Depth of an element of a Coxeter group

T. Kyle Petersen

DePaul University

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Sorting by transpositions

	1234567	cost
w :	3**7**15246	

Sorting by transpositions

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w :	3**7**15246**6**	
	↓	$7 - 2$
	3**6**1524**4**7	

Sorting by transpositions

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Knuth's "Straight Selection Sort"

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Can we lower the cost of sorting?

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w :	371**52**46	

Sorting by transpositions

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	↓	$5-4$
	37**1**2546	

Sorting by transpositions

	1234567	cost
w :	371**52**46	
	$\downarrow$	$5-4$
	37**1**2546	
	$\downarrow$	$3-1$
	1**7**3**2**546	

Sorting by transpositions

	1234567	cost
w :	371**52**46	
	↓	$5-4$
	37**1**2546	
	↓	$3-1$
	1**7**3**2**546	
	↓	$4-2$
	123**7**5**4**6	

Sorting by transpositions

	1234567	cost
w :	371**52**46	
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	123**7**5**4**6	
	↓	6 − 4
	12345**76**	

Sorting by transpositions

	1234567	cost
w :	371**52**46	
	↓	5 – 4
	37**1**2546	
	↓	3 – 1
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	↓	4 – 2
	123**7**5**4**6	
	↓	6 – 4
	12345**76**	
	↓	7 – 6
	1234567	8

Sorting by transpositions

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How did we make these choices?

Sorting by transpositions

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How did we make these choices?

Is 8 the minimal sorting cost?

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	1234567	8

How did we make these choices?

Is 8 the minimal sorting cost?

Let's return to straight selection sort. . .

Playing both sides

	1234567
w	3**7**1524**6**

Playing both sides

		1234567		
w		3**7**15246**6**		
	$(67)\cdot$	$\downarrow$	$\cdot(27)$	
$(67)w$		3**6**152**4**7		

Playing both sides

		$\mathbf{1234567}$	
w		$3\mathbf{7}15524\mathbf{6}$	
	$(67)\cdot$	$\downarrow$	$\cdot(27)$
$(67)w$		$3\mathbf{6}152\mathbf{4}7$	
	$(46)\cdot$	$\downarrow$	$\cdot(26)$
$(46)(67)w$		$341\mathbf{52}67$	

Playing both sides

		1234567	
w		3**7**1524**6**	
	$(67)\cdot$	$\downarrow$	$\cdot(27)$
$(67)w$		3**6**152**4**7	
	$(46)\cdot$	$\downarrow$	$\cdot(26)$
$(46)(67)w$		341**52**67	
	$(25)\cdot$	$\downarrow$	$\cdot(45)$
$(46)(67)w(45)$		3**4**1**2**567	

Playing both sides

		1234567	
w		3**7**1524**6**	
	$(67)\cdot$	$\downarrow$	$\cdot(27)$
$(67)w$		3**6**152**4**7	
	$(46)\cdot$	$\downarrow$	$\cdot(26)$
$(46)(67)w$		341**52**67	
	$(25)\cdot$	$\downarrow$	$\cdot(45)$
$(46)(67)w(45)$		3**4**1**2**567	
	$(24)\cdot$	$\downarrow$	$\cdot(24)$
$(24)(46)(67)w(45)$		**3**2**1**4567	

Playing both sides

		1234567	
w		3**7**1524**6**	
	(67)·	↓	·(27)
$(67)w$		3**6**152**4**7	
	(46)·	↓	·(26)
$(46)(67)w$		341**52**67	
	(25)·	↓	·(45)
$(46)(67)w(45)$		3**4**1**2**567	
	(24)·	↓	·(24)
$(24)(46)(67)w(45)$		**3**2**1**4567	
	(13)·	↓	·(13)
$(13)(24)(46)(67)w(45)$		1234567	

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We get $(13)(24)(46)(67)w(45) = e$, or $w = (67)(46)(24)(13)(45)$

	1234567
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$$\begin{array}{ll} & \mathbf{1234567} \\ \hline w: & 371\mathbf{52}46 \\ & \quad\downarrow \qquad \cdot(45) \\ & \mathbf{3}7\mathbf{1}2546 \end{array}$$

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	1234567	
w :	3715246	
	↓	$\cdot(45)$
	3712546	
	↓	$\cdot(13)$
	1732546	

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w :	371**52**46	
	$\downarrow$	$\cdot(45)$
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	$\downarrow$	$\cdot(24)$
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$$
\begin{array}{rll}
 & \mathbf{1234567} & \\
\hline
w: & 371\mathbf{52}46 & \\
 & \downarrow & \cdot(45) \\
 & \mathbf{3}7\mathbf{1}2546 & \\
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$$

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We get $(13)(24)(46)(67)w(45) = e$, or $w = (67)(46)(24)(13)(45)$

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$w:$	371**52**46	
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	37**1**2546	
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	12345**76**	
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	1234567	

