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ONE, TWO, SKIP A FEW... NINETY-NINE, ONE HUNDRED

INQUIRY-BASED ENUMERATIVE COMBINATORICS

DEPAUL UNIVERSITY

Version 1.2

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Contents

List of Figures

List of Tables

Introduction to combinatorial problem solving

"We are continually faced with a series of great opportunities brilliantly disguised as insoluble problems."

–John W. Gardner

The *Tower of Hanoi* is a famous puzzle invented by Edouard Lucas in 1883. There are three pegs and eight disks of all different sizes. Initially the disks are stacked on one peg in decreasing size from the bottom up.

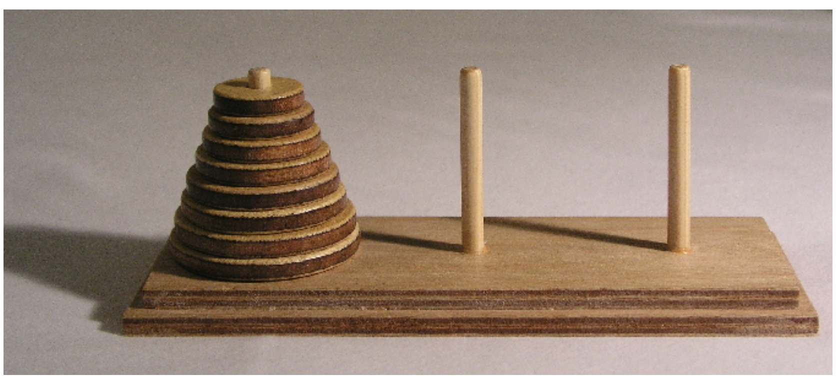


Figure 1: The Tower of Hanoi puzzle.

The object of the game is to move all eight disks onto another peg by moving only one peg at a time. However, a larger disk cannot lie on top of a smaller disk at any time.

Q: How many moves are required to solve the Tower of Hanoi puzzle?

Think about this question for yourself before turning the page.

A good way to approach a puzzle like this is to first generalize the question. For example, why should there be eight disks? What about n disks? If we suppose the puzzle can have any number of disks then we can scale the problem up and down at will. This leads to a good first step for any problem.

Data collection: Consider small cases first.

We can see right away that it's easy to solve the puzzle if there are only one or two disks, and a moment's thought leads us to solve the three disk puzzle. We see that it takes us at least 1, 3, and 7 moves, respectively, to solve these cases.

We now come to an easily overlooked part of problem solving.

Notation: Name and conquer.

As we are collecting our data, it is useful to give a name to the quantities we see. We will let T_n denote the minimum number of moves required to move n disks from one peg to another. ("T" for Tower.) So far, we have $T_1 = 1$, $T_2 = 3$, and $T_3 = 7$. There is an even smaller case we haven't considered yet: $T_0 = 0$, since it takes no moves to move no disks![1]

Now that we've handled some small cases, let's think big. How would we deal with a tower where n is large? You may have already noticed that a winning strategy for three disks is to move the top two disks onto a different peg first, then to move the big guy, then to move the smaller two back on top of the big one. In general, we can move the smallest $n-1$ disks onto a different peg (in T_{n-1} steps), move the biggest disk (in one move), then move the smaller disks back on top (in T_{n-1} more steps). With this strategy, it takes $2T_{n-1} + 1$ steps to move n disks. We don't know if this is the best possible strategy, so it only gives an upper bound on the fastest way to move n disks:

$$T_n \leq 2T_{n-1} + 1, \quad \text{for } n \geq 1.$$

We'd like to say that this is an equality. To do this, we need to show there isn't some other, faster way to move the disks.

So, is there a faster way to do it? No. In order to move the biggest disk to a new peg, all the smaller disks must be on one peg, and this requires T_{n-1} moves. (Here is where we need the fact that there are only three pegs!) We now need one move to get the big guy onto his new peg. To move the smaller disks back on top of the biggest one, we need, by definition, at least T_{n-1} more steps. Thus, we have

$$2T_{n-1} + 1 \leq T_n, \quad \text{for } n \geq 1.$$

[1] It often helps to make a table:

n	0	1	2	3	$\cdots$
T_n	0	1	3	7	$\cdots$

The only way for both of our inequalities to be true is if we in fact have equality. To summarize,

$$T_0 = 0,$$
$$T_n = 2T_{n-1} + 1, \quad \text{for } n \geq 1.$$

Such a set of equalities is called a *recurrence relation*: a way of getting new values in our sequence of numbers, given knowledge of some previous terms. We will see more of these later in the book.

The first nice thing about a recurrence is that it allows us to quickly generate terms in the sequence:

$$\begin{aligned}
T_0 &= 0, \\
T_1 &= 2 \cdot 0 + 1 = 1, \\
T_2 &= 2 \cdot 1 + 1 = 3, \\
T_3 &= 2 \cdot 3 + 1 = 7, \\
T_4 &= 2 \cdot 7 + 1 = 15, \\
T_5 &= 2 \cdot 15 + 1 = 31, \\
T_6 &= 2 \cdot 31 + 1 = 63, \text{ and so on.}
\end{aligned}$$

An obvious drawback of this approach is that if we want, say, the value of T_{100}, we need the value of T_{99}, which in turn requires T_{98}, and so on, all the way back down to $T_0 = 0$. A better solution would give some kind of useful formula for T_n that only depends on some kind of algebraic operations involving n.

Do we recognize the sequence of numbers $0, 1, 3, 7, 15, \ldots$? Aha! They are each one less than a power of 2. Specifically, it seems that

$$T_n = 2^n - 1, \quad \text{for } n \geq 0.$$

(At least it works for $n \leq 6$.) How can we verify this formula in general?

Well, we know it works when $n = 0$, since $2^0 - 1 = 1 - 1 = 0$, and if we suppose that $T_n = 2^n - 1$ for some particular $n \geq 0$, then, via the recurrence relation we find:

$$\begin{aligned}
T_{n+1} &= 2T_n + 1, \\
&= 2(2^n - 1) + 1, \\
&= 2^{n+1} - 2 + 1 = 2^{n+1} - 1, \text{ as desired.}
\end{aligned}$$

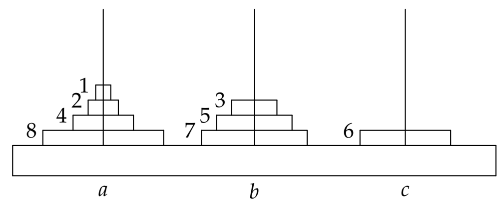
So if our formula works for some value of n, it works for the next value of n as well. Because we know it works for $n = 0$, it must work for $n = 1$, hence for $n = 2$, hence for $n = 3$, and so on.[2] It works for any value of n we like!

[2] A mental image some people like is of a line of dominoes. We prove that if you knock down domino n, then domino $n + 1$ will fall as well. The case $n = 0$ just shows that you can knock down the first domino. This kind of argument is known as a PROOF BY INDUCTION; induction is often an easy way to verify facts about mathematical structures that have some sort of recursive structure.

Now that we have complete faith in our formula, it is trivial to compute T_n. Going back to the original question, we see that $T_8 = 2^8 - 1 = 255$ moves are required to solve the Tower of Hanoi puzzle.

We've answered our original question and then some. But as mathematicians we may want a deeper understanding of the structure of the problem. We have only counted the minimal number of moves required to solve the puzzle; we haven't explicitly described how to solve it. If we really want to knock a problem out of the park, we want a *characterization of solutions*. Not only "how many are there?" but "what are they?"

To do this, it often helps to draw pictures or somehow encode the information in the problem.



disk:	1	2	3	4	5	6	7	8
peg:	*a*	*a*	*b*	*a*	*b*	*c*	*b*	*a*

$\hookrightarrow$ "*aababcba*"

Figure 2: Encoding a stage of the game.

Modeling: distill the essentials.

For the Tower of Hanoi, we can label the pegs a, b, c and write a word like *aababcba* to mean the disks $1, 2, 4, 8$ are on peg a, disks $3, 5, 7$ are on peg b, and disk 6 is on peg c, as shown in Figure 2. We know that the word *aababcba* could only encode this configuration because it tells us precisely which disks go with which pegs; on each peg the disks must be stacked biggest to smallest from the bottom up. (In some sense we now think about the disks first, rather than the pegs first.) This is a big leap forward, as far as bookkeeping goes. It becomes much easier to record a sequence of moves than drawing pictures showing what has happened. For example, to demonstrate how to solve the three disk puzzle, we could simply write down the following steps:

$$aaa \to baa \to bca \to cca \to ccb \to acb \to abb \to bbb.$$

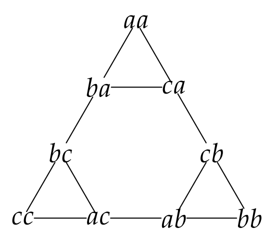
This is certainly simpler than sketching the disks and pegs![3]

Now that we've got a way to encode the possible states of the game as strings of letters, we want to understand how one string of letters gets transformed into another. In particular, we want to know the best way to transform a string of all a's to a string of all b's (or all c's). It's probably good to start with small cases again. If there is only one disk, there are three possible strings, each with one letter: a, b, and c. At any point we can move the disk from one peg to another, so we can swap any of these strings for another. Let's move on to two disks.

