Two-sided Eulerian numbers via balls in boxes

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I will shamelessly tell you what my bottom line is. It is placing balls into boxes...

Gian-Carlo Rota, *Indiscrete Thoughts*

Gian-Carlo Rota was a professor at MIT from 1959 until his death in 1999. He is arguably the father of the field today known as algebraic combinatorics. He had 49 students (notably Richard Stanley) and, as of this writing, he has 313 mathematical descendants, including many of the top names in the field today. (And me.) If balls in boxes was good enough for him, well, it should be good enough for any of us.

In this article we will first see how to use the idea of balls in boxes to obtain well-known results about the Eulerian numbers. Then we will take those ideas and study a perfectly natural (though less well-known) refinement: the “two-sided” Eulerian numbers. We will finish with a discussion of symmetries of the Eulerian numbers and a previously unpublished conjecture of Ira Gessel.

**Balls in boxes**

The symmetric group $S_n$ is the set of all permutations of length $n$, i.e., all bijections $w : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$. We typically write a permutation in one-line notation: $w = w(1)w(2) \cdots w(n)$. For a permutation $w \in S_n$, we define a *descent* to be a position $r$ such that $w(r) > w(r+1)$, and we denote by $\text{des}(w)$ the number of descents of $w$. For example, if $w = 5624713$, then there are descents in position 2 (since 6 > 2) and in position 5 (since 7 > 1).
Hence, \( \text{des}(w) = 2 \). The generating function for descents is

\[
A_n(t) = \sum_{w \in S_n} t^{\text{des}(w)+1} = \sum_{i=1}^{n} A_{n,i} t^i,
\]

which is well known as the *Eulerian polynomial*. Its coefficients \( A_{n,i} \) are the *Eulerian numbers*. From the definition given, we can see that \( A_{n,i} \) is the number of permutations in \( S_n \) with \( i - 1 \) descents. This is not the only way to define the Eulerian numbers; \( A_{n,i} \) is also the number of permutations in \( S_n \) with \( i - 1 \) *ascents* (positions \( r \) with \( w(r) < w(r + 1) \)) or the number of permutations with \( i - 1 \) *excedances* (positions \( r \) with \( w(r) > r \)). Euler himself was not interested in permutation statistics at all, but was rather investigating solutions of certain functional equations in which the polynomials \( A_n(t) \) emerged. See the original [5, Caput VII, pp. 389–390] (available digitally online—thanks Google!) or Carlitz’s survey article, [1].

A classic result in enumerative combinatorics is the following. (This result goes back at least to work of MacMahon [9, Chapter IV, §462]. See [10, Section 4.5] for a modern treatment and generalizations.)

**Theorem 1 (The generating function)** For any \( n \geq 1 \), we have

\[
\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} k^n t^k.
\]

So, for example, the reader can check the \( n = 3 \) case gives the generating function for the cubes:

\[
\frac{t + 4t^2 + t^3}{(1-t)^4} = t + 8t^2 + 27t^3 + 64t^4 + 125t^5 + \cdots.
\]

From Theorem 1 it is easy to obtain the following well known identity for Eulerian polynomials. By comparing coefficients, we also get a handy recurrence for Eulerian numbers.

**Corollary 1 (The recurrence relation)** For \( n \geq 2 \), we have

\[
A_n(t) = ntA_{n-1}(t) + t(1-t)A'_{n-1}(t),
\]
or
\[ A_{n,i} = iA_{n-1,i} + (n + 1 - i)A_{n-1,i-1}. \]

There is also a direct combinatorial proof of this recurrence. (Imagine inserting the number \( n \) into a permutation of \( S_{n-1} \). What happens if you insert it in a descent position? What happens if you insert it in an ascent position?) See, for example, Knuth’s book [8, Section 5.1.3]. In any event, these recurrences allow for easy computation of the Eulerian numbers. See Table 1 for the Eulerian numbers with \( n \leq 8 \). Another consequence of Theorem 1 that we can easily derive from our approach is Worpitzky’s identity, which is also explained in [8, Section 5.1.3].

