) at \( x = 1 \) shows that the second quantum derivative is more general than the second Peano derivative. However, the nth quantum derivative is essentially equivalent to the nth Peano derivative. Explicitly, we show that the existence of the nth quantum derivative at each point of a Lebesgue measurable set necessarily implies the existence of the nth Peano derivative at almost every point of that set. It will be clear from the proof (see especially the Sliding Lemma below) that the same result holds true for the nth symmetric quantum derivative \( D^S_n f(x) \).

The nth quantum derivative has proven to be useful in the study of hypergeometric series [GR]. One defect that it has is that it cannot be defined at the point \( x = 0 \). The nth Riemann derivative and its generalizations discussed in [A] involve sums of the form \( h^{-n} \sum_a f(x + b_n h) \), where the coefficients \( \{a_n\} \) are constants. The setting here is more complicated and more delicate techniques are needed since now the coefficients are no longer constant. For example, if we let \( h = qx - x \), the definition of \( D_2 f(x) \) becomes

\[
\lim_{h \to 0} h^{-2} \left( \frac{x}{x+h} f \left( x + 2h + \frac{h^2}{x} \right) - \frac{2x + h}{x + h} f(x + h) + f(x) \right).
\]

One technical point that arises in the proofs of two Lemmas below is the question of the measurability of certain sets. Much light is shed on this question by H. Fejzic and C. E. Weil who give an example that depends on the existence of a measurable set \( S \) for which \( S + S \) is not measurable. [FW] K. Ciesielski has shown us how such a set \( S \) can be constructed. [C] See his theorem and his proof in the appendix at the end of this paper.

2. Results

**Proposition 1.** If the nth Peano derivative \( f_n(x) \) exists and \( x \neq 0 \), then the nth quantum derivative \( D_n f(x) \) also exists and \( D_n f(x) = f_n(x) \).

**Proof.** The existence of the nth Peano derivative at \( x \) means that there are constants \( f_0(x) = f(x) \), \( f_1(x) \), \( f_2(x) \), ..., \( f_n(x) \) such that

\[
f(x + h) = \sum_{j=0}^{n} \frac{f_j(x)}{j!} h^j + o(h^n).
\]
We have
\[
D_n(q, x) = \frac{\Delta_n(q, x)}{q^{(n-1)n/2} (q-1)^n x^n} \\
= \frac{1}{q^{(n/2)} (q-1)^n x^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} q^{(k)/2} f(x + (q^{n-k} - 1) x)
\]
so it suffices to prove that
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} q^{(k)/2} (q^{n-k} - 1) = 0, j = 0, 1, \ldots, n-1
\]
and
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} q^{(k)/2} (q^{n-k} - 1)^n = [n]_q!,
\]
where \([n]_q!\) satisfies
\[
[n]_q! = (1 + q) (1 + q + q^2) \cdots (1 + q + \ldots + q^{n-1}), n = 1, 2, \ldots,
\]
and
\[
\lim_{q \to 1} [n]_q! = n!.
\]
Let
\[
B_k = \frac{(-1)^k q^{(k)/2}}{q^{(n-1)n/2} (q-1)^n} \binom{n}{k} q^{(k)/2} a_k = q^{n-k} - 1, k = 0, 1, \ldots, n.
\]
It would suffice to know that \(X_k = B_k\), where the \(n\) dimensional vector \(X\) is the solution to the Vandermonde system of equations
\[
\sum_{k=0}^{n} a_k^j X_k = \begin{cases} 
0 & j = 0, 1, \ldots, n-1 \\
[n]_q! & j = n
\end{cases}.
\]
Indeed, by Cramer’s rule, the unique solution to this system is given by

\[
X_k = \frac{[n]_q!}{\prod_{j \neq k} (a_k - a_j)} = \prod_{j \neq k} \left( (q^{n-k} - 1) - (q^{n-j} - 1) \right) = \prod_{j \neq k} (q^{n-k} - q^{n-j})
\]

\[
= \prod_{j<k} (q^{n-k} - q^{n-j}) \cdot \prod_{j>k} (q^{n-k} - q^{n-j})
\]

\[
= \frac{[n]_q!}{q^{(n-k)(n+k-1)/2} (-1)^k \prod_{i=1}^k (q^i - 1) \prod_{i=1}^{n-k} (q^i - 1)}
\]

\[
= \frac{(-1)^k q^{k(k-1)/2}}{q^{n(n-1)/2} (q-1)^n} \prod_{i=1}^k \frac{q^i - 1}{q - 1} \prod_{i=1}^{n-k} \frac{q^i - 1}{q - 1} = \frac{(-1)^k q^{k(k-1)/2}}{q^{n(n-1)/2} (q-1)^n} \left[ \begin{array}{c} n \\ k \end{array} \right].
\]

**Corollary 1.** If the ordinary nth derivative \( f^{(n)}(x) \) exists and \( x \neq 0 \), then the nth quantum derivative \( D_n f(x) \) also exists and \( D_n f(x) = f^{(n)}(x) \).

**Proof.** By Peano’s strong version of Taylor’s Theorem, the existence of \( f^{(n)}(x) \) is sufficient for the existence of \( f_n(x) \) as well as for the equation \( f_n(x) = f^{(n)}(x) \).

**Corollary 2.** For \( n \geq 2 \), the nth quantum derivative may exist at every point of a set of positive measure without the nth derivative existing at any point of that set.

**Proof.** For \( n \geq 2 \), there is a function having an nth Peano derivative and not having the ordinary nth derivative at each point of a set of positive measure. \( \square \), Theorem 5

**Proposition 2.** There is a measurable function \( f \) such that \( D_2 f(1) \) exists, but \( f \) is discontinuous at \( x = 1 \).

**Remark 1.** Referring back to the discussion of the Peano derivative in the introduction, we see that this means that \( f \) not only has no second Peano derivative at \( x = 1 \), but that \( f \) has neither a first, nor even a zeroth Peano derivative at \( x = 1 \).

