Zig-zag Eulerian polynomials: from $A_n(t)$ to $Z_n(t)$

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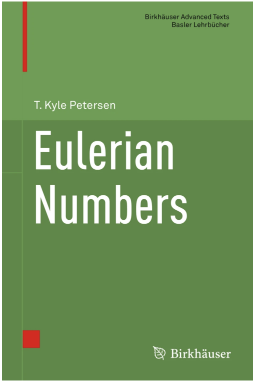
Zig-zag Eulerian polynomials

Eulerian numbers 101



Binomial Coefficients

$n\backslash k$	0	1	2	3	4	5	6	7	8	sum
0	1									1
1	1	1								2
2	1	2	1							4
3	1	3	3	1						8
4	1	4	6	4	1					16
5	1	5	10	10	5	1				32
6	1	6	15	20	15	6	1			64
7	1	7	21	35	35	21	7	1		128
8	1	8	28	56	70	56	28	8	1	256

$$B_n(t) = \sum_{S \subseteq [n]} t^{|S|} = \sum_{k=0}^{n} \binom{n}{k} t^k = (1+t)^n$$

Eulerian Numbers

$n\backslash k$	0	1	2	3	4	5	6	sum
1	1							1
2	1	1						2
3	1	4	1					6
4	1	11	11	1				24
5	1	26	66	26	1			120
6	1	57	302	302	57	1		720
7	1	120	1191	2416	1191	120	1	5040

$$A_n(t) = \sum_{\pi \in S_n} t^{\mathsf{des}(\pi)} = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle t^k$$

$$\mathsf{des}(\pi) = \{i : \pi(i) > \pi(i+1)\}$$

Eulerian Numbers

π	des(π)
123	0
$13\|2$	1
$2\|13$	1
$23\|1$	1
$3\|12$	1
$3\|2\|1$	2

$$A_3(t) = 1 + 4t + t^2 = \left\langle {3 \atop 0} \right\rangle + \left\langle {3 \atop 1} \right\rangle t + \left\langle {3 \atop 2} \right\rangle t^2$$

Eulerian Numbers

Generating functions:

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 0} (m+1)^n t^m$$

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{t-1}{t-e^{z(t-1)}}$$

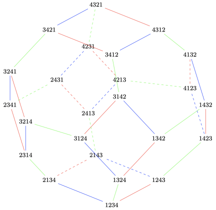
Formulas:

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle$$

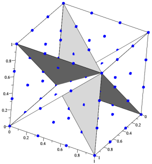
$$= \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j+1)^n$$

Eulerian Numbers

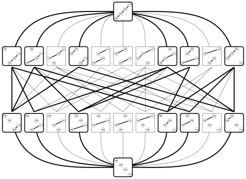
h-vector of permutahedron:



h^*-vector of $[0,1]$ hypercube:



Ranks of shard order:



Eulerian Numbers

γ-nonnegativity:

$$
\begin{aligned}
A_2(t) &= 1 + t, \\
A_3(t) &= 1 + 4t + t^2 \\
&= (1+t)^2 + 2t, \\
A_4(t) &= 1 + 11t + 11t^2 + t^3 \\
&= (1+t)^3 + 8t(1+t), \\
A_5(t) &= 1 + 26t + 66t^2 + 26t^3 + t^4 \\
&= (1+t)^4 + 22t(1+t)^2 + 16t^2 \\
A_6(t) &= 1 + 57t + 302t^2 + 302t^3 + 57t^4 + t^5 \\
&= (1+t)^5 + 52t(1+t)^3 + 136t^2(1+t) \\
&\vdots
\end{aligned}
$$

By analogy…

Narayana Numbers

π	des(π)
$\mathbf{123}$	$\mathbf{0}$
$\mathbf{13\|2}$	$\mathbf{1}$
$\mathbf{2\|13}$	$\mathbf{1}$
$23\|1$	1
$\mathbf{3\|12}$	$\mathbf{1}$
$\mathbf{3\|2\|1}$	$\mathbf{2}$

$$C_3(t) = 1 + 3t + t^2 = N(3,0) + N(3,1)t + N(3,2)t^2$$

$N(n,k)$ is the number of 231-**avoiding** permutations in S_n with k descents

Narayana Numbers

$n\backslash k$	0	1	2	3	4	5	sum
1	1						1
2	1	1					2
3	1	3	1				5
4	1	6	6	1			14
5	1	10	20	10	1		42
6	1	15	50	50	15	1	132

$$C_n(t) = \sum_{\pi \in S_n(231)} t^{\mathsf{des}(\pi)} = \sum_{k=0}^{n} N(n,k) t^k$$