**Corollary 2 (Worpitzky’s identity)** For any \( n, k \geq 1 \),
\[ k^n = \sum_{1 \leq i \leq n} A_{n,i} \binom{k+n-i}{n}. \]

For example, when \( n = 4 \), \( k = 3 \),
\[ A_{4,1} \binom{6}{4} + A_{4,2} \binom{5}{4} + A_{4,3} \binom{4}{4} + A_{4,4} \binom{3}{4} = 1 \cdot 15 + 11 \cdot 5 + 11 \cdot 1 + 1 \cdot 0 = 81. \]

One particularly elegant and elementary method for proving Theorem 1 is that of “balls in boxes”, in which we first interpret the power series as the generating function for the number of ways to put \( n \) labeled balls into \( k \) distinct boxes.

**Proof of Theorem 1 (The generating function)**

The generating function for all ways of putting \( n \) distinct balls into boxes is
\[ \sum_{k \geq 0} k^n t^k, \]
as given on the right-hand side of Theorem 1. Indeed, if there are \( k \) different boxes, there are a total \( k^n \) ways to put \( n \) labeled balls into the boxes; we
have \( k \) choices for each ball. A typical choice with \( k = 9, \ n = 6 \) might look like this:

\[
\begin{array}{cccccc}
\hline
& 5 & 6 & 2 & 1 & 4 \\
\hline
1 & 1 \\
2 & 1 & 1 \\
3 & 1 & 4 & 1 \\
4 & 1 & 11 & 11 & 1 \\
5 & 1 & 26 & 66 & 26 & 1 \\
6 & 1 & 57 & 302 & 302 & 57 & 1 \\
7 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\
8 & 1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1 \\
\hline
\end{array}
\]

Table 1: The Eulerian numbers \( A_{n,i} \)

We will use a shorthand notation for pictures like the one above, e.g.,

\[ ||56|2|14||3 \]

Here we draw a vertical bar for the divisions between the boxes, so that the number of boxes is one more than the number of bars. Also notice that by way of standardization, we list the balls in each box in increasing order. So balls 5 and 6 were placed in the third box, ball 2 in the fourth box, and so on. We call such diagram for balls in boxes a *barred permutation*.

We can partition the set of all barred permutations (arrangements of balls in boxes) according to the underlying permutations. Since we require numbers to increase within a box, we know that there *must* be a bar in each descent position of a barred permutation, but otherwise, we can insert bars into the gaps between the numbers at will. So, for example, the barred permutations corresponding to \( w = 562143 \) are obtained from the barred
by adding bars arbitrarily in the gaps between the numbers, i.e., there can be any number of bars to the left of 5, any number of bars between the 5 and the 6, at least one bar between the 6 and the 2, at least one bar between the 2 and the 1, and so on.

With the identity permutation, \( w = 12 \ldots n \), there are no required bars and we are free to add any number of bars in each of the \( n+1 \) gaps between the numbers. Thus, the generating function for barred permutations corresponding to the \( 12 \ldots n \) is the generating function for the “multi-choose” numbers \( \binom{n+1}{k} = \binom{k+n}{n} \), i.e., the number of ways to choose, with repetition, \( k \) things from an \( (n+1) \)-element set. With the convention that each barred permutation has weight \( t \)\#of bars +1 (\( = t \)\# of boxes), we express this generating function as:

\[
\sum_{k \geq 0} \binom{k + n - 1}{n} t^k = t \cdot \left( 1 + t + t^2 + \cdots \right) \left( 1 + t + t^2 + \cdots \right) \cdots \\
= t \cdot \left( 1 + t + t^2 + \cdots \right)^{n+1} \\
= \frac{t}{(1 - t)^{n+1}}.
\]

For permutations with descents, we need only multiply this generating function by a power of \( t \) to reflect the number of bars required by descent positions:

\[
\frac{t^{\text{des}(w)+1}}{(1 - t)^{n+1}} = t^{\text{des}(w)} \cdot \sum_{k \geq 0} \binom{k + n - 1}{n} t^k = \sum_{l \geq 0} \binom{l + n - 1 - \text{des}(w)}{n} t^l.
\]