**Proof.** For \( x \in [2, 4] \) define \( f \) by \( f(x) = \frac{1}{4-x} \). Since every point in the interval \( (1, 2) \) is uniquely of the form \( q^{1/n} \) for some \( q \in [2, 4] \) and some positive integer \( n \), we may extend \( f \) to the interval \( (1, 2) \) by setting

\[
f \left( q^{1/n} \right) = \frac{q^{1/n} - 1}{q - 1} f(q)
\]

for each \( q \in [2, 4] \) and each positive integer \( n \). Finally define \( f \) to be 0 on \( (-\infty, 1] \cup (4, \infty) \). Fix \( q \) and note that equation (2.2) may be rewritten as

\[
\frac{f \left( q^{1/n} \right) - 0}{q^{1/n} - 1} = \frac{f(q) - 0}{q - 1},
\]

\[
f \left( q^{1/n} \right) = \frac{q^{1/n} - 1}{q - 1} f(q)
\]
which shows that the triangles \((1,0)(q,0)(q, f (q))\) and \((1,0)(q^{1/2}, 0)(q^{1/2}, f(q^{1/2}))\) are similar, so that all of the points \(\{(q^{1/2}, f(q^{1/2})): n = 0, 1, \ldots\}\) lie on a line passing through \((1,0) = (1, f(1))\). Since \(\lim_{x \to -4} f(x) = \infty\), \(\limsup_{x \to -1} f(x) = \infty\) and \(f\) is not continuous at \(x = 1\). However, if \(p \leq 1\), \(f(1) = f(p) = f(p^2) = 0\), while if \(p \in (1, 2)\), then \(p = q^{1/2m}\) for some \(q \in [2, 4)\) and some positive integer \(m\) and

\[
\begin{align*}
\frac{f(p^2) - (1 + p) f(p) + p f(1)}{p(p - 1)^2} &= \frac{f(q^{1/2m^2}) - 0}{q^2(q^{1/2m^2}) - 1} - \frac{(q^{1/2m} + 1)(f(q^{1/2m^2}) - 0)}{(q^{1/2m} + 1)(q^{1/2m^2} - 1)} \left( q^{1/2m^2} - 1 \right),
\end{align*}
\]

where the quantity in curly brackets is 0 by virtue of equations 2.3. Thus

\[
\frac{\mathcal{D}_2 f(1) = \lim_{p \to 1} \frac{f(p^2) - (1 + p) f(p) + p f(1)}{p(p - 1)^2}}{1^2} = \frac{0}{0} = 0.
\]

We will also need another generalized \(q\)-derivative. We will call this one \(\tilde{\mathcal{D}}_n f(x)\). It will be given by means of an \(n\)th difference \(\tilde{\Delta}_n(q, x; f)\) which evaluates the function \(f\) at the \(n + 1\) points \(x, qx, q^2x, q^3x, \ldots, q^{n-1}x\). Let

\[
\tilde{\Delta}_1(q, x; f) = f(qx) - f(x),
\]

and for \(n \geq 2\) let

\[
\tilde{\Delta}_n(q, x; f) = \tilde{\Delta}_{n-1}(q^2, x; f) - \lambda_n(q) \tilde{\Delta}_{n-1}(q, x; f),
\]

where \(\lambda_1(q) = q + 1\), and, when \(n \geq 2\),

\[
\lambda_n(q) = (q^n + 1) \left( q^{n-1} + q \right) \left( q^{n-2} + q^2 \right) \ldots \left( q^{2n-2} + q^{n-2} \right).
\]

We also define difference quotients for each \(n\) by setting

\[
\tilde{D}_1(q, x; f) = \frac{f(qx) - f(x)}{qx - x},
\]

and for \(n \geq 2\),

\[
\tilde{D}_n(q, x; f) = \frac{\tilde{D}_{n-1}(q^2, x; f) - \tilde{D}_{n-1}(q, x; f)}{q^{n-1}x - qx}.
\]

Finally for \(x \neq 0\) and each \(n \geq 1\), we define the generalized derivative

\[
\tilde{D}_n f(x) = \lim_{q \to 1} \tilde{D}_n(q, x; f).
\]

**Lemma 1.** If the \(n\)th Peano derivative \(f_n(x)\) exists and \(x \neq 0\), then the \(n\)th generalized derivative \(\tilde{D}_n f(x)\) also exists and \(\tilde{D}_n f(x) = f_n(x)\).

**Proof.** First we establish the simple connection

\[
\tilde{D}_n(q, x; f) = \frac{n! \tilde{\Delta}_n(q, x; f)}{(q^{2n-1} - 1)(q^{2n-1} - q) \ldots (q^{2n-1} - q^{2n-2}) x^n}, n = 2, 3, \ldots
\]

To do this, we simplify notation by letting \(y^n = f(q^n x), n = 0, 1, \ldots, P_n(q, y) = \tilde{\Delta}_n(q, x; f)\) and \(Q_n(q, y) = \tilde{D}_n(q, x; f)\). In this notation, everything that we will need to prove for this lemma will become an identity between polynomials in \(y\).
whose coefficients are rational functions of \( q \). For example, equations (2.4), (2.6), and (2.7) become

\[
P_n (q, y) = P_{n-1} (q^2, y^2) - \lambda_{n-1} (q) P_{n-1} (q, y),
\]

(2.8)

\[
Q_n (q, y) = \frac{Q_{n-1} (q^2, y^2) - Q_{n-1} (q, y)}{q^{2n-1} - q},
\]

(2.9)

and

\[
Q_n (q, y) = \frac{P_n (q, y)}{(q^{2n-1} - 1) \left( q^{2n-1} - q \right) \cdots \left( q^{2n-1} - q^{2n-2} \right), n = 2, 3, \ldots}
\]

(2.10)

So to prove equations (2.7) it is enough to establish equations (2.10). If \( n = 2 \), both sides are equal to \( \frac{\lambda_{n-1} (q) \mu_{n-1} (q)}{\left( q^{2n-1} - 1 \right) \left( q^{2n-1} - q \right) \cdots \left( q^{2n-1} - q^{2n-2} \right)} \). Assuming the equation for \( n - 1 \), let \( \mu_n (q) \) be the denominator of \( Q_n (q, y) \) in equation (2.10) and calculate

\[
P_n (q, y) = P_{n-1} (q^2, y^2) - \lambda_{n-1} (q) P_{n-1} (q, y)
\]

\[
= \mu_{n-1} (q^2) Q_{n-1} (q^2, y^2) - \lambda_{n-1} (q) \mu_{n-1} (q) Q_{n-1} (q, y)
\]

\[
= \mu_{n-1} (q^2) \left[ Q_{n-1} (q^2, y^2) - Q_{n-1} (q, y) \right]
\]

\[
= \mu_{n-1} (q^2) \left( q^{2n-1} - q \right) Q_n (q, y) = \mu_n (q) Q_n (q, y),
\]

since

\[
\lambda_{n-1} (q) \mu_{n-1} (q) = \left( q^{2n-2} + 1 \right) \left( q^{2n-2} + q \right) \cdots \left( q^{2n-2} - q^{2n-4} \right) \left( q^{2n-2} - q^{2n-3} \right) = \left( q^{2n-1} - 1 \right) \left( q^{2n-1} - q \right) \cdots \left( q^{2n-1} - q^{2n-2} \right) = \mu_{n-1} (q^2)
\]

\[
\mu_{n-1} (q^2) \left( q^{2n-1} - q \right) = \left( q^{2n-1} - 1 \right) \left( q^{2n-1} - q^2 \right) \cdots \left( q^{2n-1} - q^{2n-2} \right) \left( q^{2n-2} - q \right) = \mu_n (q).
\]

