What does the γ-vector count?

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joint with Eran Nevo (Ben Gurion),

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October, 2010

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Gal's conjecture

An example

The Γ complex

A conjecture

Abstract simplicial complexes

A *simplicial complex* $\Delta \subset 2^V$ on a vertex set V:

- if $v \in V$ then $\{v\} \in \Delta$,
- if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

$\dim F = |F| - 1$ (so $\dim \emptyset = -1$)

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Δ is a sphere if it has a geometric realization homeomorphic to a sphere

The f- and h-vectors

Let Δ be an $(n-1)$-dimensional simplicial complex, $f_k(\Delta) =$ number of faces of dimension $k-1$

$$f(\Delta; t) := \sum_{k=0}^{n} f_k(\Delta) t^k$$

$(f_0, f_1, \ldots, f_n)$ is the *f-vector*

$$h(\Delta; t) := (1-t)^n f(\Delta; t/(1-t)) = \sum_{k=0}^{n} h_k(\Delta) t^k$$

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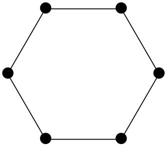
f-vectors of simplicial complexes are characterized by the *Kruskal-Katona inequalities*

The f- and h-vectors

Δ :



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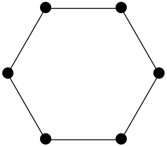
Δ : 

- $f_0 = 1$

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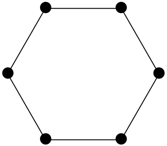

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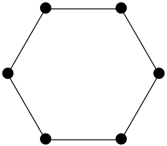
Δ : 

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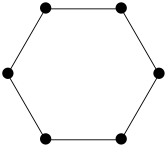

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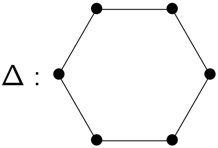

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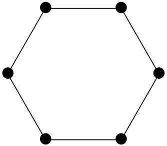
Δ : 

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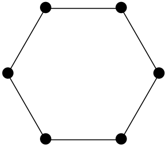
Δ : 

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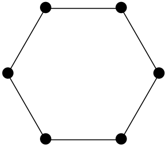

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$$\begin{aligned} f(\Delta; t) &= 1 + 6t + 6t^2 \\ &= (1 + 2t + t^2) \\ &\qquad + 4t(1+t) \\ &\qquad\qquad + t^2(1) \\ h(\Delta; t) &= 1 + 4t + t^2 \end{aligned}$$

The γ-vector

If $h(t) = \sum_{i=0}^{n} h_i t^i$ is symmetric, then there exist γ_i such that

$$h(t) = \sum_{i=0}^{n/2} \gamma_i t^i (1+t)^{n-2i},$$

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$$1 + 3t + 7t^2 + 3t^3 + t^4 = (1+t)^4 - t(1+t)^2 + 3t^2$$

the vector $(\gamma_0, \gamma_1, \ldots)$ is called the γ-*vector*

Gal's conjecture

For Δ a sphere, Dehn-Sommerville relations say $h(\Delta)$ is symmetric,

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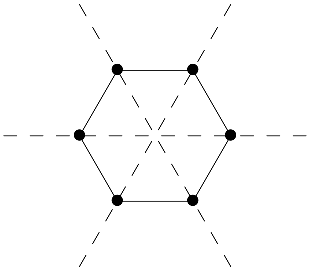

Eulerian polynomials

Let $w \in S_{n+1}$ and $d(w) := |\{i : w_i > w_{i+1}\}|$. Then,

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w	$d(w)$	$t^{d(w)}$
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132		
213		
231		
312		
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123	0	1
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$$A_2(t) = 1 + 4t + t^2 = h(\Delta(A_2); t)$$

Eulerian polynomials

We have:

$$A_1(t) = 1 + t$$

$$A_2(t) = 1 + 4t + t^2$$

$$A_3(t) = 1 + 11t + 11t^2 + t^3$$

$$A_4(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$$

$$\vdots$$

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We have:

$$\begin{aligned} A_1(t) &= 1 + t \\ &= (1 + t) \\ A_2(t) &= 1 + 4t + t^2 \\ \\ A_3(t) &= 1 + 11t + 11t^2 + t^3 \\ \\ A_4(t) &= 1 + 26t + 66t^2 + 26t^3 + t^4 \\ \\ &\vdots \end{aligned}$$

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&= (1 + t)^3 + 8t(1 + t) \\
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A_4(t) &= 1 + 26t + 66t^2 + 26t^3 + t^4 \\
&= (1 + t)^4 + 22t(1 + t)^2 + 16t^2 \\
&\vdots
\end{aligned}$$

The γ-vector for A_n

Define

$$\widehat{S}_n = \{w \in S_n : w_{n-1} < w_n, \text{ and if } w_{i-1} > w_i \text{ then } w_i < w_{i-1}\}$$

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Theorem (Foata-Schützenberger (1970))

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)},$$

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$$\gamma_i(A_n) = |\{w \in \widehat{S}_{n+1} : d(w) = i\}|$$

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Kruskal-Katona inequalities

Observe that

$$\begin{aligned} \gamma(A_1) &= (1) \\ \gamma(A_2) &= (1, 2) \\ \gamma(A_3) &= (1, 8) \\ \gamma(A_4) &= (1, 22, 16) \\ &\vdots \end{aligned}$$

are all *Kruskal-Katona vectors*

What does the γ-vector count?

Gal's conjecture

An example

The Γ complex

A conjecture

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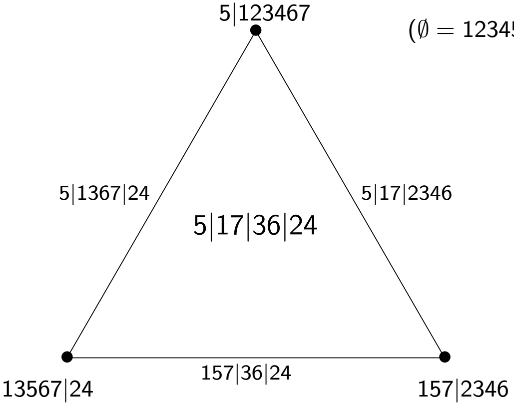
A facet of $\Gamma(A_6)$:

($\emptyset = 1234567$)



The Γ complex

Theorem (Nevo-P.)

There exists a simplicial complex $\Gamma(\Delta)$ *such that*

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- *[Nevo-P.-Tenner]* Δ *is a barycentric subdivision of a boolean sphere (builds on Brenti-Welker)*

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A conjecture - what γ counts?

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Conjecture (Nevo-P.)

If Δ *is a flag homology sphere, then* $\gamma(\Delta) = f(\Gamma(\Delta))$ *for some simplicial complex* $\Gamma(\Delta)$

Kruskal-Katona characterizes γ?

Do the Kruskal-Katona inequalities characterize γ-vectors?

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(1,3), (1,3,1), (1,3,2), (1,3,3), (1,3,3,1)	$\gamma_1 = 3$, No...

The remaining vectors are *Frankl-Füredi-Kalai vectors* = f-vectors of *balanced* complexes

Stronger inequalitites

Conjecture (Nevo-P.)

If Δ is a flag homology sphere, then $\gamma(\Delta) = f(\Gamma(\Delta))$ for some balanced (flag?-stronger) simplicial complex $\Gamma(\Delta)$

Questions?

"On γ-vectors satisfying the Kruskal-Katona inequalities," with E. Nevo, *Discrete and Computational Geometry*, to appear.

"The γ-vector of a barycentric subdivision," with E. Nevo and B. Tenner, submitted (on the arXiv).