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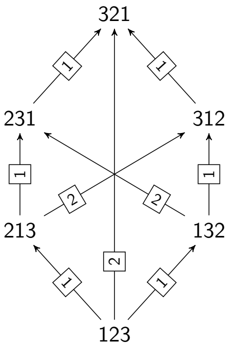
 & 123\mathbf{7}5\mathbf{4}6 & \\
 & \downarrow & \cdot(46) \\
 & 12345\mathbf{76} & \\
 & \downarrow & \cdot(67) \\
\hline
 & 1234567 &
\end{array}
$$

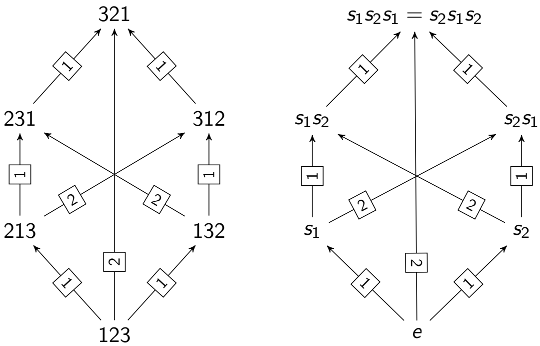
Minimizes sorting cost
(proof by induction)

Depth of an element of a Coxeter group

The Bruhat graph



The Bruhat graph



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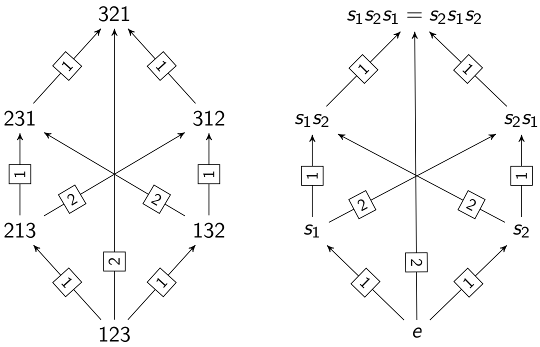
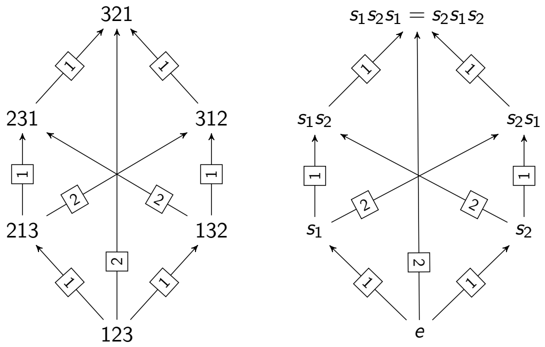


Length: $\ell_S(w)$ is maximal length of a path $e \to \cdots \to w$

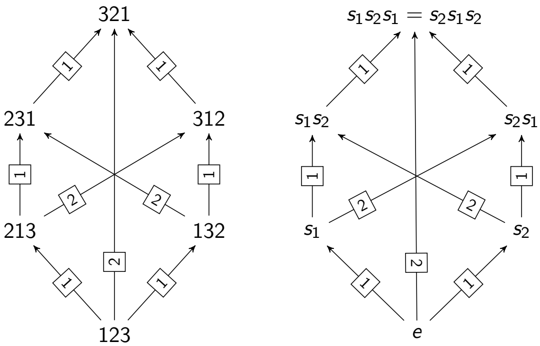
The Bruhat graph



Length: $\ell_S(w)$ is maximal length of a path $e \to \cdots \to w$

Reflection length: $\ell_T(w)$ is minimal length of a path $e \to \cdots \to w$

The Bruhat graph



Depth: $dp(w) = \min\{\text{cost of path } e \to \cdots \to w\}$
where each reflection $t = wsw^{-1}$ has cost $dp(t) = \ell_S(w) + 1$

First observations

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- $\ell_T(w) \leq dp(w) \leq \ell_S(w)$, in fact
- $(\ell_T(w) + \ell_S(w))/2 \leq dp(w) \leq \ell_S(w)$
- if $\ell_T(w) = dp(w)$, then $\ell_T(w) = \ell_S(w)$, and for the symmetric group these are known to be the 321- and 3412-avoiders (Edelman, Tenner)

The symmetric group

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Theorem (P.-Tenner)

For any permutation w,

$$dp(w) = \sum_{w_i > i} (w_i - i)$$

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$2 \cdot \sum_{w_i > i}(w_i - i) = \sum_i |w_i - i|$ is known as "total displacement" studied by Diaconis and Graham (1977)

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For any permutation w, $dp(w) = \ell_S(w)$ $(= inv(w))$ if and only if w avoids the pattern 321.