There are now nine possible strings: $aa, ab, ac, ba, bb, bc, ca, cb$, and cc. It is no longer possible to get from any string to another with just one move. For instance, aa indicates that both disks are on peg a, so it is impossible to move the bigger disk without first moving the smaller one. Hence we can't transform aa into something like ac with only one move.

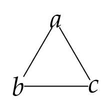
[3] This "encoding" is what's called a Bijection between the set of states of the n-disk game, and the set of (a, b, c)-strings of length n. This lets us easily see that there are 3^n possible game states with three pegs, since there are 3^n (a, b, c)-strings. The *bijective method* for enumeration is to count a set by creating a bijection (a reversible, one-to-one correspondence) with another, easier-to-count set.

After a little bit of thought however, we can sketch the following diagram[4] to indicate which one-step moves are allowed:



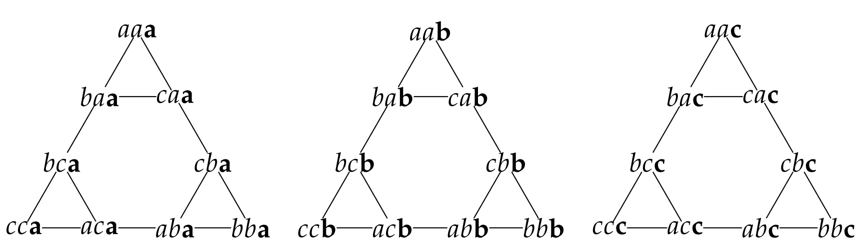
[4] This sort of diagram is called a GRAPH. In general, a graph is a collection of points, called “vertices" or “nodes", and edges between them. There is an entire branch of combinatorics devoted to the study of graphs.

Let’s compare this with the corresponding diagram for one disk:

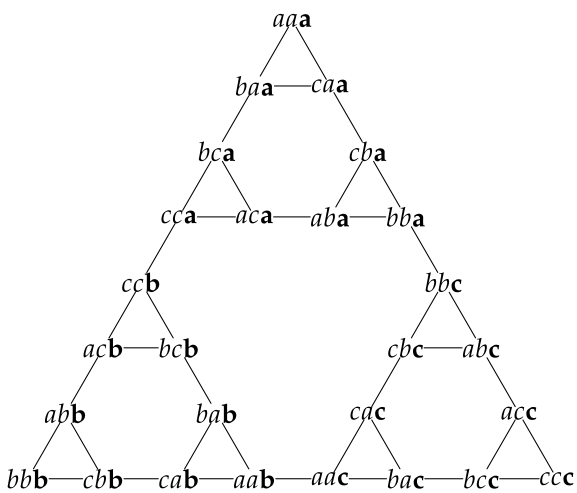


Hmm. It looks like the diagram for two disks is built out of the one-disk diagram by gluing three copies of the one-disk diagram together in a certain way. What is going on here? If we look at the smaller triangle at the top of the two-disk diagram, it looks *exactly* like the one-disk diagram if we were to add the letter a to the end of each string. But this makes sense, because these are just all the states of the game in which the biggest disk is left untouched on peg a. Similarly, the small triangle on the bottom left shows what happens when the largest disk is on peg c, and the bottom right triangle shows what happens when the largest disk is on peg b.

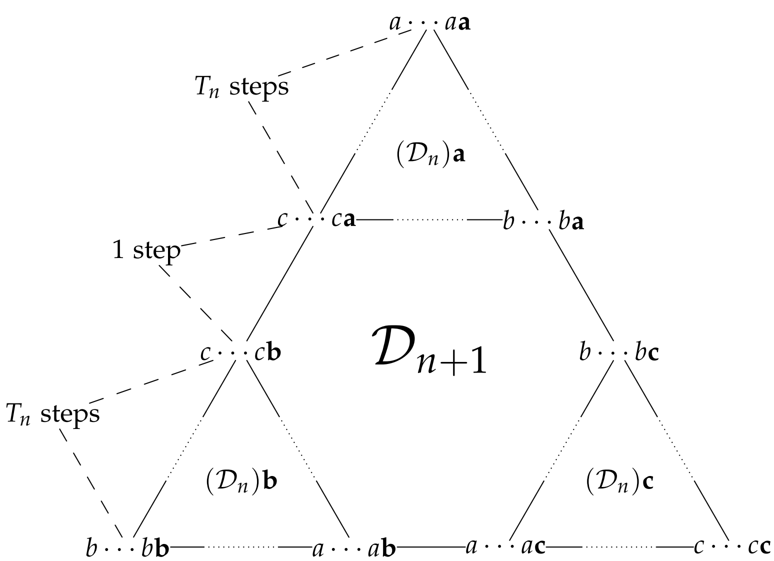
So if we want to figure out the diagram for three disks, we can get the major components of it by taking the diagram for two disks and attaching either an a, a b, or a c to the end of every string in the diagram (emphasized in bold):



All that’s left is to tie these pieces together. But when can we move that biggest disk? Only when all the smaller disks are on the same peg. This occurs at one of the corners of the triangle. We connect these corners where possible ($cca \leftrightarrow ccb$, $bba \leftrightarrow bbc$, $aab \leftrightarrow aac$) and get the following picture:



All right, now we're cooking. We are seeing a kind of *structural recurrence* happening here. If we let $\mathcal{D}_n$ denote the nth diagram, we can continue our reasoning to see how to build $\mathcal{D}_{n+1}$ from $\mathcal{D}_n$. We append the letters a, b, and c to three separate copies of $\mathcal{D}_n$, then join up the appropriate corners:



This picture now lets us read off the optimal strategy for solving the Tower of Hanoi! To move the disks in the tower, we just do the sequence of moves indicated by the edges between the game states down one side of the triangle, from $a \cdots aa$ to $b \cdots bb$, say. Why is this optimal? Because the shortest distance between two points is a straight line! By the way, notice that our numeric recurrence, $T_{n+1} = 2T_n + 1$, is built right into the picture. Awesome!

Now are we done? Only if we want to be. The great thing about mathematics is that there's always room to improvise, always new questions that can be asked. We solved the "3-peg" Tower of Hanoi puzzle. What about the "4-peg" puzzle? The "k-peg" puzzle? What if there are k pegs, and two towers of different colors, and the goal is to relocate both towers? What if there are k pegs and ℓ towers? What if you put more (or fewer) restrictions on the kinds of moves that you allow?

The possibilities are only limited by your imagination.

A word about proofs.

You may have noticed that there was no explicit "Theorem" stated in the Tower of Hanoi discussion, and nowhere was a "proof" clearly delineated. In the problems and theorems that constitute this book, you will be asked to provide solutions, to give explanations, and, occasionally, to "prove" that a certain result is true. What do I mean by "prove"?

To me, a proof is a clear, logical explanation for why something is true. Nothing more or less.

If you've taken a class about proofs, you may have mental templates for how to write proofs, based on labels like "direct proof", "contrapositive", "contradiction", "induction", and so on. That's fine, but I would prefer if you not waste energy worrying about the *form* your explanations take. Invest your energy in the *content* of your explanations, and the form will follow.[5]

[5] This idea is not too different from what Chicago architect Louis Sullivan famously wrote: "It is the pervading law of all things organic and inorganic . . . that form ever follows function. This is the law."

When working on the problems put to you in this book, focus on the following two questions. If you can answer them both satisfactorily, you will be just fine.

What is the Truth?

and

Why is it true?

1
First principles

> " 'Begin at the beginning,' the King said gravely, 'and go on till you come to the end: then stop.' "
>
> –Lewis Carroll, a.k.a., Charles Dodgson

ONE, TWO, THREE ... This is the *caveman's counting algorithm*. The only truly trivial counting problem is one for which the things you wish to count are obviously in a one-to-one correspondence with the first n counting numbers. Otherwise, most counting problems are best approached by breaking them down into smaller, more manageable pieces that can be counted "caveman style".

In this chapter, you'll explore two core counting principles: THE SUM PRINCIPLE and THE PRODUCT PRINCIPLE. The first allows you to count things separately on a case-by-case basis, provided your cases don't overlap (odds/evens, boys/girls, red/white/blue, ...). The second allows you to count the number of outcomes of a sequence of events.

Problem 1. How many even numbers are positive and less than 100?

Definition 1 (Cardinality)**.** *For a finite set S, the number of elements of S is called the* cardinality *of S, denoted by $|S|$. (For convenience, we declare that there is a unique set with 0 elements, called the "empty set" and denoted $\emptyset$.)*

Problem 2. Let $S = \{n \in \mathbb{Z} \mid 1 \leq n \leq 49\}$, i.e., the set of all positive integers from 1 to 49. What is $|S|$? Why does S have the same number of elements as the set in Problem 1?

Definition 2 (Union)**.** *The* union *of two sets S and T, written $S \cup T$, is defined to be the set of all elements of S together with all elements of T, i.e., if $x \in S \cup T$, then $x \in S$ or $x \in T$ (or both).*

Problem 3. With S as in Problem 2 and

$$T = \{n \in \mathbb{Z} \mid 0 < n < 100 \text{ and } n \text{ is even}\}$$

(just as in Problem 1), what is $|S \cup T|$?