So, returning to the example of \( w = 562143 \), we get that the generating function for its corresponding barred permutations is:

\[
\frac{t^4}{(1 - t)^7} = t^3 \cdot \sum_{k \geq 0} \binom{k + 5}{6} t^k = \sum_{l \geq 0} \binom{l + 2}{6} t^l.
\]
Summing (1) over all permutations in $S_n$ gives us, on the one hand,

$$\frac{A_n(t)}{(1 - t)^{n+1}}.$$ 

On the other hand, the sum gives the generating function for all barred permutations, which know to be $\sum_{k \geq 0} k^n t^k$. This is precisely Theorem 1.

**Proof of Corollary 1 (The recurrence relation)**

With Theorem 1 in hand, we can derive the recurrence for Eulerian polynomials given in Corollary 1 with simple calculus. Let $F_n(t) = \sum k^n t^k = A_n(t)/(1 - t)^{n+1}$, and observe that

$$F_n(t) = \sum_{k \geq 0} k^n t^k$$

$$= t \cdot \sum_{k \geq 0} k^{n-1} \cdot k t^{k-1}$$

$$= t \cdot F'_n(t)$$

$$= t \left( \frac{n A_{n-1}(t)}{(1 - t)^{n+1}} + A'_{n-1}(t) \right)$$

$$= \frac{nt A_{n-1}(t) + t(1 - t)A'_{n-1}(t)}{(1 - t)^{n+1}}.$$ 

Comparing numerators yields Corollary 1.

**Proof of Corollary 2 (Worpitzky’s identity)**

To obtain Worpitzky’s identity, we revisit (1). On the one hand, if a permutation in $S_n$ has $i - 1$ descents, (1) tells us the number of its barred permutations with $k - 1$ bars is

$$\binom{k + n - i}{n}.$$ 

There are $A_{n,i}$ such permutations, so the total number of barred permutations with $k - 1$ bars is

$$\sum_{1 \leq i \leq n} A_{n,i} \binom{k + n - i}{n}.$$
On the other hand, we already know there are \( k^n \) barred permutations with \( k - 1 \) bars (remember this is the number of ways to put \( n \) balls in \( k \) boxes), yielding
\[
\sum_{1 \leq i \leq n} A_{n,i} \binom{k + n - i}{n} = k^n,
\]
as desired.

**Balls in boxes, revisited**

We are now going to extend the method of balls in boxes to study the “two-sided” Eulerian polynomial,
\[
A_n(s,t) := \sum_{w \in S_n} s^{\text{des}(w^-1) + 1} t^{\text{des}(w) + 1} = \sum_{i,j=1}^{n} A_{n,i,j} s^i t^j,
\]
where \( w^-1 \) is the group-theoretic (compositional) inverse of \( w \), i.e., \( w(w^-1(i)) = w^-1(w(i)) = i \) for all \( i \). We think of \( A_n(s,t) \) as the generating function for the joint distribution of descents and inverse descents.

This polynomial was first studied by Carlitz, Roselle, and Scoville [2], though rather than descents and inverse descents, they looked at the equivalent notion of “jumps” (ascents) and “readings” (inverse ascents). The following results are all presented in [2, Section 7], and proved using a mixture of combinatorics and manipulatorics (i.e., manipulation of formulas using binomial identities and such). Here we take the balls in boxes approach.

**Theorem 2 (The generating function)** For \( n \geq 1 \), we have
\[
\frac{A_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}} = \sum_{k,l \geq 0} \binom{kl + n - 1}{n} s^k t^l.
\]