Suppose that \( x \neq 0 \) and \( f_n (x) \) exists. Since \( (q^j - 1)^k = O \left( (q - 1)^k \right) \), we may write \( f (q^j x) = f (x + (q^j - 1)) x = \sum_{k=0}^{\infty} f_k (x) \frac{x^k}{k!} \left( q^j - 1 \right)^k + o ((q - 1)^n) \). Thus \( \hat{D}_n (q, x; f) \) is equal to

\[
\frac{1}{(q - 1)^n} \sum_{j=1}^{n} A_0 (q) f (x) + \sum_{j=1}^{n} A_j (q) \sum_{k=0}^{n} f_k (x) \frac{x^k}{k!} \left( q^{2j-1} - 1 \right)^k + o (1),
\]

where the \( A_j \) are bounded rational functions of \( q \) when \( q \) is near 1. Interchanging the order of summation shows that the limit as \( q \to 1 \) of this expression will be \( f_n (x) \) if \( \hat{D}_n (q, x; x^j) = 0 \) for \( j = 0, 1, \ldots, n - 1 \), and \( \hat{D}_n (q, x; x^n) = n! \). Converting to polynomial language, it suffices to show that

\[
Q_n (q, q^j) = 0 \text{ for } j = 0, 1, \ldots, n - 1
\]

(2.11)

and

\[
Q_n (q, q^n) = 1.
\]

(2.12)
We begin by proving relation (2.12). For \( k \geq 1 \), let \( Q_{-1,k}(q) = 0, Q_{0,k}(q) = 1 \), and \( R_{0,k} = 1 \). For \( m \geq 0 \), let \( R_{m,1} = 1 \). For \( m, k \geq 1 \), let

\[
Q_{m,k}(q) = 1 + \sum_{l=1}^{m} \sum_{k \geq k_1 \geq \ldots \geq k_l \geq 1} q^{2^{k_1-1}} \cdots q^{2^{k_l-1}},
\]

and for \( m \geq 1, k \geq 2 \), let

\[
R_{m,k}(q) = 1 + \sum_{l=1}^{m} \sum_{k \geq k_1 \geq \ldots \geq k_l \geq 2} q^{2^{k_1-1}} \cdots q^{2^{k_l-1}}.
\]

So we have the following properties

(a) \( Q_{m,k}(q) = 1 + \sum_{s=1}^{k} q^{2^{s-1}} R_{m-1,s}(q) \) for all \( m \geq 0, k \geq 1 \)

(b) \( Q_{m,k}(q) - R_{m,k}(q) = q \cdot Q_{m-1,k}(q) \) for all \( m \geq 0, k \geq 1 \)

(c) \( R_{m,k}(q) = 1 + \sum_{s=2}^{k} q^{2^{s-1}} R_{m-1,s}(q) \) for all \( m \geq 1, k \geq 2 \)

(d) \( Q_{m,k}(q^2) = R_{m,k+1}(q) \) for all \( m \geq 0, k \geq 1 \).

We now establish that for all \( n \) and for all \( m = 0, 1, \ldots n - 1 \)

\[
Q_{n-m}(q, q^n) = Q_{m,n-m}(q).
\]

(2.13)

We fix \( n \) and do induction backwards on the variable \( m \). For \( m = n - 1 \),

\[
Q_{1}(q, q^n) = 1 + \sum_{i=1}^{n-1} q^i = 1 + \sum_{i=1}^{n-1} \sum_{1 \geq k_i \geq \ldots \geq 1} q^{2^{k_1-1}} \cdots q^{2^{k_i-1}} = Q_{n-1,1}(q).
\]

We suppose that equation (2.13) holds for \( 0 < m \leq n - 1 \). We have to prove

\[
Q_{n-m+1}(q, q^n) = Q_{m-1,n-m+1}(q).
\]

Applying equation (2.9) and the induction hypothesis, we have to prove

\[
(q^{2^{n-m}} - q)Q_{m-1,n-m+1}(q) = Q_{m,n-m}(q^2) - Q_{m,n-m}(q).
\]
Using the properties (a) – (d), we compute

\[ Q_{m,n-m}(q^2) - Q_{m,n-m}(q) = R_{m,n-m+1}(q) - Q_{m,n-m}(q) \]

\[ = \sum_{s=2}^{n-m+1} q^{s-1} R_{m-1,s}(q) - \sum_{s=2}^{n-m} q^{s-1} Q_{m-1,s}(q) \]

\[ = q^{n-m} R_{m-1,n-m+1}(q) - \sum_{s=2}^{n-m} q^{s-1} (Q_{m-1,s}(q) - R_{m-1,s}(q)) - q Q_{m-1,1}(q) \]

\[ = q^{n-m} Q_{m-1,n-m+1}(q) - q \left( \sum_{s=2}^{n-m+1} q^{s-1} Q_{m-2,s}(q) + Q_{m-1,1}(q) \right) \]

\[ = (q^{n-m} - q) Q_{m-1,n-m+1}(q). \]

In particular, equation (2.12) holds.

We can now prove relation (2.11). If \( n = 1, Q_1(q,q^0) = \frac{1-1}{q-1} = 0 \). We assume the property true for \( n - 1 \). Let \( j < n \). If \( j \leq n - 2 \), then by the induction hypothesis \( Q_{n-1}(q,q^j) = 0 \). This means that \( y - q^j \) divides \( Q_{n-1}(q,y) \). But then \( y^2 - q^{2j} \) divides \( Q_{n-1}(q^2, y^2) \) so that \( Q_{n-1}(q^2, q^{2j}) = 0 \). But then

\[ Q_n(q,q^j) = \frac{Q_{n-1}(q^2, q^{2j}) - Q_{n-1}(q, q^j)}{q^{2n-1} - q} = \frac{0 - 0}{q^{2n-1} - q} = 0. \]

In the remaining case, \( j = n - 1 \). Then by equation (2.12), \( Q_{n-1}(q, q^{n-1}) = 1 \). As above, this implies that \( Q_{n-1}(q^2, q^{2n-2}) = 1 \) also. Again we have

\[ Q_n(q, q^{n-1}) = \frac{Q_{n-1}(q^2, q^{2n-2}) - Q_{n-1}(q, q^{n-1})}{q^{2n-1} - q} = \frac{1 - 1}{q^{2n-1} - q} = 0. \]

**Lemma 2.** If \( x \neq 0 \) and \( f_{n-1}(x) \) and \( \tilde{D}_n f(x) \) exist, then \( f_{n}(x) \) also exists and \( \tilde{D}_n f(x) = f_n(x) \).