Definition 3 (Intersection)**.** *The* intersection *of two sets S and T, written $S \cap T$, is defined to be the set of all elements of both S and T, i.e., if $x \in S \cap T$, then $x \in S$ and $x \in T$.*

Problem 4. With S and T as in Problems 2 and 3, what is $|S \cap T|$?

Problem 5. For any two finite sets S and T, explain why

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

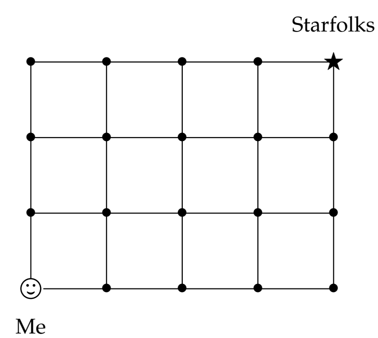


Figure 1.1: Going for coffee.

Problem 6 (Starfolks)**.** I want to get a coffee from Starfolks. The nearest one is four blocks East and three blocks North from here. Assuming I only walk East or North, how many different routes can I take to get there:

1. if the last block I walk is heading North?
2. if the last block I walk is heading East?
3. if I don't care whether the last block I walk is North or East?

(Do you notice a relationship between your answers to the different parts of this problem?)

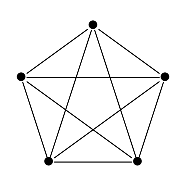


Figure 1.2: The complete graph K_5.

Problem 7. A *complete graph* on n vertices, denoted K_n, is the graph in which every vertex is connected to every other vertex. In Figure 1.2 we see K_5; the complete graph on five vertices. How many edges does K_{10} have?

Definition 4 (Disjoint sets)**.** *If $S \cap T = \emptyset$, i.e., $|S \cap T| = 0$, then we say the sets S and T are* disjoint.

Theorem 1 (Sum Principle)**.** Suppose S and T are disjoint sets. Then

$$|S \cup T| = |S| + |T|.$$

Problem 8. Suppose we roll a six-sided die and then flip two coins. How many distinct outcomes are possible?

Definition 5 (Cartesian Product)**.** *Suppose S and T are sets. Their* cartesian product*, denoted $S \times T$, is the set of all ordered pairs (s,t) such that $s \in S$ and $t \in T$:*

$$S \times T = \{(s,t) \mid s \in S, t \in T\}.$$

In general, if $S_1, S_2, \ldots, S_m$ are sets, their cartesian product is the set of all ordered m-tuples:

$$S_1 \times S_2 \times \cdots \times S_m = \{(s_1, s_2, \ldots, s_m) \mid s_1 \in S_1, s_2 \in S_2, \ldots s_m \in S_m\}.$$

Problem 9. Let $S = \{n \in \mathbb{Z} \mid -5 \leq n \leq 5\}$ and let $T = \{n \in \mathbb{Z} \mid 1 \leq n \leq 6\}$. Describe the set $S \times T$. What is $|S \times T|$?

Problem 10. Let $S_1 = \{1,2,3,4,5,6\}$, $S_2 = \{h,t\}$, and $S_3 = \{h,t\}$. Describe the set $S_1 \times S_2 \times S_3$. What is $|S_1 \times S_2 \times S_3|$?

Theorem 2 (Product Principle)**.** The following are progressively more general versions of the product principle.

1. Suppose S and T are finite sets. Then

$$|S \times T| = |S| \cdot |T|.$$

2. Suppose $S_1, S_2, \ldots, S_m$ are finite sets. Then

$$|S_1 \times S_2 \times \cdots \times S_m| = |S_1| \cdot |S_2| \cdots |S_m|.$$

3. Suppose $\mathcal{S}$ is the set of outcomes of an m-step process, where for any $i \in \{1, 2, \ldots, m\}$, there are a_i choices for step i, no matter what earlier choices were made. Then

$$|\mathcal{S}| = a_1 a_2 \cdots a_m.$$

Problem 11. How many ways can you:

1. Draw a black king from a well-shuffled deck of playing cards?

2. Draw a black king, then (after replacing the first card) draw a black card?

3. Draw a black king, then draw a black card (without replacing the first card)?

4. Draw a black card, then a black king (without replacing the first card)?

5. Draw a black king, then a red card, then a black 7, 8, or 9? (Does replacement matter here?)

Problem 12. Suppose you flip a coin five times in a row, recording the sequence of heads and tails you see, e.g., (h, h, t, h, t). How many different sequences of flips are possible?

Problem 13. It's halloween and five children arrive at your door, all hoping for candy. You have exactly five pieces of candy and you can give away some or all of the candy. Supposing you don't give any child more than one piece, how many different ways can you distribute the candy?

Definition 6 (Subset)**.** *Given a set S, we say a set T is a* subset of S, *written*

$$T \subseteq S,$$

if and only if every element of T is also an element of S, i.e., if and only if

$$S \cap T = T.$$

Notice that it is always true that $\emptyset \subseteq S$ since $S \cap \emptyset = \emptyset$, and $S \subseteq S$ since $S \cap S = S$.

Problem 14. Let $S = \{1, 2, 3, 4, 5\}$. How many subsets does S have?

Problem 15. Let n be a positive integer and suppose $S = \{i \in \mathbb{Z} \mid 1 \leq i \leq n\} = \{1, 2, 3, \ldots, n\}$. In terms of n, how many subsets does S have?

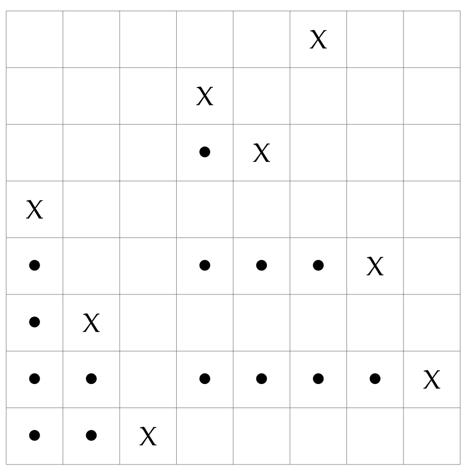
2

Permutations

> "Words differently arranged have a different meaning and meanings differently arranged have a different effect."
>
> –Blaise Pascal

53176842

$\updownarrow$



$\updownarrow$

$(0, 1, 2, 0, 1, 0, 4, 6)$

A permutation is just about the most fundamental structure in all of enumerative combinatorics. Many, many, problems can be recast as a problem about counting certain ordered arrangements of a collection of objects.

In this chapter you will start to learn to count permutations.

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	2				
3	1			6			
4	1				24		
5	1					120	
6	1						720

Table 2.1: Triangle of the numbers $P(n,k)$, or the number of k-permutations of n.

Problem 16. At a concert, you and three friends occupy seats 1, 2, 3, and 4 of row ZZ. How many different ways can you all be seated so that you have seat 1? How many different ways can you all be seated if you don't necessarily occupy seat 1?

Problem 17. Generalizing the previous problem, suppose that there are n people sitting in seats $1, 2, 3, \ldots, n$ of a certain row. How many different ways can these people be arranged?

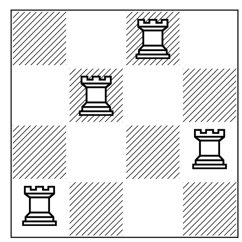


Figure 2.1: An arrangement of four non-attacking rooks on a 4-by-4 chessboard.

Problem 18. We say an arrangement of rooks on a chessboard is *non-attacking* if no two of the rooks lie in the same row or column. For example, Figure 2.1 shows an arrangement of four non-attacking rooks on a 4-by-4 chessboard. How many different arrangements of four non-attacking rooks on a 4-by-4 chessboard are there with a rook in the bottom left corner? How many arrangements are there with a rook in the first column, second row? How many such arrangements are there in total?

Problem 19. How many arrangements of n non-attacking rooks on an n-by-n chessboard are there?

Definition 7 (Factorial notation)**.** *The product of the first n consecutive natural numbers, $1 \cdot 2 \cdot 3 \cdots n$, is called "$n$ factorial", written $n!$ for short. By convention, define $0! = 1$. Notice that $n! = n \cdot (n-1)!$.*

n	0	1	2	3	4	5
$n!$	1	1	2	6	24	120

Problem 20. Now suppose ten friends go to a movie, but it is opening weekend and they can't find ten seats in a row. In one row, they find seats 1–6 unoccupied. How many different ways can six of the

friends sit together in these seats? (Ignore what happens to the four friends who don't sit in these seats.)

Problem 21. Generalizing the previous problem, suppose n friends find only k seats in a row, where $k \leq n$. Find a formula for the number of different ways k of the friends can sit together in these seats.

Problem 22. How many ways are there to arrange six non-attacking rooks on a 10-by-6 chessboard?

Problem 23. How many ways are there to arrange k non-attacking rooks on an n-by-k chessboard?