From Theorem 2 we have the following corollary (compare with [2, Equation 7.8]), which makes for easy computation of the two-sided Eulerian numbers. We see in Table 2 arrays containing the two-sided Eulerian numbers for \( n \leq 8 \).
\[ \begin{array}{c|c|c|c}
    & n = 2 & n = 3 & n = 4 \\
\hline
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
& \left[ \begin{array}{ccc} 1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right] & \left[ \begin{array}{ccc} 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{array} \right] & \left[ \begin{array}{ccc} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array} \right] \\
\end{array} \]

\[ \begin{array}{c|c|c|c}
    & n = 5 & n = 6 & n = 7 \\
\hline
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
20 & 20 & 20 & 20 \\
54 & 54 & 54 & 54 \\
6 & 6 & 6 & 6 \\
0 & 0 & 0 & 0 \\
& \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 20 & 6 & 0 \\
0 & 54 & 6 & 0 \\
0 & 6 & 20 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right] & \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 21 & 210 & 70 \\
0 & 1 & 70 & 210 \\
0 & 0 & 1 & 35 \\
0 & 0 & 0 & 1 \\
\end{array} \right] & \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\
0 & 35 & 21 & 0 \\
0 & 21 & 210 & 70 \\
0 & 1 & 70 & 210 \\
0 & 0 & 1 & 35 \\
0 & 0 & 0 & 1 \\
\end{array} \right] \\
\end{array} \]

\[ \begin{array}{c|c|c|c}
    & n = 8 \\
\hline
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
84 & 84 & 84 & 84 \\
126 & 126 & 126 & 126 \\
36 & 36 & 36 & 36 \\
1773 & 1773 & 1773 & 1773 \\
405 & 405 & 405 & 405 \\
8436 & 8436 & 8436 & 8436 \\
4761 & 4761 & 4761 & 4761 \\
0 & 9 & 9 & 9 \\
0 & 405 & 405 & 405 \\
0 & 0 & 9 & 9 \\
0 & 0 & 0 & 1 \\
& \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\
0 & 84 & 126 & 36 \\
0 & 126 & 1773 & 1980 \\
0 & 36 & 1980 & 8436 \\
0 & 1 & 405 & 4761 \\
0 & 0 & 9 & 405 \\
0 & 0 & 0 & 1 \\
\end{array} \right] & \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\
0 & 84 & 126 & 36 \\
0 & 126 & 1773 & 1980 \\
0 & 36 & 1980 & 8436 \\
0 & 1 & 405 & 4761 \\
0 & 0 & 9 & 405 \\
0 & 0 & 0 & 1 \\
\end{array} \right] & \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\
0 & 84 & 126 & 36 \\
0 & 126 & 1773 & 1980 \\
0 & 36 & 1980 & 8436 \\
0 & 1 & 405 & 4761 \\
0 & 0 & 9 & 405 \\
0 & 0 & 0 & 1 \\
\end{array} \right] \\
\end{array} \]

Table 2: The two-sided Eulerian numbers \([A_{n,i,j}]_{1 \leq i,j \leq n}, n = 1, \ldots, 8\)
Corollary 3 (The recurrence relation) For \( n \geq 2 \),

\[
nA_n(s, t) = (n^2 st + (n - 1)(1 - s)(1 - t))A_{n-1}(s, t)
+ nst(1 - s)\frac{d}{ds}A_{n-1}(s, t) + nst(1 - t)\frac{d}{dt}A_{n-1}(s, t)
+ st(1 - s)(1 - t)\frac{d^2}{dsdt}A_{n-1}(s, t),
\]

or

\[
nA_{n,i,j} = (ij + n - 1)A_{n-1,i,j}
+ [1 - n + j(n + 1 - i)]A_{n-1,i-1,j} + [1 - n + i(n + 1 - j)]A_{n-1,i,j-1}
+ [n - 1 + (n + 1 - i)(n + 1 - j)]A_{n-1,i-1,j-1}.
\]

Just as Worpitzky’s identity can be derived from Theorem 1 and Equation (1), Theorem 2 and Equation (1) yields the following identity of binomial coefficients.

Corollary 4 (A Worpitzky-like identity) For any positive integers \( k, l, n \),

\[
\binom{kl + n - 1}{n} = \sum_{1 \leq i,j \leq n} A_{n,i,j} \binom{k + n - i}{n} \binom{l + n - j}{n}.
\]

We remark that this identity can also be proved (and considerably generalized) using the method of bipartite \( P \)-partitions, as follows from \[7, \text{Corollary 10}\].

To prove these “two-sided” results, we will put balls in boxes another way. First, we introduce a two-dimensional analogue of barred permutations.

