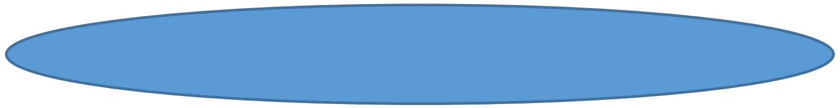
Alice and Bob have lunch

T. Kyle Petersen, PhD
Department of Mathematical Sciences

Alice and Bob, I

Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich



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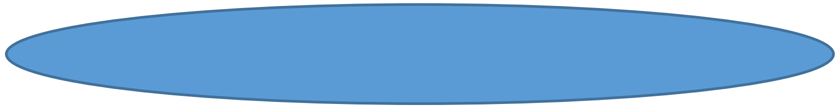

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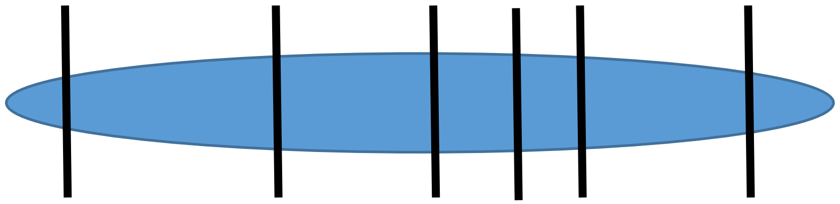

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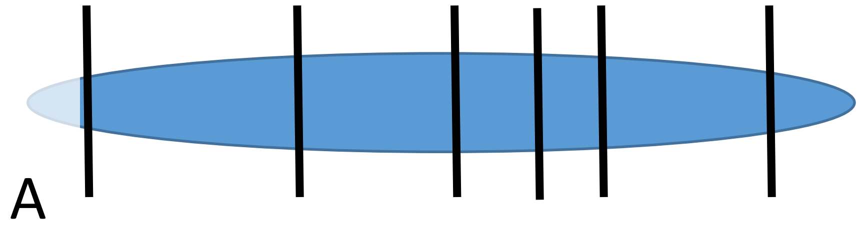

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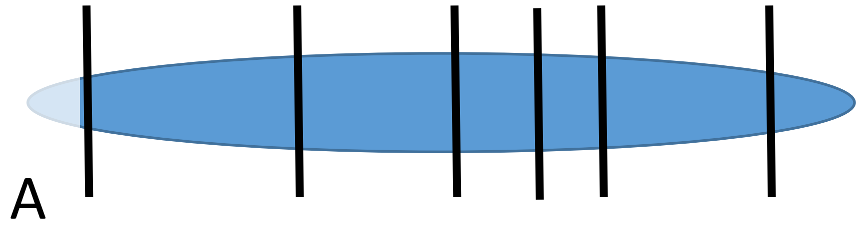

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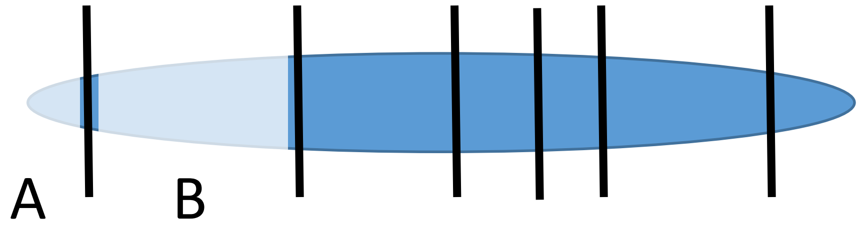

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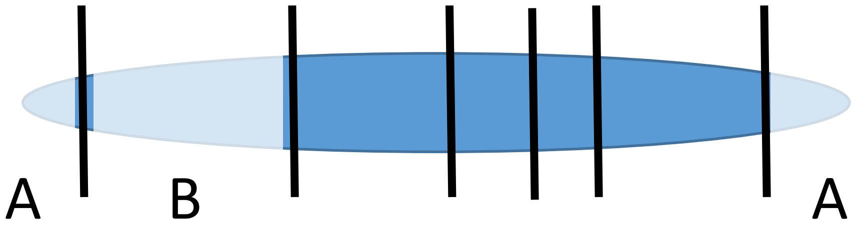