123	132	213	231	312	321
124	142	214	241	412	421
125	152	215	251	512	521
134	143	314	341	413	431
135	153	315	351	513	531
145	154	415	451	514	541
234	243	324	342	423	432
235	253	325	352	523	532
245	254	425	452	524	542
345	354	435	453	534	543

Figure 2.2: The 3-permutations of $\{1, 2, 3, 4, 5\}$.

Definition 8 (Permutation)**.** *A* k-permutation *of a set S is an ordered list of precisely k elements of S. Let $P(n, k)$ denote the number of k-permutations of a set with n elements. By convention $P(n, 0) = 1$, i.e., there is precisely 1 0-permutation of any set. We see the 3-permutations of* $\{1, 2, 3, 4, 5\}$ *in Figure 2.2.*

The set of n-permutations of n forms the SYMMETRIC GROUP, denoted by S_n. If you have taken a class on group theory, you know that this is a very important finite group.

Theorem 3. For any $n \geq k \geq 0$,

$$P(n, k) = \qquad\qquad \text{(fill in a formula; see Problem 21)}$$

Problem 24. Use your formula for $P(n, k)$ to fill out the rest of Table 2.1.

Problem 25. Describe any patterns you see in the triangle in Table 2.1.

Problem 26. Prove that $P(n, n) = P(n, k)P(n - k, n - k)$, both by using the formula in Theorem 3, and by using the meaning of k-permutations.

Problem 27. Prove that $P(n, k) = P(n - 1, k) + kP(n - 1, k - 1)$, both by using the formula in Theorem 3, and by using the meaning of k-permutations.

Definition 9 (Inversion sequence)**.** *An* inversion sequence *of length n is an element of the cartesian product*

$$\{0\} \times \{0, 1\} \times \{0, 1, 2\} \times \cdots \times \{0, 1, \ldots, n - 1\}.$$

In other words, $s = (s_1, s_2, \ldots, s_n)$ is an inversion sequence of length n if and only if $s_i \in \{0, 1, \ldots, i - 1\}$ for all $1 \leq i \leq n$. For example, $(0, 1, 0, 3)$ *is an inversion sequence of length* 4*.*

Problem 28. How many inversion sequences of length 4 are there?

Problem 29. How many inversion sequences of length n are there?

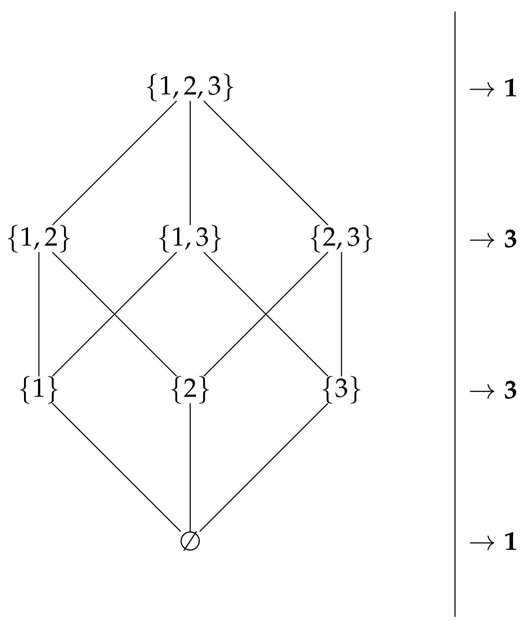
Problem 30. How many inversion sequences of length 10 begin with four zeroes? For example, $(0, 0, 0, 0, 2, 5, 2, 7, 4, 9)$ is such a sequence.

Problem 31. How many inversion seqences of length n begin with k zeroes?

3
Subsets

"If you don't make mistakes, you're not working on hard enough problems. And that's a big mistake."

–Frank Wilczek



Binomial coefficients are among the most important whole numbers you will ever know. Most middle-school students have seen Pascal's triangle, and we will see that this triangle of numbers is one of the most flexible and durable objects in mathematics.

This chapter defines binomial coefficients and explores a few of their properties. In particular, we will find a nice formula for binomial coefficients, prove Pascal's recurrence, and prove a symmetry relation for binomial coefficients.

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1			1			
4	1				1		
5	1					1	
6	1						1

Table 3.1: Triangle of the binomial coefficients $\binom{n}{k}$, or the number of k-subsets of an n-element set.

Definition 10 (Binomial coefficients)**.** *A k-*subset *of a set S is an unordered collection of k elements of S. Let $\binom{n}{k}$ denote the number of k-subsets of an n-element set. (Aloud we read $\binom{n}{k}$ as "n choose k".) By convention $\binom{n}{0} = 1$, i.e., there is precisely one way to choose nothing.*

The binomial coefficient $\binom{n}{k}$ is sometimes denoted $C(n,k)$, and k-subsets are sometimes called "k-combinations".

Problem 32. Five friends are outside a hip LA nightclub, and they all want to get inside. However, the bouncer will only take two of them. How many different ways can the bouncer choose two of them?

Problem 33. Generalizing the previous problem, suppose n friends are outside a hip LA nightclub, but the bouncer will only admit 2 of them. How many different ways can the bouncer choose 2 of the n friends? (Hint: be careful not to double count!)

Problem 34. Now the bouncer consents to admitting 3 of them. How many different ways can the bouncer choose 3 of the n friends? 4 of the friends?

Problem 35. Generalizing the previous problem, suppose the bouncer will now take k of the n friends, where $k \leq n$. Find a formula for the number of ways the bouncer can choose k of the n friends. (Hint: it may help to relate this problem to the result of Problem 21.)

Problem 36. Using the meanings of k-subset and k-permutation, explain why

$$P(n,k) = k!\binom{n}{k}.$$

Theorem 4. For any $n \geq k \geq 0$,

$$\binom{n}{k} = \qquad \text{(fill in a formula; see Problem 36)}$$

Problem 37. Fill out the rest of Table 3.1.

Problem 38. Describe any patterns you see in Table 3.1.

Problem 39. Prove that

$$\binom{n}{k} = \binom{n}{n-k},$$

both by using the formula in Theorem 4 and by using the meaning of $\binom{n}{k}$.

Theorem 5 (Pascal's identity). [1] For any integers $n \geq k \geq 1$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

(Try to prove this both by using the formula in Theorem 4 and by using the meaning of $\binom{n}{k}$.)

[1] This is why we sometimes call the array in Table 3.1 *Pascal's triangle*.

Problem 40. What are the row sums in Table 3.1? That is, for any $n \geq 0$, find a formula for

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

Explain your formula using the meaning of $\binom{n}{k}$ and the result of Problem 15.

Problem 41. What are the diagonal sums in Table 3.1? That is, for any $n \geq 0$, find the numbers

$$\sum_{k \geq 0} \binom{n-k}{k} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots.$$

(The sum ends after about $n/2$ terms because $\binom{a}{b} = 0$ if $b > a$.) You don't need to have an explicit formula for these numbers, but see if you can observe any patterns.

Problem 42. A university committee is composed of 2 men and 3 women. Two of the committee members must serve as co-chairs. How many ways can the co-chairs be chosen? Explain your answer in two different ways.

Problem 43 (Starfolks, II). I'm headed to Starfolks again. It's still four blocks East and three blocks North, but there's a diagonal street as indicated in Figure 3.1. How many ways can I get to Starfolks by walking only seven blocks? Explain your answer in two different ways.

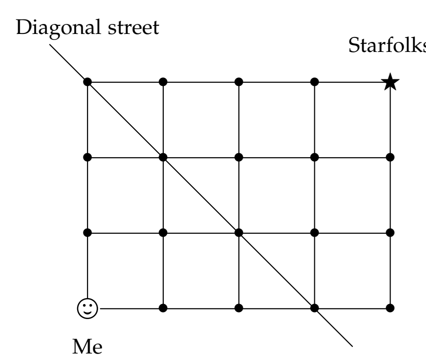


Figure 3.1: Going for coffee again.

Problem 44. Can you explain why, for any k and m less than or equal to n,

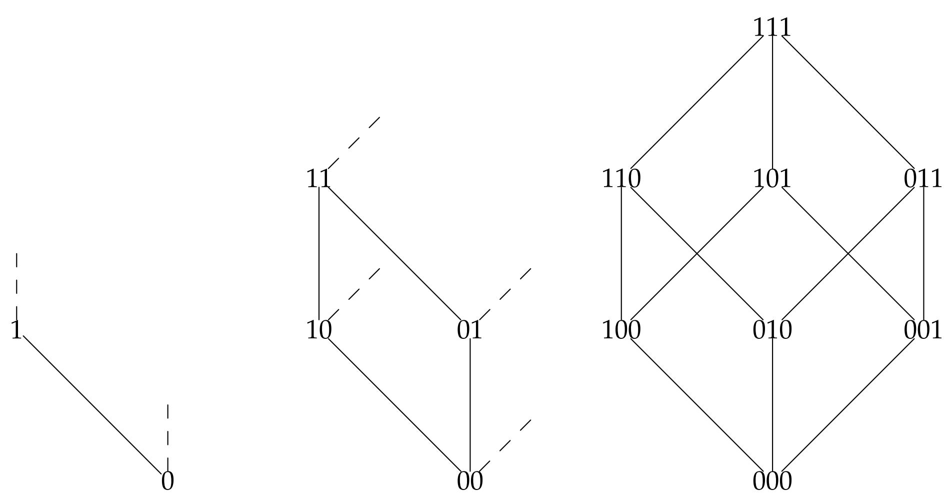
$$\binom{n}{k} = \sum_{j=0}^{k} \binom{n-m}{j}\binom{m}{k-j}?$$

For example, with $n = 5$, $k = 2$, and $m = 2$,

$$\binom{5}{2} = \binom{3}{0}\binom{2}{2} + \binom{3}{1}\binom{2}{1} + \binom{3}{2}\binom{2}{0} = 1 \cdot 1 + 3 \cdot 2 + 3 \cdot 1 = 10.$$

4
The Binomial Theorem

> "Mathematics is the cheapest science. Unlike physics or chemistry, it does not require any expensive equipment. All one needs for mathematics is a pencil and paper."
>
> –George Polyá



But why are they called binomial coefficients? We'll see why in this chapter, which shows how we can start to encode combinatorial results algebraically.