Permutations in \( S_n \) can be represented visually as an array of \( n \) indistinguishable balls so that no two balls lie in the same column or row. If \( w(i) = j \), we put a ball in column \( i \) (from left to right) and row \( j \) (from
bottom to top). For example, with $w = 562143$, we draw:

```
  6  5  4  3  2  1
/ \ / \ / \ / \ / \ /
3  2  1
/ \ / \ / \ / \ / \ /
4  3  2  1
/ \ / \ / \ / \ / \ /
5  4  3  2  1
/ \ / \ / \ / \ / \ /
6  5  4  3  2  1
```

The advantage of such an array is that $w^{-1}$ is easily seen in this picture. Rather than reading the heights of the balls from left to right to get $w$, we can read the column numbers of the balls from bottom to top to get $w^{-1}$. So in the example above, $w^{-1} = 436512$. (Check this! Why is it generally true?)

The analogue of a barred permutation, which we call a *two-sided* barred permutation, is any way of inserting both horizontal and vertical lines into the array of balls, with the requirement that there be at least one vertical line between balls that form a descent in $w$, and at least one horizontal line between balls that form a descent in $w^{-1}$. Other horizontal and vertical lines can be added arbitrarily. For example:

```

```

(2)

In a sense, this two-sided barred permutation corresponds to two ordinary barred permutations, one for $w$: $|[56]|2|[14]|3$, and one for $w^{-1}$: $|4|3|[6]|5|12$. 

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Proof of Theorem 2 (The generating function)

Fix an arrangement of balls corresponding to a permutation \( w \). We will give a two-sided barred permutation the weight

\[ s^{\# \text{of horizontal bars} + 1} t^{\# \text{of vertical bars} + 1}, \]

so the example seen in (2) would contribute \( st^9 \). Since the horizontal and vertical bars can be inserted independently of one another, we see that the generating function for the number of two-sided barred permutations corresponding to a fixed permutation \( w \in S_n \) is the product of the generating function for barred permutations for \( w^{-1} \) (in the variable \( s \)) with the generating function for barred permutations of \( w \) (in the variable \( t \)). Thus (1) gives:

\[ s^{\text{des}(w^{-1}) + 1} t^{\text{des}(w) + 1} \frac{1 - s}{(1 - s)^{n+1}} \frac{1 - t}{(1 - t)^{n+1}}. \]  

Adding up (3) over all permutations in \( S_n \) we get the generating function for all two-sided barred permutations is

\[ \frac{A_n(s, t)}{(1 - s)^{n+1}(1 - t)^{n+1}}. \]  

Now consider forming two-sided barred permutations with the bars first. Given \( k - 1 \) vertical bars and \( l - 1 \) horizontal bars, we get a \( k \)-by-\( l \) grid of boxes in which to place our balls, with the convention that if more than one ball goes into a particular row or column, we arrange the balls diagonally from bottom left to top right. For example, the following arrangement of 7 balls in a \( 5 \times 4 \) grid of boxes yields the following two-sided barred permutation
(with underlying permutation 1723465):

With $n$ unlabeled balls and $kl$ distinct boxes, this means there are a total of

$$\binom{kl+n-1}{n}$$

two-sided barred permutations with $k-1$ vertical lines and $l-1$ horizontal lines. (We are essentially choosing $n$ of the $kl$ boxes, with repetition allowed.) The generating function for the number of two-sided barred permutations is thus

$$\sum_{k,l \geq 0} \binom{kl+n-1}{n} s^k t^l,$$

and comparing with (4) yields Theorem 2.