**Proof.** Fix \( x \neq 0 \). We may assume that \( f(x), f'(x), ..., f_{n-1}(x) \) and \( \tilde{D}_n f(x) \) are all 0. Let \( \varepsilon > 0 \) be given. Then if \( q \in (1/2, 2) \) is sufficiently close to 1, abbreviating \( \tilde{\Delta}_n(q,x,f) \) to \( \tilde{\Delta}_n(q) \), we have

\[ \left| \tilde{\Delta}_{n-1}(q^2) - \lambda_{n-1}(q) \tilde{\Delta}_{n-1}(q) \right| < \varepsilon \left| q - 1 \right|^n |x|^n. \]

Iterate this relation by replacing \( q \) by \( q^{1/2} \) and then multiplying by \( \prod_{j=0}^{i-1} \lambda_{n-1}(q^{1/2^j}) \), \( i = 1, 2, ..., M. \) Letting \( d = \varepsilon \left| x \right|^n \), we get

\[ (2.14) \quad \left| \tilde{\Delta}_{n-1}(q^2) - \lambda_{n-1}(q) \tilde{\Delta}_{n-1}(q) \right| < d \left| q - 1 \right|^n \]

\[ |\lambda_{n-1}(q)| \left| \tilde{\Delta}_{n-1}(q) - \lambda_{n-1}(q^{1/2}) \tilde{\Delta}_{n-1}(q^{1/2}) \right| < d \lambda_{n-1}(q) \left| 1 - q^{1/2} \right| \left| 1 - q \right|^n \]
\[ \left| \lambda_{n-1}(q) \lambda_{n-1}(q^{1/2}) - \lambda(q^{1/2}) \Delta_{n-1}(q^{1/2}) \right| < |1 - q|^n \]

\[ \frac{d \lambda_{n-1}(q) \lambda_{n-1}(q^{1/2})}{1 - q} \left| 1 - q \right|^n \]

\[ \cdots \]

\[ \left| \lambda_{n-1}(q) \lambda_{n-1}(q^{1/2}) \cdots \lambda_{n-1}(q^{2^{n-1}/2^n}) - \lambda(q^{2^{n-1}/2^n}) \Delta_{n-1}(q^{2^{n-1}/2^n}) \right| < |1 - q|^n \]

\[ \left| \lambda_{n-1}(q) \lambda_{n-1}(q^{1/2}) \cdots \lambda_{n-1}(q^{2^{n-1}/2^n}) \right| \frac{1 - q^{1/2^n}}{1 - q} \left| 1 - q \right|^n. \]

Adding all these inequalities, it follows from the triangle inequality that

\[ \left| \Delta_n(q^2) - \lambda_n(q) \lambda_n(q^{1/2}) \cdots \lambda_n(q^{2^{n-1}/2^n}) \Delta_n(q^{2^{n-1}/2^n}) \right| < d \left| 1 - q \right|^n \sum_{i=0}^M \left| \lambda_n(q^{1/2}) \cdots \lambda_n(q^{2^{n-1}/2^n}) \right| \left| 1 - q^{1/2^n} \right|^n. \]

The ratio test shows that the last sum is uniformly bounded,

\[ \limsup_{i \to \infty} \sup_{q \in [1,2]} \frac{\left| \lambda_{n-1}(q) \lambda_{n-1}(q^{1/2}) \cdots \lambda_{n-1}(q^{2^{n-1}/2^n}) \right|}{\left| \lambda_{n-1}(q^{1/2}) \cdots \lambda_{n-1}(q^{2^{n-1}/2^n}) \right|} \left| 1 - q^{1/2^n} \right|^n = \left( \frac{2^{n-1}}{2^n} \right) = \frac{1}{2} < 1. \]

So we may rewrite inequality (2.15) as

\[ \left| \Delta_n(q^2) - \left( \prod_{j=0}^M \lambda_{n-1}(q^{1/2^j}) \right) \Delta_n(q^{2^{n-1}/2^n}) \right| \leq C |x|^n |1 - q|^n. \]

Now applying the identity \( 1 + q^{2^k} = \frac{1 - q^{2^{k+1}}}{1 - q^{2^{k}}} \) to definition (2.5) and then using the identity \( \prod_{j=0}^M \frac{\alpha_j}{\alpha_{j+1}} = \frac{\alpha_0}{\alpha_{M+1}} \), we calculate

\[ \prod_{j=0}^M \lambda_{n-1}(q^{1/2^j}) = \prod_{j=0}^M q^{(2^n-2^{-1})2^{-j}} \left( \frac{1 - q^{2^{n-1}-j}}{1 - q^{2^{n-2}-j}} \right) \prod_{j=0}^{n-3} \left( \frac{1 - q^{2^{n-2}2^{-j}+1}}{1 - q^{2^{n-2}-2^{-j}}} \right) \]

\[ = q^{(2^n-2^{-2})2^{n-1}-j} \prod_{j=0}^{n-3} \left( \frac{1 - q^{2^n-2^{-j}+1}}{1 - q^{2^{n-2}-2^{-j}}} \right). \]

From this it follows that for fixed \( q \), as \( M \to \infty \) there are constants \( C = C(n, q) \) and \( d = d(n, q) \) so that

\[ \left| \prod_{j=0}^M \lambda_{n-1}(q^{1/2^j}) \right| > \left| \prod_{j=0}^M \lambda_{n-1}(q^{1/2^j}) \right| \left| 1 - q^{1/2^n} \right|^{n-1} \]

\[ \left| \prod_{j=0}^M \lambda_{n-1}(q^{1/2^j}) \right| > C |x|^n |1 - q|^n. \]
Since \( f_{n-1}(x) = 0 \), from Lemma 1 we have

\[
\lim_{M \to \infty} \left( \prod_{j=0}^{M} \lambda_{n-1}(q^{1/2^j}) \right) \Delta_{n-1}(q^{1/2^j}) = f_{n-1}(x) = 0,
\]

so that letting \( M \to \infty \) in inequality (2.16) shows that \(|\Delta_{n-1}(q^2)| = o(|q - 1|^n)\), or, equivalently,

\[
|\Delta_{n-1}(q,x; f)| = o(|q - 1|^n).
\]

From this we similarly deduce

\[
\Delta_{n-2}(q,x; f) = o(|q - 1|^n), \Delta_{n-3}(q,x; f) = o(|q - 1|^n), ...
\]

and finally \( \Delta_1(q,x; f) = o(|q - 1|^n) \), i.e. \( f(qx) = o(|q - 1|^n) \). This proves the lemma.