The main result we will prove is the *Binomial Theorem*. Then we will show how you can use this theorem to deduce several results for binomial coefficients in a way that is relatively sweat-free.

Problem 45. Take your time with the algebra on this one so you don't make a mistake. (Or use a computer program to help you avoid mistakes!)

1. Write down all the subsets of the set $\{1,2,3\}$.
2. Carefully expand the product
$$(1+x_1t)(1+x_2t)(1+x_3t),$$
and write down the coefficients of t, t^2, and t^3.
3. Write down all the subsets of the set $\{1,2,3,4\}$.
4. Carefully expand the product
$$(1+x_1t)(1+x_2t)(1+x_3t)(1+x_4t),$$
and write down the coefficients of t, t^2, t^3, and t^4.
5. What happens if you set $x_1 = x_2 = x_3 = x_4 = 1$?

Theorem 6 (Binomial Theorem)**.** For any $n \geq 0$, we have
$$(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k.$$

Problem 46. Using the Binomial Theorem, find and prove a formula for
$$\sum_{k=0}^{n} \binom{n}{k}.$$
Can you explain your result in terms of the subsets of an n element set?

Problem 47. Using the Binomial Theorem, find and prove a formula for
$$\sum_{k=0}^{n} 2^k \binom{n}{k}.$$

Problem 48. Using the Binomial Theorem, find and prove a formula for
$$\sum_{k=0}^{n} (-2)^k \binom{n}{k}$$

Problem 49. Using the Binomial Theorem, prove that
$$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}.$$
(Hint: what is the derivative of $(1+t)^n$?) If all 2^n subsets are equally likely, what is the average size of a subset?

Problem 50. Using the Binomial Theorem, prove that the number of subsets of n with an odd number of elements equals the number of subsets with an even number of elements, i.e.,

$$\sum_{k\geq 0}\binom{n}{2k}=\sum_{k\geq 0}\binom{n}{2k+1},$$

or

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots.$$

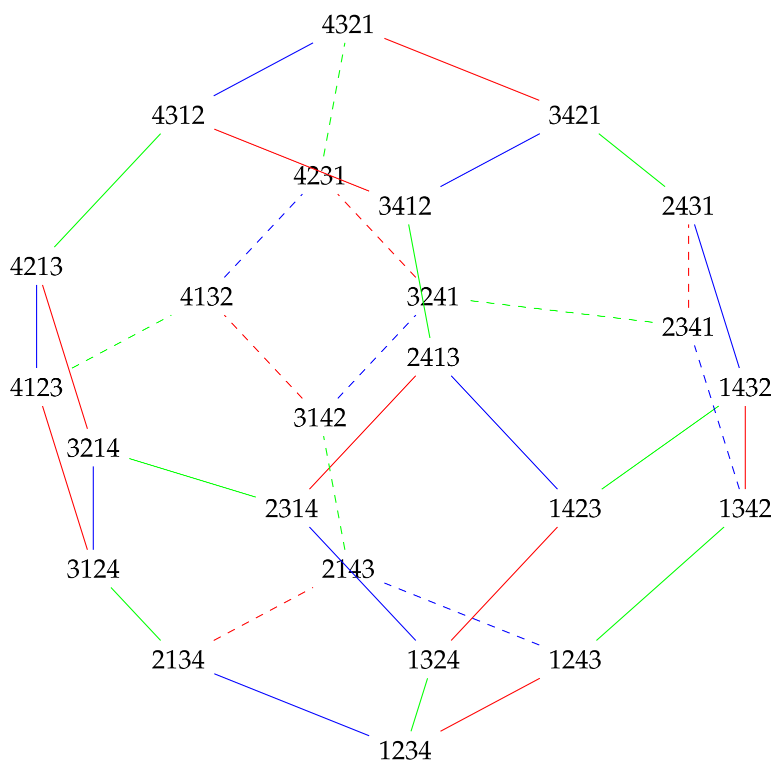
Problem 51. Using the Binomial Theorem, and the fact that $(1+t)^n=(1+t)^{n-m}(1+t)^m$ for any $0\leq m\leq n$, show that,

$$\binom{n}{k}=\sum_{j=0}^{k}\binom{n-m}{j}\binom{m}{k-j}.$$

5
Inversions

"Mathematics is not a deductive science—that's a cliche. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork."

–Paul Halmos



The Binomial Theorem is only the first example we will see of how combinatorial information is encoded algebraically. This chapter will explore how to encode some information about permutations in a similar fashion.

Table 5.1: Triangle of the numbers $I(n,k)$, or number of permutations of n with k inversions.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
1	1										
2	1	1									
3	1			1							
4	1						1				
5	1										1

Definition 11 (Inversions). *An* inversion *of a permutation* $w = w_1 \cdots w_n$ *is a pair* (i, j) *such that* $i < j$ *and* $w_i > w_j$*. For example,* $w = 31542$ *has five inversions:* $(1,2), (1,5), (3,4), (3,5),$ *and* $(4,5)$*. We let* $\mathrm{inv}(w)$ *denote the number of inversions of a permutation* w*, and let* $I(n,k)$ *denote the number of permutations* w *of* $\{1, 2, \ldots, n\}$ *with* $\mathrm{inv}(w) = k$*.*

Problem 52. Compute the number of inversions for all permutations of $\{1,2,3\}$ and $\{1,2,3,4\}$.

Problem 53. How many permutations of $\{1,2,3,4,5\}$ have exactly four inversions?

Definition 12 (Sorting length)**.** *The* sorting length *of a permutation is the number of adjacent transpositions required to sort a permutation* $w = w_1 \cdots w_n$ *into the permutation* $12 \cdots n$*. For example,* 31542 *has sorting length at most* 5 *since the permutation can be sorted in five moves as follows:* $31542 \to 13542 \to 13524 \to 13254 \to 12354 \to 12345$*. (In fact, the sorting length of* 31542 *is exactly* 5*.)*

Problem 54. Show that applying an adjacent transposition to a permutation either increases the number of transpositions by one, or decreases the number of transpositions by one.

Problem 55. Describe a sorting algorithm (using adjacent transpositions) that decreases the number of inversions after each swap.

Theorem 7 (Characterization of sorting length)**.** The sorting length of a permutation is equal to the number of inversions of the permutation.

Problem 56. Recall the definition of an inversion sequence from Definition 9. How many inversion sequences of length 5 have entries whose sum is 4? For example, $s = (0,1,2,0,1)$ is one such sequence.

Problem 57. Generalizing the previous problem, show that the number of inversion sequences of length n whose entries add up to k is $I(n,k)$.

Problem 58. For any fixed choice of n, show:

1. there is only one permutation with no inversions,
2. there is a unique permutation with the most inversions (what is this permutation and how many inversions does it have?), and
3. Table 5.1 has symmetric rows: $I(n,k) = I(n, \binom{n}{2} - k)$.

Problem 59. Show that for $k < n$,

$$I(n,k) = I(n,k-1) + I(n-1,k)$$

Problem 60. Show that

$$\begin{aligned} I(n,k) &= I(n-1,k-n+1) + I(n-1,k-n+2) + \cdots + I(n-1,k) \\ &= \sum_{j=1}^{n} I(n-1,k-n+j) \end{aligned}$$

(Note that the formula is valid for all n and k since $I(n-1,k) = 0$ if $k < 0$ or $k > \binom{n-1}{2}$.)

Problem 61. Fill out the rest of Table 5.1.

Problem 62. This problem is similar to Problem 45. Make sure to do the algebra carefully.

1. Write down all the inversion sequences of length three.
2. Carefully expand the product

   $$x_0(x_0 + x_1 q)(x_0 + x_1 q + x_2 q^2),$$

   and write the result in terms of powers of q.
3. Write down all the inversion sequences of length four.
4. Carefully expand the product

   $$x_0(x_0 + x_1 q)(x_0 + x_1 q + x_2 q^2)(x_0 + x_1 q + x_2 q^2 + x_3 q^3),$$

   and write result in terms of powers of q.
5. What happens if you set $x_0 = x_1 = x_2 = x_3 = 1$?