**Proof of Corollary 3 (The recurrence relation)**

We now proceed to derive the recurrence for $A_n(s, t)$ given in Corollary 3. Let $F_n(s, t) = A_n(s, t)/(1 - s)^{n+1}(1 - t)^{n+1}$, and note the following identity of binomial coefficients:

$$n\binom{kl+n-1}{n} = kl\binom{kl+n-2}{n-1} + (n-1)\binom{kl+n-2}{n-1}.$$
Thus, we can see that
\[ nF_n(s, t) = \sum_{k,l \geq 0} n\binom{kl + n - 1}{n}s^k t^l \]
\[ = \sum_{k,l \geq 0} kl\binom{kl + n - 2}{n-1}s^k t^l + \sum_{k,l \geq 0} (n-1)\binom{kl + n - 2}{n-1}s^k t^l \]
\[ = st \frac{d^2}{dsdt} F_{n-1}(s, t) + (n-1)F_{n-1}(s, t). \quad (5)\]

Now, with a little calculus, we have
\[ \frac{d^2}{dsdt} F_{n-1}(s, t) = \frac{d}{ds} \left[ \frac{nA_{n-1}(s, t)}{(1-s)^n(1-t)^{n+1}} + \frac{d}{dt}A_{n-1}(s, t) \right] \]
\[ = \frac{n^2A_{n-1}(s, t)}{(1-s)^{n+1}(1-t)^{n+1}} + \frac{n\frac{d}{ds}A_{n-1}(s, t)}{(1-s)^n(1-t)^{n+1}} \]
\[ + \frac{n\frac{d}{dt}A_{n-1}(s, t)}{(1-s)^{n+1}(1-t)^n} + \frac{d^2}{dsdt}A_{n-1}(s, t). \]

Comparing numerators in (5) yields
\[ nA_n(s, t) = (n^2st + (n-1)(1-s)(1-t))A_{n-1}(s, t) + nst(1-s)\frac{d}{ds}A_{n-1}(s, t) \]
\[ + nst(1-t)\frac{d}{dt}A_{n-1}(s, t) + st(1-s)(1-t)\frac{d^2}{dsdt}A_{n-1}(s, t), \]
as desired.

**Proof of Corollary 4 (The Worpitzky-like identity)**

To get Corollary 4, we use (1) and (3) to see that for any permutation \( w \in S_n \) with \( \text{des}(w^{-1}) = i - 1, \text{des}(w) = j - 1 \), the generating function for its barred permutations is
\[ \frac{s^i t^j}{(1-s)^{n+1}(1-t)^{n+1}} = \sum_{k \geq 0} \binom{k + n - i}{n}s^k \cdot \sum_{l \geq 0} \binom{l + n - j}{n}t^l \]
\[ = \sum_{k,l \geq 0} \binom{k + n - i}{n}\binom{l + n - j}{n}s^k t^l. \]
There are $A_{n,i,j}$ such permutations, so summing over $w \in S_n$ yields

$$
\sum_{1 \leq i,j \leq n} A_{n,i,j} \sum_{k,l \geq 0} \binom{k + n - i}{n} \binom{l + n - j}{n} s^k t^l
$$

$$
= \sum_{k,l \geq 0} \sum_{1 \leq i,j \leq n} A_{n,i,j} \binom{k + n - i}{n} \binom{l + n - j}{n} s^k t^l
$$
as the generating function for all two-sided barred permutations for $S_n$. But we already know there are $\binom{k l + n - 1}{n}$ two-sided barred permutations of weight $s^k t^l$, so we have

$$
\binom{k l + n - 1}{n} = \sum_{1 \leq i,j \leq n} A_{n,i,j} \binom{k + n - i}{n} \binom{l + n - j}{n},
$$
as desired.

**Symmetry and more**

It is well-known that the Eulerian numbers are *symmetric* for fixed $n$: $A_{n,i} = A_{n,n+1-i}$, or $A_n(t) = t^{n+1} A_n(1/t)$. (If $w$ has $i$ descents it has $n - 1 - i$ ascents.) Another property exhibited by the Eulerian polynomials is *unimodality*, i.e., that the Eulerian numbers in a given row increase up to a certain maximum and then decrease:

$$
A_{n,1} \leq A_{n,2} \leq \cdots \leq A_{n,[n/2]} \geq \cdots \geq A_{n,n-1} \geq A_{n,n}.
$$