Lemma 3. For each \( n \) and \( q \), there are constants \( C_0, C_1, ..., C_{2n-1-n} \) such that

\[
(2.18) \quad \tilde{D}_n(q,x; f) = \sum_{i=0}^{2n-1-n} C_i D_n(q,q^i x).
\]

For fixed \( n \), these constants are uniformly bounded in \( q \) near \( q = 1 \).

Proof. Fix \( n \). We will show that there are polynomials \( q, C_0', C_1', ..., C_{2n-1-n}' \) such that

\[
(2.19) \quad \tilde{\Delta}_n(q,x; f) = \sum_{i=0}^{2n-1-n} C_i' \Delta_n(q,q^i x).
\]

This is sufficient, because it will then follow from definitions (1.1) and (2.7) and the fact that

\[
q^{2n-1} - q^{2n-1} = q^{2n-1} \left( q^{2n-1-2^{i-1}} - 1 \right) \simeq c(j) (q - 1) \text{ as } q \to 1
\]

that there is a rational function \( \frac{a(q)}{b(q)} \) with \( a(1) \neq 0 \) and \( b(1) \neq 0 \) such that identity (2.18) holds with \( C_j(q) = \frac{a(q)}{b(q)} C_j'(q) \) for all \( j \). The polynomials \( a(q) \) and \( C_j(q) \) are uniformly bounded above in \( q \) near \( q = 1 \) and since \( b(1) \neq 0 \) the polynomial \( b(q) \) is uniformly bounded below in \( q \) near \( q = 1 \).

To prove equation (2.19), write \( \Delta_n(q,x) = f(q^n x) + a_{n-1}(q) f(q^{n-1} x) + ... + a_0(q) f(x) \) as \( p(q,y) = y^n + a_{n-1}(q) y^{n-1} + ... + a_0(q) \), using once more the polynomial notation used in the proof of Lemma 1 above. Then for each nonnegative integer \( i \), we have the identification of \( \Delta_n(q,q^i x) \) with \( y^i p(q,y) \). Recalling that \( \tilde{\Delta}_n \) was identified with a polynomial \( P_n \) of degree \( 2^n-1 \) in \( y \), we see that the lemma amounts to proving that \( P_n(q^i y) = (C_0 + C_1 y + ... + C_{2n-1-n} y^{2^n-1-n}) p(y) \). By the division algorithm, there are polynomials \( Q \) and \( R \) with \( \deg(Q) = 2^n-1 - n \) and \( \deg(R) < \deg(p) = n \) so that

\[
R = P_n - Qp.
\]

Note that the coefficients of \( Q \) are polynomials in \( q \) since the definitions (1.2), (2.4) and (2.5) show the coefficients of \( P_n \) and \( p \) to be polynomials in \( q \) and \( p \) is monic.

But \( P_n(q^j) = 0 \), for \( j = 0,1,...,n-1 \) by equations (2.11) and (2.10), and that \( p(q^j) = 0 \), for \( j = 0,1,...,n-1 \) follows easily from equations (2.1). Since \( R \) is
a polynomial of degree smaller than \( n \) with \( n \) distinct zeros, \( R \) must be 0 which establishes equation (2.19). Lemma 3 is proved.

**Lemma 4 (The Sliding Lemma).** Let \( a_0, \ldots, a_N \) be distinct real numbers and \( A_0(q), \ldots, A_N(q) \) bounded measurable functions such that \( a_0 \neq 0 \) and \( |A_0(q)| > C > 0 \) when \( q \) is near 1. Let \( b \) be a non-negative real number. If

\[
\sum_{i=0}^{N} A_i(q)f(q^{a_i}x) = O((q - 1)^b)
\]

for all \( x \in E \), then for any real number \( a \),

\[
\sum_{i=0}^{N} A_i(q)f(q^{a_i-a}x) = O((q - 1)^b)
\]

for almost every \( x \in E \). If “\( O \)” is replaced by “\( o \)” in the hypothesis, then the conclusion also holds with “\( o \)” in place of “\( O \)”.

**Proof.** We begin with some measure theoretic preliminaries. Let \( a \neq 0 \) and \( I = (p, q) \) with \( 0 < p < q < +\infty \). The Mean Value Theorem guarantees that there exists a constant \( C(I) \) such that for all \( x, y \in I \),

\[
C(I)|x - y| \leq |x^a - y^a| \leq \left( \frac{q}{p} \right)^{|a-1|} C(I)|x - y|,
\]

where \( C(I) = \inf_{z \in I} |a|z^{a-1} \). Let \( E \subset I \) and denote outer Lebesgue measure by \( m^* \). We have

\[
m^*(E^a) = \inf \left\{ \sum_{n=1}^{+\infty} p_n - q_n ; E^a \subset \bigcup_{n=1}^{+\infty} (p_n, q_n) \subset I^a \right\}
\]

\[
= \inf \left\{ \sum_{n=1}^{+\infty} p_n - q_n ; E \subset \bigcup_{n=1}^{+\infty} (p_n, q_n)^{1/a} \subset I \right\}
\]

\[
= \inf \left\{ \sum_{n=1}^{+\infty} |p_n^a - q_n^a| ; E \subset \bigcup_{n=1}^{+\infty} (p_n, q_n) \subset I \right\},
\]

so by inequalities (2.22)

\[
C(I)m^*(E) \leq m^*(E^a)
\]

\[
\leq \left( \frac{q}{p} \right)^{|a-1|} C(I)m^*(E).
\]

Let \( G \) be a cover of \( E \). This means that \( E \) is contained in the Lebesgue measurable set \( G \) and for every measurable set \( A \), \( m^*(E \cap A) = |G \cap A| \). We now show that \( G^a \) is a cover of \( E^a \). It is clear that \( G^a \) is measurable and \( E^a \subset G^a \). So it’s enough to prove that \( m^*(E^a) \geq |G^a| \). Let \( p_{k,n} = p + k(q - p) / n \) for \( n \) a natural number and \( k = 0, 1, \ldots, n \) and \( I_{k,n} = (p_{k-1,n}, p_{k,n}) \). It follows that

\[
m^*(E^a) = \sum_{k=1}^{n} m^*((E \cap I_{k,n})^a),
\]
whence by inequality (2.23)

\[ m^*(E^n) \geq \sum_{k=1}^{n} C(I_{k,n})m^*(E \cap I_{k,n}). \]

Since \( G \) is a cover of \( E \)

\[ m^*(E^n) \geq \sum_{k=1}^{n} C(I_{k,n})|G \cap I_{k,n}|, \]

so by inequality (2.24)

\[ m^*(E^n) \geq \sum_{k=1}^{n} \left( \frac{p_{k-1,n}}{p_{k,n}} \right)^{[a-1]} |(G \cap I_{k,n})^a|. \]