Theorem 8 (Rodrigues' Theorem)**.** For any $n \geq 1$, we have

$$\begin{aligned} (1)(1+q)\cdots(1+q+\cdots+q^{n-1}) &= \prod_{i=0}^{n-1}(1+q+\cdots+q^i) \\ &= \sum_{k=0}^{\binom{n}{2}} I(n,k) q^k. \end{aligned}$$

Problem 63. Use Rodrigues' Theorem to find and prove a formula for

$$\sum_{k=0}^{\binom{n}{2}} (-1)^k I(n,k)$$

6
Recurrences

"I have had my results for a long time: but I do not yet know how I am to arrive at them."

–Carl Friedrich Gauss

The sequence of *Fibonacci numbers*,

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

is one of the most famous sequences in all of mathematics. It is defined by the initial values $f_0 = 1, f_1 = 1$, and the identity $f_n = f_{n-1} + f_{n-2}$ that holds for all $n \geq 2$.

Other sequences satisfy similar identities, called *recurrence relations*. In this chapter we will see several different recurrence relations arising from enumeration problems. (The Pascal identity for binomial coefficients is another sort of recurrence relation. It is a *two-dimensional* recurrence since it generates a two-dimensional array of numbers rather than a one-dimensional line of numbers like a sequence.)

Problem 64. Generate the first few terms for the sequences defined by the following recurrences. Give an explicit formula for the terms of the sequence if possible.

1. $a_0 = 1$, $a_n = 2a_{n-1}$ for $n \geq 1$.

2. $a_0 = 1$, $a_n = na_{n-1}$ for $n \geq 1$.

3. $a_0 = 0$, $a_n = a_{n-1} + n$ for $n \geq 1$.

4. $a_0 = 1$, $a_1 = 1$, $a_n = a_{n-1} + \sum_{i=0}^{n-1} a_i$ for $n \geq 2$.

5. $a_0 = 1$, $a_n = \sum_{i=0}^{n-1} a_i a_{n-1-i}$ for $n \geq 1$.

Definition 13 (Compositions)**.** *A* composition *of n is an ordered list of positive integers whose sum is n, e.g., $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition of n if and only if all $\alpha_i > 0$ are positive integers and $\alpha_1 + \cdots + \alpha_k = n$. Notice that $(2, 2, 1)$ and $(1, 2, 2)$ are two* different *compositions of 5.*

Problem 65. This problem explores some properties of compositions.

1. How many compositions of 3 are there?

2. How many compositions of 4 are there?

3. Let c_n denote the number of compositions of n. Express c_n in terms of c_{n-1}.

4. Write a formula for c_n in terms of n.

5. A composition made up of k positive integers, e.g., $\alpha = (\alpha_1, \ldots, \alpha_k)$ is said to have k *parts*. How many compositions of n have k parts?

Problem 66 (Handshake problem)**.** When the math club executive board meets, each of the members shakes hands with every other board member exactly once.

1. If there are three members, how many handshakes occur?

2. If a fourth person joins the board, how many more handshakes occur?

3. Let T_n denote the number of handshakes if the executive board has n members. Express T_n in terms of T_{n-1}.

4. Write a formula for T_n in terms of n. (The numbers T_n are called the *triangular numbers*. Can you think why?)

5. Can you relate the handshake problem to the complete graph K_n defined in Problem 7?

Problem 67. Suppose a set of n lines is drawn so that no lines are parallel and no three lines intersect at any one point. (Such a set is called *generic*, or is said to be in *general position*.) Into how many regions is the plane divided by n such lines?

How many of the regions are unbounded? bounded? For example, with $n = 3$ lines there are six unbounded regions and one bounded region.

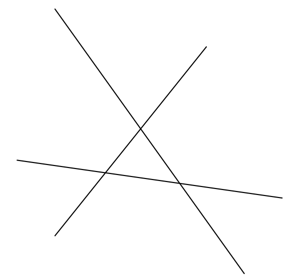


Figure 6.1: Three lines in general position.

Problem 68 (Domino tilings)**.** A *domino tiling* is a way to cover a rectangle with 1×2 or 2×1 rectangles so that the rectangles cover the larger rectangle with no overlapping and no hanging over the edges.

How many domino tilings of a 2×10 rectangle are there? We see two such tilings in Figure 6.2.

What about a $2 \times n$ rectangle?

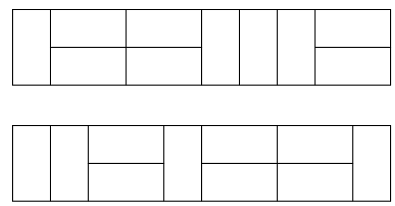


Figure 6.2: Two domino tilings of a 2×10 rectangle.

Problem 69. Suppose now that dominoes come in two colors. How many 2-colored domino tilings of a $2 \times n$ rectangle are there?

Problem 70. How many ways can you write n as a sum of ones and twos? In other words, how many compositions of n use only parts of size 1 and size 2? (See Problem 65 for the definition of a composition)

Problem 71. How many compositions of n have only odd parts? (See Problem 65 for the definition of a composition.)

Problem 72. Show that the sequence satisfying recurrence number 4 of Problem 64 also satisfies the following recurrence (ignoring a_0):

$$a_1 = 1, a_2 = 3, a_n = 3a_{n-1} - a_{n-2} \text{ for } n \geq 3.$$

Can you relate the sequence $a_1, a_2, a_3, \ldots$ to another sequence you've seen?

Problem 73 (Starfolks, III)**.** Okay, now I'm in a corner of town that abuts the highway, so my grid of streets is incomplete, as seen in Figure 6.3. Starfolks is five blocks East and five blocks South of me, but the highway runs on a straight diagonal line between me and my coffee.

Without walking along (or crossing) the highway, how many ways are there to get to Starfolks?

What if Starfolks was n blocks East and South of me? (Hint: try to show the recurrence from Problem 64, part 5, holds.)

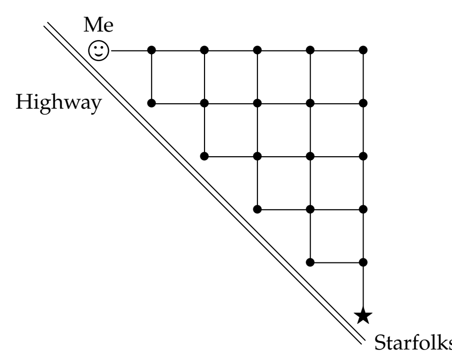


Figure 6.3: Going for coffee one more time.

Problem 74. A *fan graph* of order n is a graph whose vertices are labeled $0, 1, \ldots, n$, with an edge from 0 to every other vertex, and

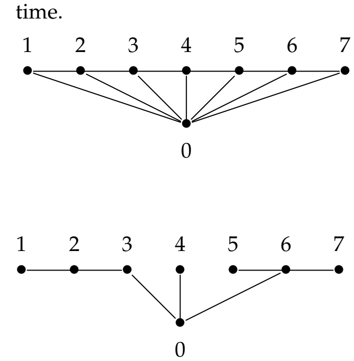


Figure 6.4: A fan graph of order 7 and one of its spanning trees.

edges between vertices i and $i+1$ for all $i = 1, \ldots, n-1$. See Figure 6.4.

How many spanning trees does a fan graph of order 10 have? Of order n?

A *spanning tree* for a graph is a connected subgraph with no loops that uses all the original vertices.

7
Generating functions

"The art of doing mathematics consists in finding that special case which contains all the germs of generality."

–David Hilbert

You may recall from a calculus class the *geometric series*:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^k + \cdots = \sum_{k \geq 0} z^k.$$

The coefficients in the (MacLaurin) series expansion of a function $f(z)$ define a sequence of numbers. Generally, if $f(z)$ has MacLaurin series

$$\sum_{k \geq 0} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots ,$$

we can relate the function f with the sequence $a_0, a_1, a_2, \ldots$. So the function $1/(1-z)$ encodes the rather boring sequence $1, 1, 1, \ldots$.

Incredibly, this is a two-way street. Given a sequence, there is a function that encodes the sequence, called its *generating function*. In this chapter we will explore some properties of generating functions. We will practice both (1) how to extract a sequence from a generating function, and (2) how to find the generating function for a sequence. The first task is fairly mechanical, whereas the second can be very tricky (and very interesting!) in general.

Definition 14 (Generating function/formal power series)**.** *Given a sequence of numbers $a_0, a_1, a_2, \ldots, a_k, \ldots$, we define its* formal power series *by:*

$$f(z) = \sum_{k \geq 0} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots .$$

We also refer to f as the generating function *for the sequence.*

Note 1: We use the letter z for power series in the definition, but feel free to use letters like $q, r, s, t, u, v, w, x, y$, or whatever suits your taste.

Note 2: The sequence does not need to be an infinite sequence. For example, the Binomial Theorem shows that the generating function for the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n-1}, \binom{n}{n}$ is the function $(1+t)^n$, and Rodrigues' Theorem shows the sequence of inversion numbers $I(n,0), I(n,1), \ldots, I(n, \binom{n}{2})$ is encoded with $(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$. Indeed polynomial generating functions have great interest in their own right.