This property is familiar to most good distributions, where the bulk of the mass is in the middle and the rest is spread out symmetrically. The canonical example is the binomial distribution, $\binom{n}{k}$ with fixed $n$. Similarly, the two-sided Eulerian numbers appear to increase in the direction of the middle of the main diagonal, with a maximum at $A_{n,[n/2],[n/2]}$. However, things are somewhat delicate. For example, when $i,j < n/2$ it is not always true that $A_{n,i+1,j} \geq A_{n,i,j} \leq A_{n,i,j+1}$, i.e., moving toward the diagonal can sometimes lead to a smaller number. The first example of this occurs for $n = 8$. See Table 2.
γ-nonnegativity

Both the symmetry and unimodality of the Eulerian polynomial follow from a stronger result, first proved in 1970, via an elegant combinatorial argument, by Foata and Schützenberger [6, Théorème 5.6]. See also Carlitz and Scoville [3].

**Theorem 3** For $n \geq 1$, there exist nonnegative integers $\gamma_{n,i}$, $i = 1, \ldots, \lceil n/2 \rceil$, such that

$$A_n(t) = \sum_{1 \leq i \leq \lfloor n/2 \rfloor} \gamma_{n,i} t^i (1 + t)^{n+1-2i}.$$ 

For example, when $n = 4, 5$ we have:

$$A_4(t) = t + 11t^2 + 11t^3 + t^4$$
$$= t(1 + t)^3 + 8t^2(1 + t),$$
$$A_5(t) = t + 26t^2 + 66t^3 + 26t^4 + t^5$$
$$= t(1 + t)^4 + 22t^2(1 + t)^2 + 16t^3.$$ 

In other words, the polynomials $A_n(t)$ can be expressed as a positive sum of symmetric binomial terms with the same center of symmetry. Since the binomial distribution is symmetric and unimodal, so is the Eulerian. One might say the Eulerian distribution is “super binomial”.

Foata and Schützenberger’s proof goes something like this. First, define an action on permutations we call “valley hopping” as follows. We visualize permutations as arrays of balls again, but now we connect the dots to form a kind of mountain range. Some balls sit at peaks, others sit in valleys, and the rest are somewhere in between. If a ball is not at a peak or in a valley, it is free to jump straight across a valley to the nearest point on a slope at the same height.

Valley hopping naturally partitions $S_n$ into equivalence classes according to whether one permutation can be obtained from another through a sequence of hops. For example, the permutation $w = 863247159$ would be drawn as...
follows:

There are $2^6$ permutations in its equivalence class, formed by choosing which of the six free balls will be on the left sides of their respective valleys and which will be on the right. Notice that when a free ball is on the right side of a valley, it is not in a descent position, while if it is on the left side of a valley, it is in a descent position. Moreover, this property holds true regardless of the positions of the other free balls.

As for the non-free balls, we know that peaks are always in descent positions while valleys are never in descent positions. If a permutation has $i - 1$ peaks, then it must have $i$ valleys, and the remaining $n + 1 - 2i$ balls are free. Thus, we can conclude that for a fixed $w \in S_n$ with $i - 1$ peaks,

$$\sum_{u \sim w} t^{\text{des}(u)+1} = t^i (1 + t)^{n+1-2i}.$$  

For example, the equivalence class for $w = 863247159$ would contribute

$$t^2 (1 + t)^6.$$  

Since the union of all equivalence classes is $S_n$, we see that the Eulerian polynomial is a sum of terms of the form $t^i (1 + t)^{n+1-2i}$, proving Theorem 3. Moreover, the coefficient $\gamma_{n,i}$ equals the number of distinct equivalence classes with $i - 1$ peaks.
Gessel’s conjecture

For the two-sided Eulerian polynomials, Gessel has conjectured a generalization of Theorem 3. But before discussing the conjecture, we first remark on some symmetries seen in Table 2.

Observation 1 (Symmetries) We have the following identities for any $n, i, j$.

1. $A_{n,i,j} = A_{n,j,i}$, or $A_n(s, t) = A_n(t, s)$,

2. $A_{n,i,j} = A_{n,n+1-i,n+1-j}$, or $A_n(s, t) = (st)^{n+1}A_n(1/s, 1/t)$, and

3. $A_{n,i,j} = A_{n,n+1-j,n+1-i}$, or $A_n(s, t) = (st)^{n+1}A_n(1/t, 1/s)$.