Note: $C_n(1) = |S_n(231)| = \frac{1}{n+1}\binom{2n}{n}$, the nth **Catalan number**

$$1, 1, 2, 5, 14, 42, 132, 429, \ldots$$

Narayana Numbers

Generating functions:

$$C_{n+1}(t) = C_n(t) + t\sum_{i=0}^{n-1} C_i(t)C_{n-i}(t)$$

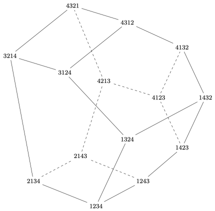
$$\sum_{n\geq 0} C_n(t)z^n = \frac{1+z(t-1)-\sqrt{1-2z(t+1)+z^2(t-1)^2}}{2tz}$$

Formulas:

$$\begin{aligned} N(n,k) &= \frac{2n-k-1}{k+1}N(n-1,k-1)+N(n-1,k) \\ &= \binom{n-1}{k}\binom{n+1}{k+1}-\binom{n}{k}\binom{n}{k+1} \\ &= \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k} \end{aligned}$$

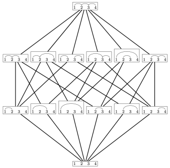
Narayana Numbers

h-vector of associahedron:



h^*-vector of order polytope: $\mathcal{O}([2] \times [n])$

Ranks of noncrossing partition lattice:



Narayana Numbers

γ-nonnegativity:

$$
\begin{aligned}
C_2(t) &= 1+t, \\
\hline
C_3(t) &= 1+3t+t^2 \\
&= (1+t)^2+t, \\
\hline
C_4(t) &= 1+6t+6t^2+t^3 \\
&= (1+t)^3+3t(1+t), \\
\hline
C_5(t) &= 1+10t+20t^2+10t^3+t^4 \\
&= (1+t)^4+6t(1+t)^2+2t^3 \\
\hline
C_6(t) &= 1+15t+50t^2+50t^3+15t^4+t^5 \\
&= (1+t)^5+10t(1+t)^3+10t^2(1+t) \\
&\vdots
\end{aligned}
$$

Zig-zag Eulerian polynomials

Another analogy

Inspired by Stanley's "A survey of alternating permutations":

How well does the analogy hold up if we replace 231-avoiding permutations with zig-zag (alternating) permutations?

Zig-zag permutations

A permutation is **zig-zag** if its descents are every other position:

$$u_1 < u_2 > u_3 < \cdots \text{ or } u_1 > u_2 < u_3 > \cdots$$

Let U_n, D_n denote the sets of up-downers and down-upppers

U_4	D_4
1324	4231
1423	4132
2314	3241
2413	3142
3412	2143

$|U_n| = |D_n| = E_n$, the nth **Euler number**, whose sequence begins

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, \ldots$$

Zig-zag permutations

Question I asked some DePaul undergrads in 2012: Is there an analogue of the Eulerian numbers for the set U_n?

$$U_n(t) = \sum_{u \in U_n} t^{f(u)} = \sum_{k=0}^{n} u(n,k) t^k$$

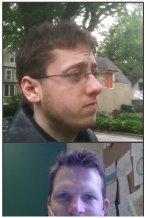
$n \backslash k$	0	1	2	3	4	5	sum
1	1						1
2	1						1
3	1	1					2
4	1	3	1				5
5	1	7	7	1			16
6	1	?	?	?	1		61
7	1	?	?	?	?	1	272

Zig-zag permutations

Answer from students: not entirely sure, but here's some other cool stuff we see!