Note 3: Formal power series obey the same arithmetic operations as polynomials. For convenience we list them here:

- $c \cdot \sum_{k \geq 0} a_k z^k = \sum_{k \geq 0} (c \cdot a_k) z^k$ for any constant c,
- $\left(\sum_{k \geq 0} a_k z^k \right) + \left(\sum_{l \geq 0} b_l z^l \right) = \sum_{m \geq 0} (a_m + b_m) z^m$,
- $\left(\sum_{k \geq 0} a_k z^k \right) \left(\sum_{l \geq 0} b_l z^l \right) = \sum_{j \geq 0} \left(\sum_{k+l=j} a_k b_l \right) z^j = \sum_{j \geq 0} \left(\sum_{k=0}^{j} a_k b_{j-k} \right) z^j.$

Logically, one can take two points of view for these rules. With analysis (calculus) in mind, we could only wish to consider functions whose series expansions converge absolutely in some neighborhood of the origin, in which case all three properties hold easily. Alternatively, we can take an algebraic approach in which we define the ring of formal power series. Here the second and third properties *define* the ring operations of addition and multiplication. The first property is then a very special case of the multiplication.

Problem 75. Taking the geometric series as a starting point, what sequences are defined by the following generating functions?

1. $\dfrac{1}{1+z}$
2. $\dfrac{1}{1-2z}$
3. $\dfrac{1}{z-3}$

4. $\dfrac{1}{1-z^2}$

5. $\dfrac{1}{(1-z)^2}$

Problem 76. Suppose

$$f(z) = \sum_{k\geq 0} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots .$$

What sequence is encoded by the derivative of f, i.e., if $f'(z) = \sum_{k\geq 0} b_k z^k$, what is the sequence $b_0, b_1, b_2, \ldots, b_k, \ldots$?

Problem 77. Using your observations from the previous problem, along with usual differentiation rules from calculus, find the sequences defined by the following series, where $n \geq 1$.

1. $\dfrac{d^n}{dz^n}\left[\dfrac{1}{1-z}\right]$

2. $\dfrac{1}{(1-z)^n}$

Can you find these sequences in Tables 2.1 and 3.1?

Problem 78. What sequence is defined by the following generating function?

$$\frac{1}{1-5z+6z^2}$$

Problem 79. Suppose α and β are nonzero real numbers. What sequences are defined by the following generating functions?

1. $\dfrac{1}{1-\alpha z}$

2. $\dfrac{1}{1-\beta z}$

3. $\dfrac{1}{(1-\alpha z)(1-\beta z)}$

4. Find A and B such that

$$\frac{1}{(1-\alpha z)(1-\beta z)} = \frac{A}{(1-\alpha z)} + \frac{B}{(1-\beta z)}$$

and derive another expression for the sequence given in part 3.

Problem 80. Recall that the Fibonacci numbers are defined by $f_0 = 1, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Find a formula for the generating function for the Fibonacci sequence:

$$F(z) = \sum_{k\geq 0} f_k z^k = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + \cdots$$

Problem 81. Use your expression from Problem 80 along with the Binomial Theorem to prove

$$f_n = \sum_{k\geq 0} \binom{n-k}{k}.$$

Problem 82. Using the formula from Problem 80, along with the results of Problem 79, find a (non-recursive) formula for the nth Fibonacci number.

Problem 83. Find an expression for the generating function for the sequence $a_1, a_2, a_3, \ldots$ defined by Problem 72. Can you derive the same generating function using the recurrence given in part 4 of Problem 64?

Problem 84. Let $C_0, C_1, C_2, \ldots$ be the sequence defined by part 5 of Problem 64, i.e., $C_0 = 1$ and $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$ for $n \geq 1$. Find an expression for the generating function of this sequence.

Problem 85. Find a formula for the generating function for the sequence of squares:

$$g_2(z) = \sum_{k\geq 0} k^2 z^k = z + 4z^2 + 9z^3 + 16z^4 + \cdots + k^2 z^k + \cdots .$$

Problem 86. Find a formula for the generating function for the sequence of cubes:

$$g_3(z) = \sum_{k\geq 0} k^3 z^k = z + 8z^2 + 27z^3 + 64z^4 + \cdots + k^3 z^k + \cdots .$$

Problem 87. Let $g_n(z)$ denote the generating function for the sequence of nth powers:

$$g_n(z) = \sum_{k\geq 0} k^n z^k = z + 2^n z^2 + 3^n z^3 + 4^n z^4 + \cdots + k^n z^k + \cdots .$$

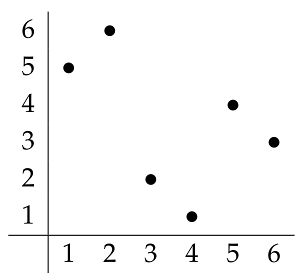
Find a formula for $g_n(z)$ in terms of $g_{n-1}(z)$.

Problem 88. With $g_n(z)$ as in Problem 87, let $A_n(z) = (1-z)^{n+1} g_n(z)$. Find a formula for $A_n(z)$ in terms of $A_{n-1}(z)$, and use this recurrence (perhaps with the aid of a computer) to make a table of the coefficients of these polynomials for $n = 1, 2, \ldots, 8$.

What happens when you set $z = 1$ in $A_n(z)$?

8
Eulerian numbers

"I am interested in mathematics only as a creative art."
–G. H. Hardy



$w = 562143$

LEONHARD EULER CERTAINLY didn't have permutations on his mind when in 1787 he wrote about the numbers bearing his name in this chapter. (He was describing the solution to a certain differential equation.) Nowadays, however, these numbers are usually defined combinatorially: in terms of permutations.

In this chapter we will study Eulerian numbers. Among other things, we will see that these numbers satisfy a Pascal-like recurrence.

$n\backslash k$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	4	1				
4	1			1			
5	1				1		
6	1					1	
7	1						1

Table 8.1: Triangle of the numbers $A(n,k)$, or the number of permutations of n with $k-1$ descents.

Definition 15 (Descents)**.** *A* descent *of a permutation $w = w_1 \cdots w_n$ is an index $i \in \{1, 2, \ldots, n-1\}$ such that $w_i > w_{i+1}$. We let* des(w) *denote the number of descents in w. For example, if $w = 51243$,* des$(w) = 2$ *since $w_1 = 5 > 1 = w_2$ and $w_4 = 4 > 3 = w_5$.*

Problem 89. How many permutations of 5 have no descents? exactly one descent? two descents? three descents? four?

Definition 16 (Eulerian numbers)**.** *Recall S_n denotes the set of permutations of n. Let $A(n,k)$ denote the number of permutations in S_n with exactly $k-1$ descents:*

$$A(n,k) = |\{w \in S_n | \operatorname{des}(w) = k-1\}|.$$

We call $A(n,k)$ an Eulerian number*.*

Problem 90. Find a correspondence (bijection) between those permutations of n with k descents and those permutations of n with $n-1-k$ descents to show that

$$A(n,k+1) = A(n,n-k).$$

Problem 91. Suppose $w = w_1w_2w_3w_4w_5w_6$ has descents in positions 2 and 5. Consider the set of permutations formed by inserting the number 7 into w in all possible ways:

$$7w_1w_2w_3w_4w_5w_6, \quad w_17w_2w_3w_4w_5w_6, \quad w_1w_27w_3w_4w_5w_6, \ldots$$

Which of these new permutations have 2 descents? 3 descents?

Problem 92. Let $\mathcal{A}(n,k)$ denote the set of all permutations of n with $k-1$ descents. (So $|\mathcal{A}(n,k)| = A(n,k)$.) If $w \in S_n$, let $\widehat{w}$ denote the permutation formed from w by removing the largest letter, n. For example, if $w = 314625$, then $\widehat{w} = 31425$.

1. If $w \in \mathcal{A}(n,k)$, how many descents can $\widehat{w}$ have?
2. How many permutations w in $\mathcal{A}(n,k)$ have $\widehat{w} \in \mathcal{A}(n-1,k)$?
3. How many permutations w in $\mathcal{A}(n,k)$ have $\widehat{w} \in \mathcal{A}(n-1,k-1)$?

Theorem 9 (Eulerian recurrence)**.** For $n \geq 2$ and $1 \leq k \leq n$ we have

$$A(n,k) = kA(n-1,k) + (n+1-k)A(n-1,k-1).$$

Problem 93. Fill out the rest of Table 8.1.

Problem 94. Use Theorem 9 and the recurrence from Problem 88 involving the polynomials $A_n(z)$ to show that

$$A_n(z) = \sum_{w \in S_n} z^{\mathrm{des}(w)+1} = \sum_{i=1}^{n} A(n,i) z^i,$$

i.e.,

$$\frac{\sum_{w \in S_n} z^{\mathrm{des}(w)+1}}{(1-z)^{n+1}} = \frac{\sum_{i=1}^{n} A(n,i) z^i}{(1-z)^{n+1}} = \sum_{k \geq 0} k^n z^k. \tag{8.1}$$

Theorem 10 (Worpitzky's Identity)**.** For any $n, k \geq 1$,

$$k^n = \sum_{i=1}^{n} A(n,i) \binom{k+n-i}{n}.$$

(For example, $k^3 = \binom{k+2}{3} + 4\binom{k+1}{3} + \binom{k}{3}$. You can use Equation 8.1 along with the series for $z^i/(1-z)^{n+1}$ to prove this in general.)