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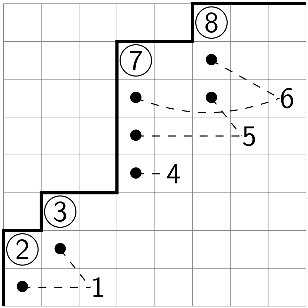

The distribution of depth

	$k=0$	1	2	3	4	5	6	7	8	9	10	11	12
$n=1$	1												
2	1	1											
3	1	2	3										
4	1	3	7	9	4								
5	1	4	12	24	35	24	20						
6	1	5	18	46	93	137	148	136	100	36			
7	1	6	25	76	187	366	591	744	884	832	716	360	252
8	1	7	33	115	327	765	1523	2553	3696	4852	5708	5892	$\cdots$

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Knuth: "The generating function for total displacement does not appear to have a simple form."

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Knuth: "The generating function for total displacement does not appear to have a simple form."

Gardner: "We are continually faced with a series of great opportunities brilliantly disguised as insoluble problems."

The distribution of depth

Theorem (Guay-Paquet, P.)

$$\sum_{n\geq 0}\sum_{w\in S_n} t^{dp(w)} z^n = \cfrac{1}{1-\cfrac{z}{1-\cfrac{tz}{1-\cfrac{2tz}{1-\cfrac{2t^2z}{1-\cfrac{3t^2z}{1-\cfrac{3t^3z}{1-\cfrac{4t^3z}{1-\cfrac{4t^4z}{1-\cdots}}}}}}}}}$$

The distribution of depth

A refined version of this continued fraction expansion also appears in Clarke, Steingrímsson, Zeng (1997)

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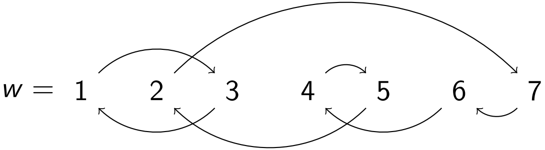
Proof relies on a map from permutations to Motzkin paths that takes depth to area under the path. . .

Map to Motzkin paths



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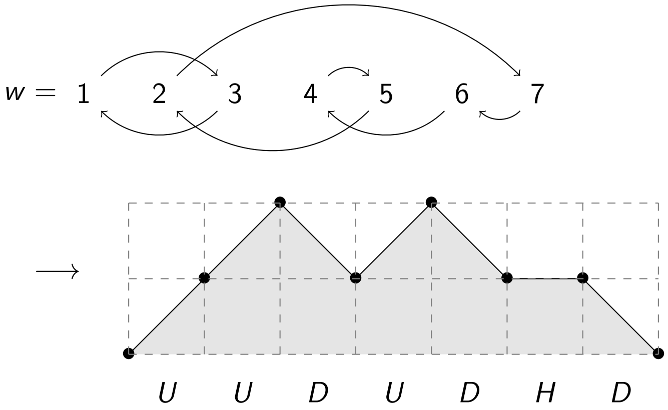
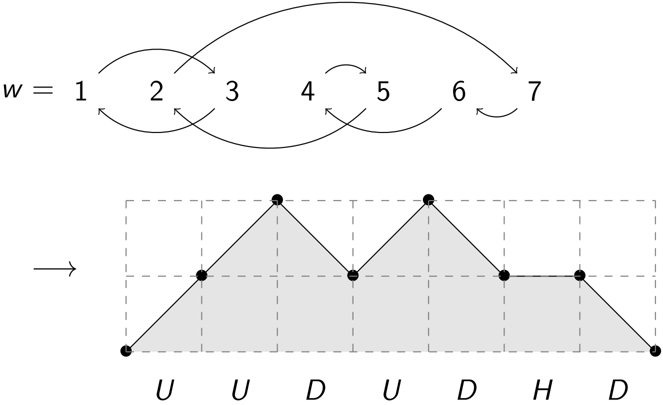


Map to Motzkin paths



Map first appears in work of Foata and Zeilberger (1990)

Map to Motzkin paths

Consequences of the map to Motzkin paths:

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- maximum depth $=$ maximum area $= \lfloor n^2/4 \rfloor$

Map to Motzkin paths

Consequences of the map to Motzkin paths:

- maximum depth = maximum area = $\lfloor n^2/4 \rfloor$
- number of permutatons with maximal depth is

$$|\{w \in S_n : dp(w) = \lfloor n^2/4 \rfloor\}| = \begin{cases} (k!)^2 & \text{if } n = 2k \\ n(k!)^2 & \text{if } n = 2k+1 \end{cases}$$

Depth of an element of a Coxeter group

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2. Can we find combinatorial characterizations for depth in other Coxeter groups, e.g., the hyperoctahedral group B_n?

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2. Can we find combinatorial characterizations for depth in other Coxeter groups, e.g., the hyperoctahedral group B_n?
3. In type B_n, the elements for which $dp(w) = \ell_S(w)$ are

$$2, 5, 14, 42, 132, 429$$

for $n = 1, \ldots, 6$.

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$$|\{w \in B_n : dp(w) = \ell_S(w)\}| = C_{n+1}?$$