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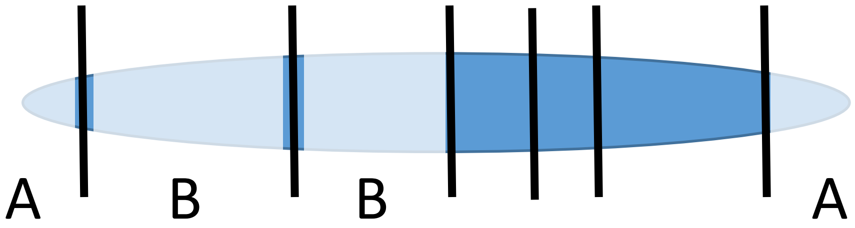

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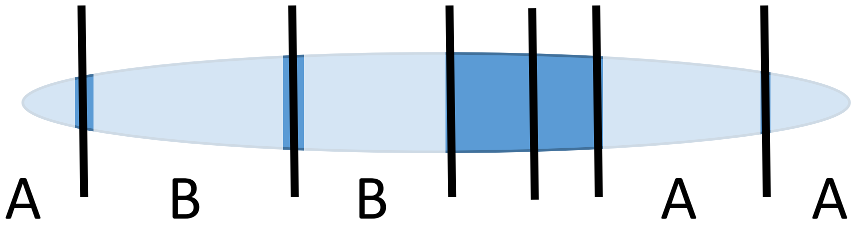

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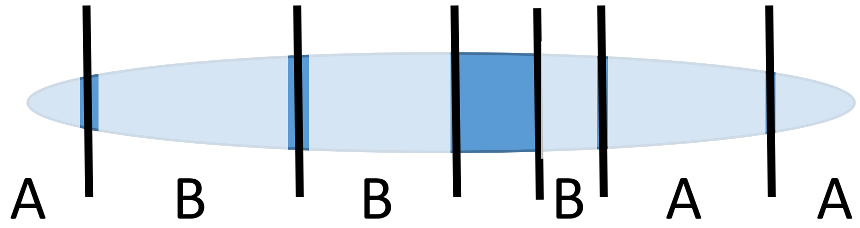

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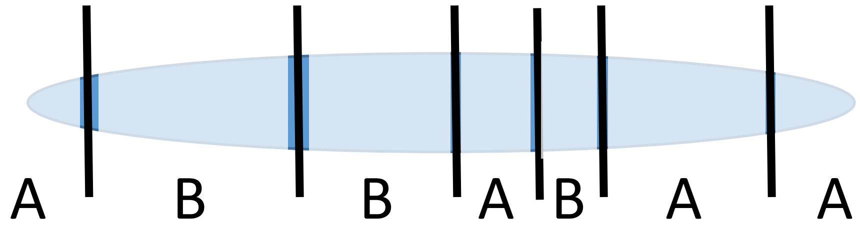

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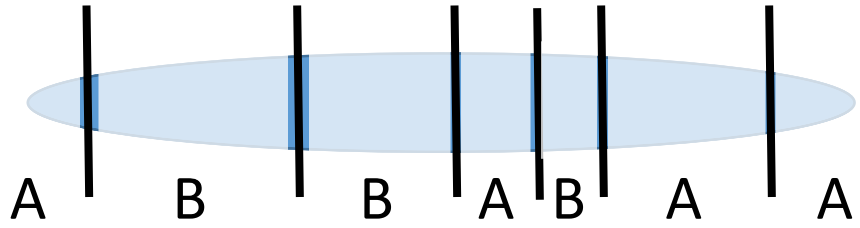

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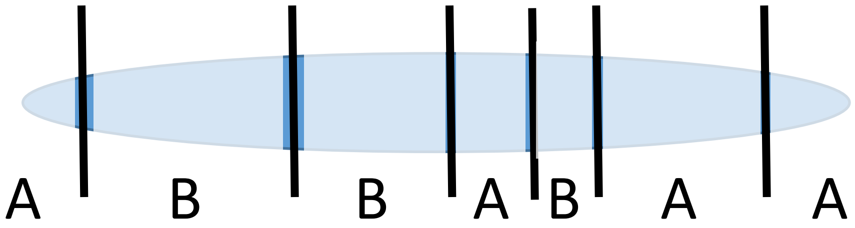