Bound the fractions from below and then let \( n \to \infty \),

\[ m^*(E^n) \geq \left( \frac{p}{p + \frac{q-b}{n}} \right)^{[a-1]} |G^a| \]
\[ \geq |G^a|. \]

Now let \( a, b \) be real numbers, with \( a \) not zero, and \( I_q \) the interval with endpoints \( q \) and \( 2q - 1 \). Let \( x \neq 0 \) be a point of outer density of \( E \) and define \( A_q = \{ p \in I_q : q^b p^a x \in E \} = (q^{-b} x^{-1} E)^{1/a} \cap I_q \). We next prove that for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that:

(2.25) \[ m^*(A_q) > |I_q|(1 - \varepsilon) \text{ if } 0 < |q - 1| < \delta. \]

We can suppose that \( x = 1 \) and \( E \subset (1/2, 3/2) \). Let \( G \subset (1/2, 3/2) \) be a cover of \( E \). As we showed above, \( G^a \) is a cover of \( E^a \), so that

\[ m^*(A_q) = \int_{I_q} \chi_G(q^b p^a)dp, \]

where \( \chi_G \) is the characteristic function of \( G \). Letting \( s = q^b p^a \),

\[ m^*(A_q) = \frac{1}{|a| q^{b/a}} \int_{q^b I_q^a} \chi_G(s)s^{1/a-1}ds \]
\[ \geq \frac{k(q)}{|a|} \int_{q^b I_q^a} \chi_G(s)ds \]

where \( k(q) = \inf_{s \in q^b I_q^a} s^{1/a-1} \to 1 \) as \( q \to 1 \). Since \( x \) is a point of density of \( G \),

\[ \frac{k(q)}{|a|} \int_{q^b I_q^a} \chi_G(s)ds \simeq \frac{|I_q^a|}{|a|} \simeq |q - 1| = |I_q|. \]

This establishes relation (2.25).

Passing to the proof, let:

\[ E_j = \left\{ x \in E : \left| \sum_{i=0}^{N} A_i(q^b x) \right| \leq j|q - 1| \text{ whenever } |q - 1| \leq 1/j \right\} \]

(The following argument, modeled on work of Fejzić and Weil avoids the question of whether the sets \( E_j \) are Lebesgue measurable.\cite{FW}. There is a strongly related example in \cite{FW} that depends on the existence of a measurable set \( S \) for which \( S+S \) is not measurable. To see how such a set \( S \) can be constructed, see the Theorem of
Let it’s easy to prove that $B_0 = \emptyset$ and define a point of outer density of $E$ as a subset of $\bigcup_j E_j$ of outer density of every $E_j$. K. Ciesielski in the appendix at the end of this paper. Then $E = \bigcup_j E_j$. For each positive integer $j$, the subset $N_j$ of $E_j$ consisting of those points of $E_j$ that are not points of outer density of $E_j$ is measurable and has measure 0, regardless of whether $E_j$ is itself measurable. [M], p. 290 Thus if a property holds at every point of outer density of every $E_j$, then the subset of $E$ where that property fails to hold is a subset of $\bigcup_j N_j$, and thus has measure 0. So, fixing $j$ and letting $x \neq 0$ be a point of outer density of $E_j$, it suffices to prove that relation (2.21) holds at $x$. We define

$$A_k = \{ p \in I_q : q^{a_k - a} p^{-a_0} x \in E_j \}, k = 0, \ldots, N$$

and

$$B_\ell = \{ p \in I_q : q^{-a} p^{a_\ell - a_0} x \in E_j \}, \ell = 1, \ldots, N.$$  

Also define

$$A_k = \{ p \in I_q : \sum_{i=0}^{N} A_i(p) f(p^{a_i} q^{a_k - a} p^{-a_0} x) \leq j 2^b |q - 1|^b \}, k = 0, \ldots, N$$

and

$$B_\ell = \{ p \in I_q : \sum_{i=0}^{N} A_i(q) f(q^{a_i} q^{-a} p^{a_\ell - a_0} x) \leq j |q - 1|^b \}, \ell = 1, \ldots, N.$$  

Let $\varepsilon \in (0, (2N + 1)^{-1})$. By relation (2.25), there exists $\delta \in (0, 1/(2j))$ such that if $0 < |q - 1| < \delta$ then $m^*(A_k^*) > |q - 1|(1 - \varepsilon)$ and $m^*(B_\ell^*) > |q - 1|(1 - \varepsilon)$. Using that $|p - 1| < 1/j$, it follows that $A_k^* \subset A_k$. Similarly, but using that $|q - 1| < 1/j$, it’s easy to prove that $B_\ell^* \subset B_\ell$. Therefore

$$m \left( \bigcap_k A_k \right) \cap \bigcap_\ell B_\ell > |q - 1|(1 - (2N + 1)\varepsilon) > 0.$$  

Let $p \in (\bigcap_k A_k) \cap (\bigcap_\ell B_\ell)$. Then

$$\left| A_0(p) \sum_{k=0}^{N} A_k(q) f(q^{a_k - a} x) \right| = \left| A_0(p) \sum_{k=0}^{N} A_k(q) f(q^{a_k} q^{-a_0} a q_x) \right| =$$

\[
\left| \sum_{\ell=0}^{N} A_\ell(p) \sum_{k=0}^{N} A_k(q) f(q^{a_k} q^{-a_0} a q_x) - \sum_{\ell=0}^{N} A_\ell(p) \sum_{k=0}^{N} A_k(q) f(q^{a_k} q^{-a_0} a q_x) \right| \leq
\]

\[
\sum_{k=0}^{N} |A_k(q)| \left| \sum_{\ell=0}^{N} A_\ell(p) f(q^{a_k} q^{-a_0} a q_x) \right| + \sup_{\ell} \| A_\ell \|_\infty \sum_{k=0}^{N} |A_k(q)| \left| \sum_{\ell=0}^{N} A_\ell(p) f(q^{a_k} q^{-a_0} a q_x) \right| \leq
\]

\[
\sup_k \| A_k \|_\infty (N + 1) j 2^b |q - 1|^b + \sup_\ell \| A_\ell \|_\infty N j |q - 1|^b.
\]

Dividing by $|A_0(p)|$, we finish the proof.  

**Lemma 5.** Let $a_0, \ldots, a_N$ be distinct real numbers and $A_0(q), \ldots, A_N(q)$ bounded measurable functions such that $a_0 \neq 0$ and $|A_0(q)| > C > 0$ when $q$ is near 1. If

$$\sum_{i=0}^{N} A_i(q) f(q^{a_i} x) = O(1) \tag{2.26}$$

for all $x \in E$, then $f(x)$ is bounded in a neighborhood of almost every point of $E$.  