Power series for up-down min-max permutations

Fiacha Heneghan and T. Kyle Petersen



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Zig-zag Eulerian numbers

Breaking news! (Coons and Sullivant 2023)
Define **big returns** of u: $\mathrm{ret}_1(u) = \{i : u^{-1}(i) > 1 + u^{-1}(i+1)\}$

$$U_n(t) = \sum_{u \in U_n} t^{\mathrm{ret}_1(u)} = \sum_{k=0}^{n} u(n,k)t^k$$

u	$\mathrm{ret}_1(u)$
1324	0
142**3**	1
23**1**4	1
24**13**	2
341**2**	1

$$U_4(t) = 1 + 3t + t^2 = u(4,0) + u(4,1)t + u(4,2)t^2$$

Zig-zag Eulerian numbers

$n\backslash k$	0	1	2	3	4	5	sum
1	1						1
2	1						1
3	1	1					2
4	1	3	1				5
5	1	7	7	1			16
6	1	14	31	14	1		61
7	1	26	109	109	26	1	272

Theorem (Coons and Sullivant '23)

The polynomial $U_n(t)$ is the h^*-polynomial of a Gorenstein* polytope, and hence it is palindromic and unimodal.

Research questions

	Eulerian	Narayana	Zig-zag
formulas	Y	Y	?
recurr.	Y	Y	**y**
gen. func.	Y	Y	**y**
statistics (CLT)	Y	Y	?
P-Eulerian	Y	Y	**Y**
h^* of polytope	Y	Y	Y
h-poly of sphere	Y	Y	?
γ-nonn.	Y	Y	**Y**
weak order	Y	Y	?
shard poset	Y	Y	?

(still need better combinatorial understanding)

Zig-zag Eulerian polynomials

P-Eulerian polynomials

For any finite poset P of $[n] = \{1, 2, \ldots, n\}$, there is a P-**Eulerian polynomial**

$$A_P(t) = \sum_{w \in \mathcal{L}(P)} t^{\mathsf{des}(w)}$$

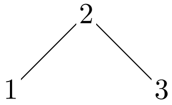
where $\mathcal{L}(P) \subseteq S_n$ is the set of *linear extensions* P:

$$w \in \mathcal{L}(P) \Leftrightarrow (i <_P j \Rightarrow w^{-1}(i) < w^{-1}(j))$$

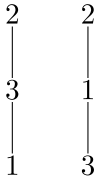
P-Eulerian polynomials

Example:

P:



$\mathcal{L}(P)$:



so $\mathcal{L}(P) = \{132, 312\}$, and

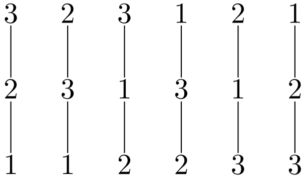
$$A_P(t) = 2t$$

P-Eulerian polynomials

Example (antichain):

P:

$\mathcal{L}(P)$:



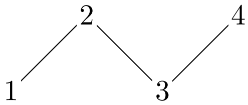
so $\mathcal{L}(P) = S_3$, and

$$A_P(t) = 1 + 4t + t^2 = A_3(t)$$

P-Eulerian polynomials

Example (zig-zag poset):

Z_4:



$$\begin{aligned}\mathcal{L}(Z_4) &= \{w : w^{-1}(1) < w^{-1}(2) > w^{-1}(3) < w^{-1}(4)\} \\ &= \{w : w^{-1} \in U_4\} \\ &= \{1324, 1342, 3124, 3142, 3412\}\end{aligned}$$

$$A_{Z_4}(t) = 4t + t^2 = \sum_{w \in \mathcal{L}(Z_4)} t^{\mathsf{des}(w)} = \sum_{u \in U_n} t^{\mathsf{ret}(u)}$$

where $\mathsf{ret}(u) = \mathsf{des}(u^{-1}) = \{i : u^{-1}(i) > u^{-1}(i+1)\}$

P-Eulerian polynomials

So Z_n is nice because $\mathcal{L}(Z_n) = \{w : w^{-1} \in U_n\}$ and

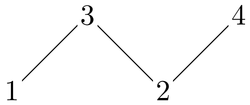
$$A_{Z_n}(t) = \sum_{u \in U_n} t^{\mathsf{ret}(u)}$$