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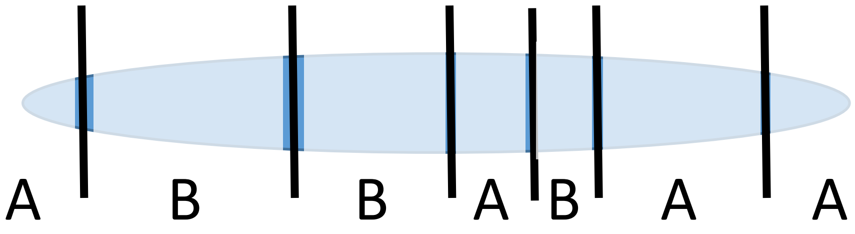

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Alice and Bob, II

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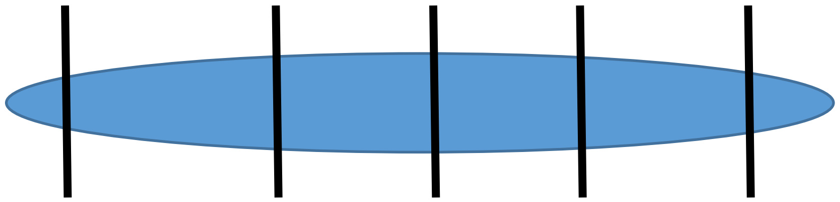

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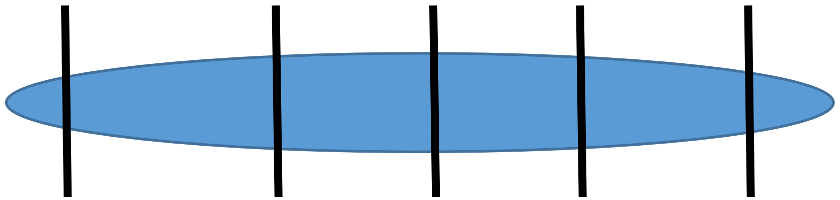

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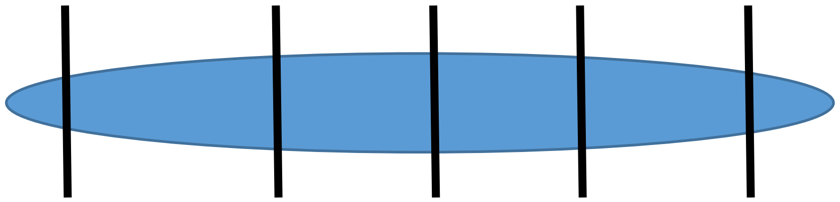

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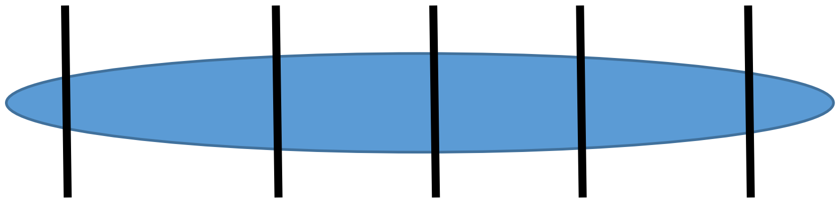

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Alice and Bob, III

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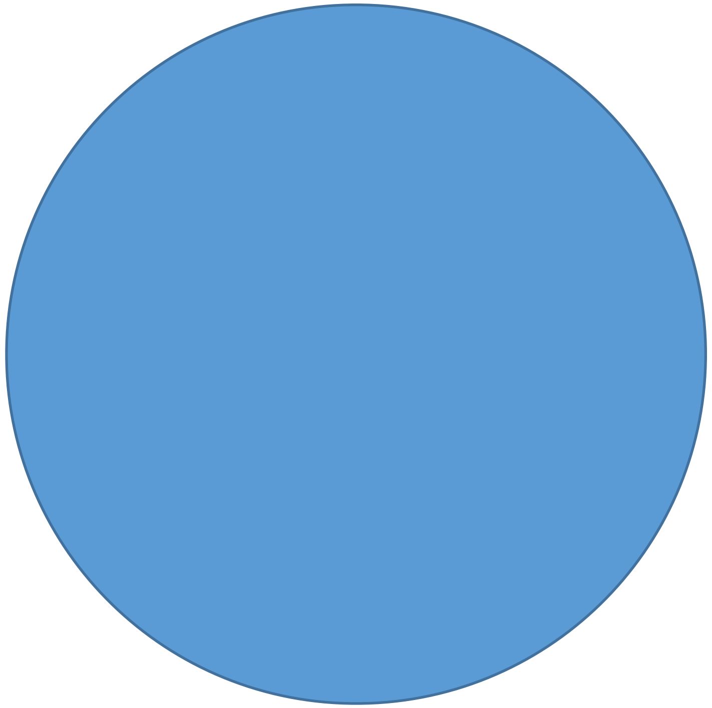