For almost every $x$ and $E$ by definition of $E$ we avoid the question of whether $E_j$ is measurable. Working at a point $x \neq 0$ of outer density of $E_j$ and making use of a cover of $E_j$, just as in the proof of the Sliding Lemma, we produce for each $q$ sufficiently close to 1, a corresponding number $p$ such that

\begin{equation}
\text{(2.27)} \quad \text{all of the points } p^{-a_0}q^{a_0}x \text{ and } p^{a_1-a_0}q^{a_0}x, i = 1, 2, ..., N \text{ are in } E_j.
\end{equation}

Thus all $y$ sufficiently close to $x$ can be written in the form $y = q^{a_0}x$ where there is a number $p$ such that condition (2.27) holds. For such a $y$ we have

\begin{equation}
A_0(p)f(y) = \left\{ \sum_{i=0}^{N} A_i(p) f(p^{a_1} (p^{-a_0}q^{a_0}x)) \right\} - \sum_{i=1}^{N} A_i(p) \left\{ f(p^{a_1-a_0}q^{a_0}x) \right\}.
\end{equation}

By definition of $E_j$, every term in curly brackets is bounded by $j$. Since $A_0(p)$ is bounded below, we may divide through by it and then apply the triangle inequality to get a bound for $|f(y)|$.

**Lemma 6.** If for every $x$ in the measurable set $E$,

\[ \limsup_{q \to 1} \left| \tilde{D}_n(q, x; f) \right| < \infty, \]

then

\begin{equation}
\text{(2.28)} \quad \limsup_{q \to 1} \left| \tilde{D}_{n-1}(q, x; f) \right| < \infty
\end{equation}

for almost every $x \in E$.

**Proof.** Fix $x \neq 0, x \in E$ such that $f$ is bounded in a neighborhood of $x$. So there is a constant $K$ such that for all $q$ sufficiently close to 1 there hold

\begin{equation}
\left| \tilde{D}_n(q, x; f) \right| \leq K \text{ and } |f(qx)| \leq K.
\end{equation}

Because of Lemma 5, it suffices to show that inequality (2.28) holds at $x$. From (2.29) we have

\[ \left| \tilde{D}_{n-1}(q^2) - \lambda_{n-1}(q) \tilde{D}_{n-1}(q) \right| \leq KC |q - 1|^n |x|^n. \]

Repeating the iteration process that was used in the proof of Lemma 2 leads to

\[ \left| \tilde{D}_{n-1}(q^2) - \left( \prod_{j=0}^{M} \lambda_{n-1}(q^{1/2^j}) \right) \tilde{D}_{n-1} \left( q^{1/2^M} \right) \right| \leq KC' |q - 1|^n |x|^n. \]

But $\left| \tilde{D}_{n-1}(q^2) \right| \leq C''K$, so

\[ \left| \left( \prod_{j=0}^{M} \lambda_{n-1}(q^{1/2^j}) \right) \tilde{D}_{n-1} \left( q^{1/2^M} \right) \right| \leq C''K. \]
is uniformly bounded as \( q \) varies between \( b \) and \( b^{1/2} \) for some fixed \( b \) close to 1. From inequalities (2.17), it follows that

\[
\tilde{\Delta}_{n-1} \left( q^{1/2M} \right) \frac{1}{(1 - q^{1/2M})^{n-1}}
\]

and, as a consequence of equation (2.7), also \( \tilde{D}_{n-1} \left( q^{1/2M} \right) \) is uniformly bounded as \( q \) varies between \( b \) and \( b^{1/2} \) for some fixed \( b \) close to 1. As \( M \) varies through the positive integers, \( q^{1/2M} \) takes on all values between \( b \) and 1. To prove relation (2.28), make two appropriate choices of \( b \), one larger than 1 and one less than 1.

\begin{lemma}
If for every \( x \in E \), we have

\[ f(x + t) = f(x) + tf_1(x) + \ldots + \frac{t^{n-1}}{(n-1)!} f_{n-1}(x) + \frac{t^n}{n!} \omega(x, t), \]

where \( \omega(x, t) = O(1) \) as \( t \to 0 \), then the Peano derivative \( f_n(x) \) exists for almost every \( x \in E \). [MZ], Lemma 7
\end{lemma}

\begin{lemma}
If \( x \neq 0 \), \( f_{n-1}(x) \) exists, and

\[
\limsup_{q \to 1} |\tilde{D}_n(q, x; f)| < \infty,
\]

then

\[ f(x + t) = f(x) + tf_1(x) + \ldots + \frac{t^{n-1}}{(n-1)!} f_{n-1}(x) + \frac{t^n}{n!} \omega(x, t) \]

where \( \omega(t) \) is bounded as \( t \to 0 \).

This lemma follows by exactly the same argument as was used to prove Lemma 2.

\begin{theorem}
If for every \( x \) in the measurable set \( E \),

\( \limsup_{q \to 1} |D_n(q, x)| < \infty \),

then the \( n \)th Peano derivative exists for almost every \( x \in E \).
\end{theorem}

\begin{corollary}
If \( D_n f(x) \) exists for every \( x \) in the measurable set \( E \), then the \( n \)th Peano derivative exists at almost every point of \( E \).
\end{corollary}

\begin{corollary}
If \( D_n^S f(x) \) exists for every \( x \) in the measurable set \( E \), then the \( n \)th Peano derivative exists at almost every point of \( E \).
\end{corollary}

The first corollary clearly follows from Theorem 1, while the second follows from the Sliding Lemma and Theorem 1, so we pass to the proof of Theorem 1.

\begin{proof}
Because of Lemma 3 and the Sliding Lemma, we may replace condition (2.30) by

\[
\limsup_{q \to 1} |\tilde{D}_n(q, x)| < \infty \text{ for almost every } x \in E.
\]

Now apply Lemma 6 \( n - 1 \) times to see that

\( \limsup_{q \to 1} |\tilde{D}_k(q, x; f)| < \infty, k = n - 1, n - 2, \ldots, 1, \)
holds almost everywhere on \( E \). In particular,
\[
\limsup_{q \to 1} \left| \tilde{D}_1(q, x; f) \right| = \limsup_{q \to 1} \left| \frac{f(qx) - f(x)}{qx - x} \right| < \infty
\]
holds almost everywhere on \( E \). But the change of variable \( h = qx - x \) shows that
this is equivalent to
\[
\limsup_{h \to 0} \left| \frac{f(x + h) - f(x)}{h} \right| < \infty,
\]
so \( f'(x) \) exists almost everywhere on \( E \) by a theorem of Denjoy.\(^{[D]}\), p. 270 of \([S]\)