But these polynomials are not especially nice...

$$\begin{aligned}
A_{Z_2}(t) &= 1 \\
A_{Z_3}(t) &= 2t \\
A_{Z_4}(t) &= 4t + t^2 \\
A_{Z_5}(t) &= 2t + 12t^2 + 2t^3 \\
A_{Z_6}(t) &= 4t + 36t^2 + 20t^3 + t^4 \\
&\vdots
\end{aligned}$$

P-Eulerian polynomials

Example (natural zig-zag poset):
Let $\varepsilon = s_2 s_4 \cdots$, the product of all even adjacent transpositions

εZ_4: 

$$\begin{aligned}\mathcal{L}(\varepsilon Z_4) &= \{w : w^{-1}(1) < w^{-1}(3) > w^{-1}(2) < w^{-1}(4)\}\\ &= \varepsilon\{1324, 1342, 3124, 3142, 3412\}\\ &= \{1234, 1243, 2134, 2143, 2413\}\end{aligned}$$

$$A_{\varepsilon Z_4}(t) = 1 + 3t + t^2 = \sum_{u \in U_4} t^{\mathsf{des}(\varepsilon u^{-1})} = \sum_{u \in U_4} t^{\mathsf{ret}(u\varepsilon)}$$

P-Eulerian polynomials

$$A_{\varepsilon Z_n}(t) = \sum_{u \in U_n} t^{\mathsf{ret}(u\varepsilon)}$$

$$\begin{aligned}
A_{\varepsilon Z_2}(t) &= 1 \\
\hline
A_{\varepsilon Z_3}(t) &= 1 + t, \\
\hline
A_{\varepsilon Z_4}(t) &= 1 + 3t + t^2 \\
&= (1+t)^2 + t, \\
\hline
A_{\varepsilon Z_5}(t) &= 1 + 7t + 7t^2 + t^3 \\
&= (1+t)^3 + 4t(1+t), \\
\hline
A_{\varepsilon Z_6}(t) &= 1 + 14t + 31t^2 + 14t^3 + t^4 \\
&= (1+t)^4 + 10t(1+t)^2 + 5t^2 \\
&\vdots
\end{aligned}$$

And these polynomials **are** nice: a lemma of Brändén shows $A_{\varepsilon Z_n}(t)$ is gamma-nonnegative

Zig-zag Eulerian polynomials

Zig-zag Eulerian polynomials

Our definition of the **zig-zag Eulerian polynomial** is to simplify notation: $Z_n(t) = A_{\varepsilon Z_n}(t)$

Proposition (P.-Zhuang)

For any $u \in U_n$

$$\mathit{ret}(u\varepsilon) = \mathit{ret}_1(u)$$

Two proofs: bijective and using "relaxed" P-partitions

Zig-zag Eulerian polynomials

Theorem (P.-Zhuang)

For $n \geq 1$, $Z_n(t)$ is γ-nonnegative and has the following combinatorial descriptions:

$$\begin{aligned} Z_n(t) &= \sum_{u \in U_n} t^{\mathsf{ret}_1(u)} \quad \textit{(up-down)} \\ &= \sum_{v \in D_n} t^{\mathsf{ret}_1(v)} \quad \textit{(down-up)} \\ &= \sum_{w \in J_n} t^{\mathsf{ret}(w)} \quad \textit{(Jacobi)} \end{aligned}$$

Zig-zag Eulerian polynomials

More on P-Eulerian polynomials

Theorem (Stanley)

$$\frac{tA_P(t)}{(1-t)^{n+1}} = \sum_{m\geq 0} \Omega_P(m)t^m$$

where $\Omega_P(m)$ is the order polynomial, which counts the number of P-partitions bounded by m.