Definition 17 (Major index)**.** *The* major index *of a permutation is the sum of the positions of the descents. For example,* $w = 613524$ *has major index 5 since its descents occur in positions 1 and 4. Let* $\mathrm{maj}(w)$ *denote the major index of* w*.*

Problem 95. Compute the major index for all permutations of $\{1,2,3\}$ and $\{1,2,3,4\}$.

Problem 96. Show that major index has the same distribution as inversion number:

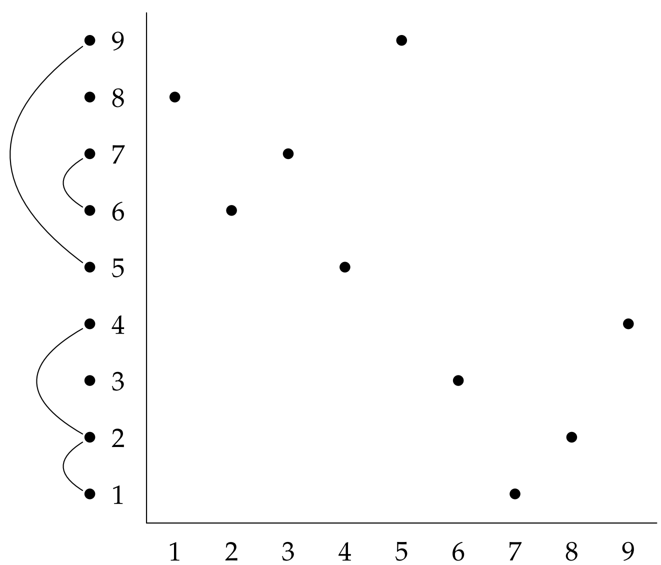
$$\sum_{w \in S_n} q^{\mathrm{maj}(w)} = \sum_{w \in S_n} q^{\mathrm{inv}(w)},$$

i.e.,

$$|\{w \in S_n \mid \mathrm{maj}(w) = k\}| = I(n,k).$$

9
Catalan numbers

> "An expert problem solver must be endowed with two incompatible qualities, a restless imagination and a patient pertinacity."
>
> –Howard W. Eves



The Fibonacci numbers are pretty cool, but in modern algebraic combinatorics, the most interesting sequence is the sequence of *Catalan numbers*, named for E. Catalan:

$$1, 1, 2, 5, 14, 42, 132, 429, \ldots$$

The remarkable ability of these numbers to pop up in surprising locations has led some to joke that a combinatorics paper is not complete until the Catalan numbers have made an appearance.

$n\backslash k$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	3	1				
4	1			1			
5	1				1		
6	1					1	
7	1						1

Table 9.1: Triangle of the numbers $N(n,k)$, or the number of 132-avoiding permutations of n with $k-1$ descents.

Definition 18 (132-avoiding permutation). *A permutation* $w = w_1w_2\cdots w_n$ *in* S_n *is said to* contain the pattern 132 *if there is a triple of indices* $i < j < k$ *such that* $w_i < w_k < w_j$*. Otherwise we say* w *is* 132-avoiding*. Let* $S_n(132)$ *denote the set of 132-avoiding permutations.*

Problem 97. List all the elements of $S_n(132)$, for $n \leq 4$. For each n, count the number of permutations with 0, 1, 2, and 3 descents.

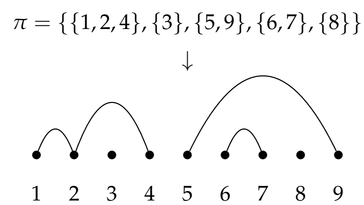


Figure 9.1: String diagram for a non-crossing partition of 9 elements.

Definition 19 (Non-crossing partition). *A* non-crossing partition *of* n *is a collection of subsets of* $\{1,2,\ldots,n\}$ *(called blocks)*

$$\pi = \{B_1, B_2, \ldots, B_k\}$$

such that:

(Partition) $B_1 \cup B_2 \cup \cdots \cup B_k = \{1,2,\ldots,n\}$ *and* $B_i \cap B_j = \emptyset$ *if* $i \neq j$,

(Non-crossing) If $\{a,c\} \subseteq B_i$ *and* $\{b,d\} \subseteq B_j$ *with* $a < b < c < d$*, then* $i = j$.

We denote the set of all non-crossing partitions of n *by* $NC(n)$*. Non-crossing partitions are often represented visually by their* string diagrams*, as in Figure 9.1.*

Problem 98. List all the elements of $NC(n)$, for $n \leq 4$. For each n, count the number of partitions with the 1 block, 2 blocks, 3 blocks, and 4 blocks.

Problem 99. Find a correspondence (bijection) between $S_n(132)$ and $NC(n)$.

Definition 20 (Narayana numbers)**.** *Let $N(n,k)$ denote the number of 132-avoiding permutations of n with $k-1$ descents:*

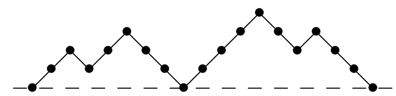
$$N(n,k) = |\{w \in S_n(132) | \operatorname{des}(w) = k-1\}|.$$

We call $N(n,k)$ a Narayana number.

Problem 100. Show that $N(n,k)$ also counts noncrossing partitions of n with k blocks.

Problem 101. Use a correspondence (bijection) to show that $N(n,k+1) = N(n,n-k)$.

Definition 21 (Dyck paths)**.** *A* Dyck path *of order n is a lattice path from $(0,0)$ to $(2n,0)$ that takes steps Northeast, from (a,b) to $(a+1,b+1)$, or Southeast, from (a,b) to $(a+1,b-1)$, and never goes below the x-axis. We let $D(n)$ denote the set of all Dyck paths of order n. Dyck paths are easily encoded by their* height sequence *as in Figure 9.2.*



$(0,1,2,1,2,3,2,1,0,1,2,3,4,3,2,3,2,1,0)$

Figure 9.2: A Dyck path (with four peaks) and its height sequence.

Problem 102. Show that $N(n,k)$ counts the number of Dyck paths of order n with k peaks. (A peak occurs when a Northeast step is followed immediately by a Southeast step.)

Problem 103. Prove that

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

(Hint: $\frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \binom{n+1}{k}\binom{n-1}{k-1} - \binom{n}{k}\binom{n}{k-1}$.)

Problem 104. Fill out the rest of Table 9.1.

Definition 22 (Catalan numbers)**.** *Let C_n denote the number of 132-avoiding permutations of n, i.e., $C_n = |S_n(132)|$.*

Problem 105. Prove combinatorially that

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

(Hint: $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$.)

Problem 106. Show that the Catalan numbers satisfy the recurrence given in part 5 of Problem 64, i.e.,

$$C_0 = 1, C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \text{ for } n \geq 1.$$

What is the generating function for the Catalan numbers? (You did this in an earlier problem.)

Problem 107. It is straightforward from the definitions that

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

Come up with an expression for the following two-variable generating function:

$$C(y,z) = 1 + \sum_{n\geq 1} \sum_{k=1}^{n} N(n,k) y^k z^n.$$

Note that if you set $y = 1$, you should recover the Catalan generating function.

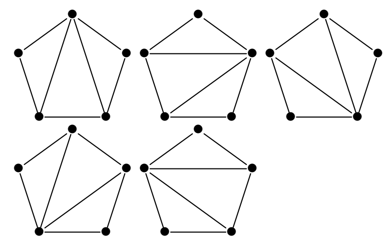


Figure 9.3: Triangulations of a pentagon.

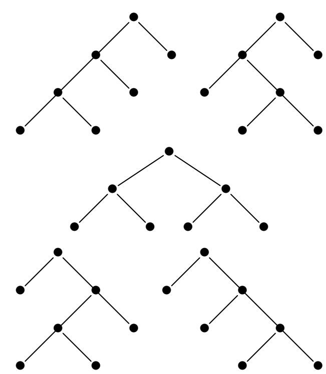


Figure 9.4: Decreasing binary trees.

Problem 108 (Catalania = Catalan Mania)**.** Show that the following are enumerated by the Catalan numbers:

1. 123-avoiding permutations
   (no indices $i < j < k$ with $w_i < w_j < w_k$)
2. 213-avoiding permutations
   (no indices $i < j < k$ with $w_j < w_i < w_k$)
3. 231-avoiding permutations
   (no indices $i < j < k$ with $w_k < w_i < w_j$)
4. 312-avoiding permutations
   (no indices $i < j < k$ with $w_j < w_k < w_i$)
5. 321-avoiding permutations
   (no indices $i < j < k$ with $w_k < w_j < w_i$)
6. Triangulations of a polygon. See Figure 9.3.
7. Balanced parenthesizations, e.g., ()(()(())()).
8. Decreasing binary trees. See Figure 9.4.
9. $2 \times n$ arrays of the numbers $1, 2, \ldots, 2n$ such that the numbers increase across rows and down columns, e.g.,

1	2	3
4	5	6

1	2	4
3	5	6

1	2	5
3	4	6

1	3	4
2	5	6

1	3	5
2	4	6