It is a fun exercise to come up with combinatorial arguments to verify symmetries 1 and 2. (Hint: how does flipping a permutation array upside-down affect descents and inverse descents?) Symmetry 3 follows from 1 and 2.

It is possible to show that any polynomial in two variables (of degree $n$ in each) satisfying symmetries 1 and 2 can be written uniquely in the basis

$$\{(st)^i(s+t)^j(1+st)^{n+1-j-2i}\}_{0 \leq j+2i \leq n+1}.$$  

Thus, the two-sided Eulerian polynomials can be expressed in this basis, and in unpublished correspondence, Gessel has conjectured that such an expression is nonnegative.

Conjecture 1 (Gessel’s conjecture) For $n \geq 1$, there exist nonnegative integers $\gamma_{n,i,j}$, $0 \leq i - 1, j, j + 2i \leq n + 1$, such that

$$A_n(s, t) = \sum_{i,j} \gamma_{n,i,j}(st)^i(s+t)^j(1+st)^{n+1-j-2i}.$$  

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For example, when \( n = 4, 5 \), we have

\[
A_4(s, t) = st + 10(st)^2 + 10(st)^3 + (st)^4 + s^2t^3 + s^3t^2
\]

\[
= st(1 + st)^3 + 7(st)^2(1 + st) + (st)^2(s + t)
\]

\[
A_5(s, t) = st + 20(st)^2 + 54(st)^3 + 20(st)^4 + (st)^5 + 6s^2t^3 + 6s^3t^2 + 6s^4t^3 + 6s^4t^3
\]

\[
= st(1 + st)^4 + 16(st)^2(1 + st)^2 + 16(st)^3 + 6(st)^2(s + t)(1 + st)
\]

In terms of the arrays \([A_{n,i,j}]\), we see:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 10 & 1 & 0 \\
0 & 1 & 10 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
+ 7 \cdot
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 10 & 1 & 0 \\
0 & 1 & 10 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 20 & 6 & 0 & 0 \\
0 & 6 & 54 & 6 & 0 \\
0 & 0 & 6 & 20 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
+ 16 \cdot
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 20 & 6 & 0 & 0 \\
0 & 6 & 54 & 6 & 0 \\
0 & 0 & 6 & 20 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

It is possible that one might use a “manipulatorics” approach to prove Conjecture 1. (Perhaps an inductive proof using the recurrence in Corollary 3?) However, a more satisfying proof might be one that generalizes Foata and Schützenberger’s proof of Theorem 3.

If we look at both descents and inverse descents for the valley hopping equivalence class of \( w = 12 \cdots n \) (i.e., the class with no peaks), we get a
distribution of $st(1 + st)^{n-1}$. This is encouraging, but for the class of $w = 863247159$ shown in (6), we get

$$\sum_{u \sim w} s^{{\text{des}(u^{-1})}+1} t^{{\text{des}(u)}+1} = s^3 t^2 (1 + t)^2 (1 + st)^4,$$

which is not symmetric in $s$ and $t$. So valley hopping as done by Foata and Schützenberger does not immediately give us a way to prove Gessel’s conjecture.

How should we partition $S_n$ so that we get groupings whose distribution of descents and inverse descents is given by $(st)^i(s+t)^j(1 + st)^{n+1-j-2i}$?

**Generalization**

We finish by remarking that $S_n$ is an example of a finite reflection group, or Coxeter group. The notion of a descent can be generalized to any Coxeter group, and there is a “Coxeter-Eulerian” polynomial that enjoys many of the same properties of the classical Eulerian polynomial, including an analogue of Theorem 3. Moreover, this polynomial has geometric meaning, as the “$h$-polynomial” of something called the Coxeter complex. See [11, 4].

The two-sided Eulerian polynomial generalizes to Coxeter groups as well, and seems to enjoy many of the same properties of $A_n(s,t)$. In particular, the analogue of Conjecture 1 appears to hold in any finite Coxeter group. It would be interesting to have a general approach to the problem.

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**References**