In fact,

$$\begin{aligned}\Omega_P(m+1) &= i(\mathcal{O}(P);m)\\ &= i(\mathcal{C}(P);m)\end{aligned}$$

where $i(\mathcal{P};m)$ is the *Ehrhart polynomial* which counts integer points in the mth dilation of the polytope $\mathcal{P}$

Order polynomials

Whenever P is naturally labeled

$$\Omega_P(m) = |\{(a_1, \ldots, a_n) \in [m]^n : i <_P j \Rightarrow a_i \leq a_j\}|$$

Write

$$\begin{aligned}\Omega_n(m) &= \Omega_{\varepsilon Z_n}(m) \\ &= |\{(a_1, \ldots, a_n) \in [m]^n : a_1 \leq a_3 \geq a_2 \leq \cdots\}|\end{aligned}$$

$n\backslash m$	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8
2	1	3	6	10	15	21	28	36
3	1	5	14	30	55	91	140	204
4	1	8	31	85	190	371	658	1086
5	1	13	70	246	671	1547	3164	5916

Order polynomial generating function

Classic: $H_n(t) = A_n(t)/(1-t)^{n+1} = \sum_{m\geq 1} m^n t^m$, so

$$H_{n+1}(t) = tH'(t), \quad A_{n+1}(t) = t(1-t)A_n'(t) + (n+1)tA_n(t)$$

Order polynomial generating function

Classic: $H_n(t) = A_n(t)/(1-t)^{n+1} = \sum_{m\geq 1} m^n t^m$, so

$$H_{n+1}(t) = tH'(t), \quad A_{n+1}(t) = t(1-t)A_n'(t) + (n+1)tA_n(t)$$

New: $G_n(t) = Z_n(t)/(1-t)^{n+1} = \sum_{m\geq 1} \Omega_n(m) t^m$, so

$$G_{n+1}(t) = \ldots ??, \quad Z_{n+1}(t) = \ldots ??$$

Order polynomial generating function

Classic: $H_n(t) = A_n(t)/(1-t)^{n+1} = \sum_{m\geq 1} m^n t^m$, so

$$H_{n+1}(t) = tH'(t), \quad A_{n+1}(t) = t(1-t)A_n'(t) + (n+1)tA_n(t)$$

New: $G_n(t) = Z_n(t)/(1-t)^{n+1} = \sum_{m\geq 1} \Omega_n(m)t^m$, so

$$G_{n+1}(t) = \ldots ??, \quad Z_{n+1}(t) = \ldots ??$$

Helps to refine:

$$G_n(p,q,x) = \sum_{m\geq 1} \Omega_n(p,q;m)x^m = \frac{pqxU_n(px,qx)}{(1-qx)^{n+1}},$$

where $Z_n(t) = tU_n(t,t)$.

Order polynomial generating function

Theorem (P.-Zhuang)

For all $n \geq 0$,

- $G_{n+1}(p,q,x) = \dfrac{p}{p-q}\left[G_n(q,p,x) - G_n(q,q,x)\right]$, *and thus*
- $U_{n+1}(s,t) = \dfrac{1-t}{s-t}\left[\left(\dfrac{1-t}{1-s}\right)^{n+1} sU_n(t,s) - tU_n(t,t)\right]$.

Corollary (Derivative expressions)

For all $n \geq 0$,

- $G_{n+1}(x) = \dfrac{d}{dq}\left[G_n(1,q,x)\right]_{q=1}$ *and thus*
- $U_{n+1}(t) = t(1-t)\dfrac{d}{dt}\left[U_n(s,t)\right]_{s=t} + (nt+1)U_n(t)$.

Details for the refinement

Let

$$\mathsf{ret}_1^-(u) = |\mathsf{Ret}_1(u) \cap \{1, 2, \ldots, u(1)\}|$$
$$\mathsf{ret}_1^+(u) = |\mathsf{Ret}_1(u) \cap \{u(1)+1, \ldots, n-1\}|$$

Then

$$U_n(s,t) = \sum_{u \in U_n} s^{\mathsf{ret}_1^-(u)} t^{\mathsf{ret}_1^+(u)} \left(\frac{1-t}{1-s}\right)^{u(1)}$$

With $U_4 = \{1324, 1423, 2314, 2413, 3412\}$, we find

$$U_4(s,t) = \frac{1-t}{1-s} + \frac{t(1-t)}{(1-s)} + \frac{s(1-t)^2}{(1-s)^2} + \frac{st(1-t)^2}{(1-s)^2} + \frac{s(1-t)^3}{(1-s)^3}$$