Proceed inductively. First use the existence of \( f'(x) \), condition (2.31) with
\( k = 2 \), Lemma 8, and 7 to establish the existence of \( f_2(x) \) almost everywhere on \( E \).
Then use the existence of \( f_2(x) \), condition (2.31) with \( k = 3 \), Lemma 8, and 7 to
establish the existence of \( f_3(x) \) almost everywhere on \( E \). After \( n - 2 \) similar steps,
Theorem 1 is proved.

\[\square\]

3. Appendix

**Theorem 2 (K. Ciesielski).** There is a Lebesgue measurable set \( S \) such that
\( S + S \) is not Lebesgue measurable.\(^{[C]}\)

**Proof.** Let \( K = \{a_1a_2... : \) every \( a_i \in \{0, 1, 3, 4\} \} \), where \( a_1a_2... \) is a base 5
decimal expansion, be the Cantor set obtained by removing the open middle fifth
of \([0, 1]\), then removing the open middle fifth of each of the 4 intervals \([0, 1/5]\),
\([1/5, 2/5]\), \([3/5, 4/5]\) and \([4/5, 1]\), et cetera. Then \(|[0, 1] \setminus K| = 1/5 + 4(1/5)^2 +
4^2(1/5)^3 + ... = 1\), so \(|K| = 0\). Every \( x = \alpha_x1x_2... \in (0, 2) \) has \( \tau \) representations
of the form \( y + z \) where \( y = .y_1y_2... \) and \( z = .z_1z_2... \) are in \( K \), and \( \tau \) is the
cardinality of the continuum \( \mathbb{R} \). Indeed, when \( \alpha = 0 \), there are two possible cases.
First, if infinitely many \( x_i \neq 0 \), each such \( x_i \) may be written as \( y_i + z_i \), with
\( y_i, z_i \in \{0, 1, 3, 4\}\)in two different ways, while \( y_i = z_i = 0 \) otherwise. Second,
if \( n \) is the largest index with \( x_n \neq 0 \), for each \( i > n \), choose \( y_i \) to be either 1
or 3 and set \( z_i = 4 - y_i \); while if \( 1 \leq i \leq n - 1 \), there are \( y, z_i \in \{0, 1, 3, 4\} \)
such that \( .x_1x_2...x_n - 5^{-n} = .y_1y_2...y_n + .z_1z_2...z_n \). In all cases, there are \( 2^{8n} = \tau \)
representations of \( x \). If \( \alpha \neq 0 \), then \( x = .444...4x_mx_m+1... \), where \( x_m < 4 \). This case
is treated similarly, since there exist \( y_m, z_m \in \{0, 1, 3, 4\} \) such that \( .444...4x_m =
.444...4y_m + .444...4z_m \) and as above, there exist \( \tau \) distributions of the tail of \( x \)
between \( y \) and \( z \).

It is enough to show that there is a set \( S \subset K \) so that \( S \) is linearly independent
over the rational numbers \( \mathbb{Q} \) and \( S + S \) is a Bernstein set, that is, both \( S + S \)
and its complement in \([0, 2]\) have nonempty intersection with every perfect subset of \([0, 2]\);
because then \( S \), a subset of a measure 0 set will also have measure 0 and hence be
measurable, but \( S + S \) is not measurable, since neither \( S + S \) nor its complement
can have positive measure (since such sets contain perfect subsets).

To construct \( S \), we will work in \( \mathbb{R}/\mathbb{Z} \), which we will identify with \([0, 2]\). Note,
however, that since \( S \) will be contained in \([0, 1]\), the “+” sign in \( S + S \) coincides
with ordinary addition with possibly the single exception of \( 1 + 1 = 0 \). Let \( \{P_\xi : \xi < \tau\} \)
be a well ordering of all the perfect subsets of \([0, 2]\). (We know that there are
exactly \( \tau \) perfect sets, each of cardinality \( \tau \).\(^{[AW]}\)) For each \( x \in [0, 2] \), let \( K(x) =
\{u \in S : u + v = x \text{ for some } v \in S\} \). Pick \( p_1 \neq 0 \in P_1 \). Since \( \#(K(p_1)) = \tau \),
there is \( a_1 \in K(p_1) \setminus \langle p_1 \rangle \), where \( \langle \cdot \rangle \) denotes the \( \mathbb{Q} \)-span in \( \mathbb{R}/\mathbb{Z} \) of a set \( \cdot \). Then
$S_1 = \{a_1, b_1 = p_1 - a_1\}$ is a linearly independent set and $S_1 + S_1$ intersects $P_1$. Having chosen $S_\omega \subset K$ for each ordinal $\omega < \xi$, such that for each $\alpha \leq \omega$, there is a pair of elements in $S_\omega$ whose sum is in $P_\alpha$, set $S_\xi = \bigcup_{\omega < \xi} S_\omega \cup \{a_\xi, b_\xi\}$ where $a_\xi$ and $b_\xi$ are chosen as follows. Since $\#(P_\xi) > \#(\bigcup_{\omega < \xi} S_\omega)$ and $\#(K(p_\xi)) > \#(\bigcup_{\omega < \xi} S_\omega \cup \{p_\xi\})$, we can find $p_\xi \in P_\xi$ independent from $\bigcup_{\omega < \xi} S_\omega$ and then find $a_\xi \in K(p_\xi)$ independent from $\bigcup_{\omega < \xi} S_\omega \cup \{p_\xi\}$. Let $b_\xi = p_\xi - a_\xi$. It is clear that $S_\xi$ is a linearly independent set and $S_\xi + S_\xi$ intersects $P_\omega$ for all $\omega \leq \xi$. Then $S = \bigcup_{\xi < \kappa} S_\xi$ is linearly independent and intersects every $P_\alpha$. It remains to show that $(S + S)^c$ also has nonempty intersection with every perfect subset of $[0, 2)$. Fix any $s \in S$. The perfect subsets of $[0, 2)$ can also be well ordered as $\{s + P_\xi : \xi < \zeta\}$. For each $\alpha$, we must find a point in $s + P_\alpha$ which is not in $S + S$. From the construction of $S$, we know that $a_\alpha + b_\alpha \in (S + S) \cap P_\alpha$, so that the point $s + a_\alpha + b_\alpha \in s + P_\alpha$. But if $s + a_\alpha + b_\alpha \in S + S$, then $s + a_\alpha + b_\alpha - u - v = 0$, where all five of $s, a_\alpha, b_\alpha, u,$ and $v \in S$. But a sum of an odd number of independent elements can not be zero.

References


[C] K. Ciesielski, Measure zero sets whose algebraic sum is non-measurable, Real Analysis Exchange, 26(2000/01), 919–922.


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