(refines Entriger numbers)

Column generating function

Proposition (P.-Zhuang)

For $n \geq 2$, $m \geq 1$,

$$\Omega_n(m) = \Omega_n(m-1) + \Omega_{n-2}(m) + \sum_{i\ odd} \Omega_i(m-1)\Omega_{n-1-i}(m)$$
$$= \Omega_n(m-1) + \sum_{i\ even} \Omega_i(m-1)\Omega_{n-1-i}(m)$$

Corollary (Stanley ('73), Bóna-Ju (2006), P.-Zhuang)

With $F_m(y) = \sum_{n\geq 0} \Omega_n(m) y^n$, *we have* $F_1(y) = 1/(1-y)$ *and for* $m \geq 2$

$$F_m(y) = \frac{1}{F_{m-1}(-y) - y} \quad (\textit{known})$$
$$= \frac{y + 2F_{m-1}(y)}{2 - y^2 - yF_{m-1}(y)} \quad (\textit{new})$$

Column generating function

It follows that $F_m(y) = P_m(y)/Q_m(y)$, where

$$\begin{bmatrix} 1 & y \\ -y & 1-y^2 \end{bmatrix} \begin{bmatrix} P_m(y) \\ Q_m(y) \end{bmatrix} = \begin{bmatrix} P_{m+2}(y) \\ Q_{m+2}(y) \end{bmatrix}.$$

e.g.,

$$F_2(y) = \frac{1+y}{1-y-y^2} = 1 + 2y + 3y^2 + 5y^3 + 8y^4 + 13y^5 + \cdots ,$$

But still...

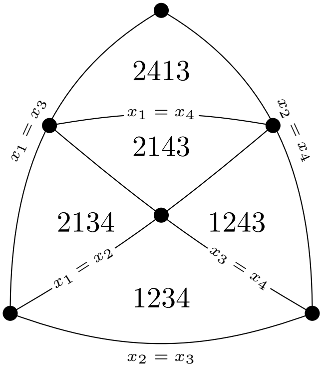
	Eulerian	Narayana	Zig-zag
formulas	Y	Y	?
recurr.	Y	Y	**y**
gen. func.	Y	Y	**y**
statistics (CLT)	Y	Y	?
P-Eulerian	Y	Y	**Y**
h^* of polytope	Y	Y	Y
h-poly of sphere	Y	Y	?
γ-nonn.	Y	Y	**Y**
weak order	Y	Y	?
shard poset	Y	Y	?

it's mostly about what we don't know...

Zig-zag Eulerian polynomials

Interpretations of $\Omega_n(m)$

Integer points in a "Coxeter cone":



$Z_n(t)$ is the "h-polynomial" of the simplicial cone

Interpretations of $\Omega_n(m)$

Stanley (1986), Ehrhart polynomial of "order polytope" or "chain polytope":

$\mathcal{O}(Z_n)$	$\mathcal{C}(Z_n)$
$\mathbf{x} \in \mathbb{R}^n$:	$\mathbf{y} \in \mathbb{R}^n$:
$0 \leq x_i \leq 1$	$0 \leq y_i \leq 1$
$x_1 \leq x_2 \geq x_3 \leq \cdots$	$y_i + y_{i+1} \leq 1$

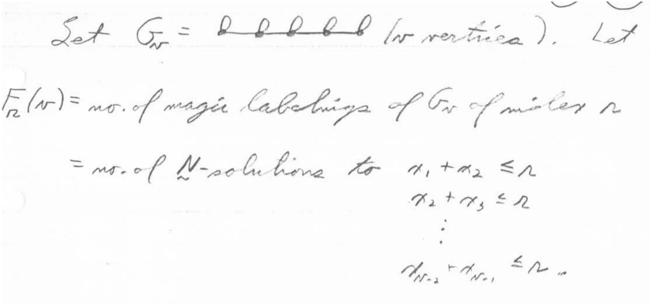
$$\Omega_n(m+1) = i(\mathcal{O}(Z_n); m) = i(\mathcal{C}(Z_n); m)$$

so $Z_n(t)$ is the "h^*-polynomial" for both of these

h^* is symmetric; Kirillov (2000) conjectured unimodality; proved by Chen and Zhang (2016), also Coons and Sullivant (2023); (γ-nonnegativity of $Z_n(t)$ gives third proof)

Interpretations of $\Omega_n(m)$

Stanley (1973, unpublished), Bóna-Ju (2006), "magic labelings of graphs":


Let G_N = (N vertices). Let

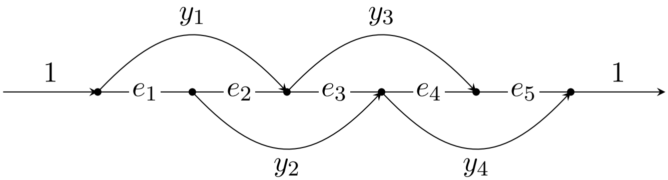
$F_n(N)$ = no. of magic labelings of G_N of index n

= no. of $\mathbb{N}$-solutions to
$x_1 + x_2 \le n$
$x_2 + x_3 \le n$
$\vdots$
$x_{N-2} + x_{N-1} \le n$.

counting integer points in $\mathcal{C}(Z_n)$ again; also Bóna, Ju, Yoshida (2007), "edge polytopes of graphs"

Interpretations of $\Omega_n(m)$

González d'León, Hanusa, Morales, and Yip (2021), "flow polytope of $G(2, n+2)$":



$G(2, n+2)$ is integrally equivalent to $\mathcal{C}(Z_n)$, i.e.,
$\Omega_n(m+1) = i(G(2, n+2); m)$, so $Z_n(t)$ is h^*-polytope

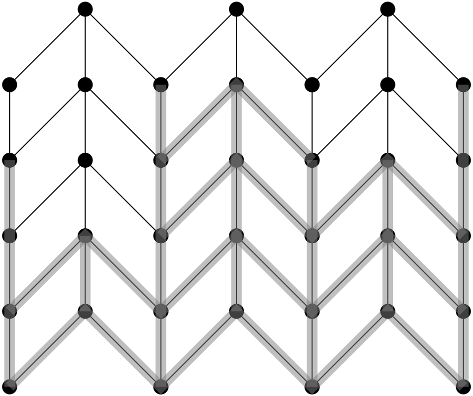
Interpretations of $\Omega_n(m)$

Coons and Sullivant (2020), "toric vanishing ideal for Cavender-Farris-Neyman model for phylogenetic trees with a molecular clock":

to any binary tree T with $n+1$ leaves, there is a polytope $\mathcal{P}_T$ with $\Omega_n(m+1) = i(\mathcal{P}_T; m)$

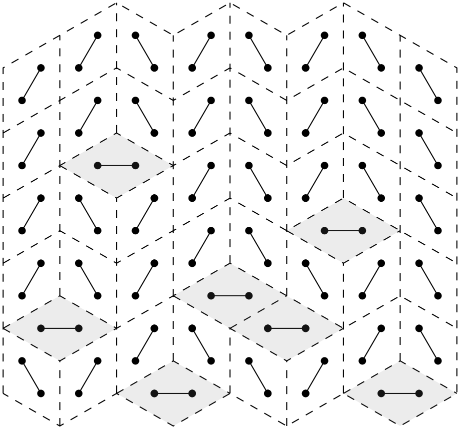
Interpretations of $\Omega_n(m)$

Berman and Köhler (1976), "order ideals of $Z_n \times I_m$":



Interpretations of $\Omega_n(m)$

Cyvin and Gutman (1988), "Kekulé structures of benzenoid hydrocarbons":
(i.e., perfect matchings of a hexagonal lattice)



Future work

	Eulerian	Narayana	Zig-zag	Zig-?
formulas	Y	Y	?	?
recurr.	Y	Y	?	**y**
gen. func.	Y	Y	?	**y**
statistics (CLT)	Y	Y	?	?
P-Eulerian	Y	Y	Y	**Y**
h^* of polytope	Y	Y	Y	**Y**
h-poly of sphere	Y	Y	?	?
γ-nonn.	Y	Y	Y	**Y**
weak order	Y	Y	?	(maybe)
shard poset	Y	Y	?	?