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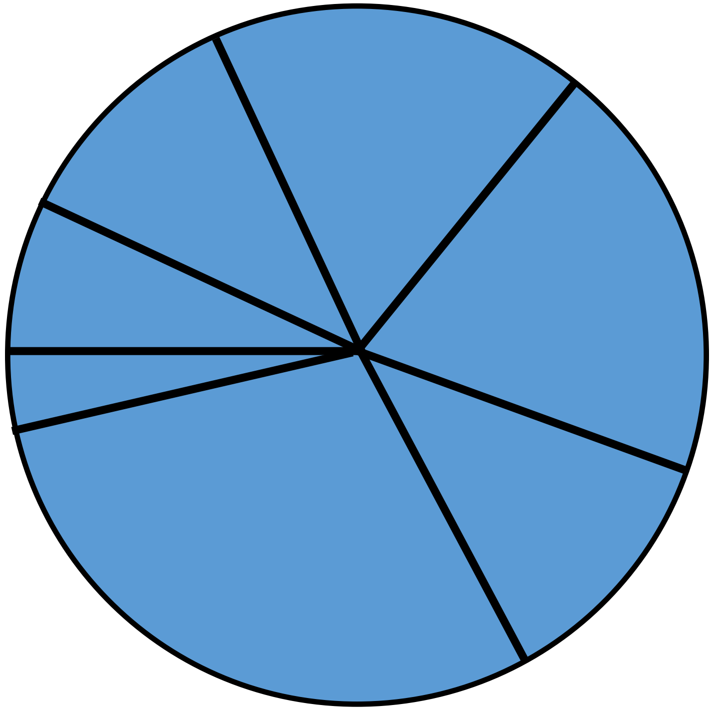

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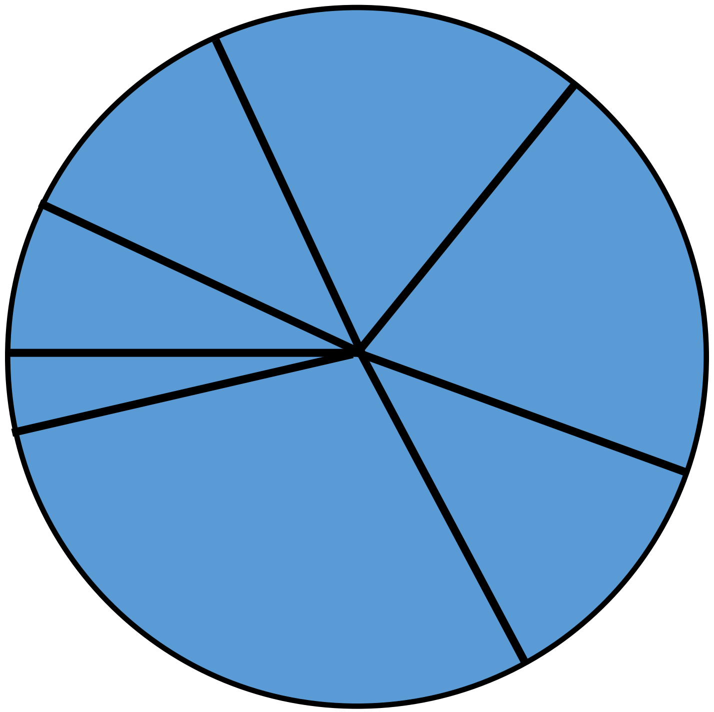

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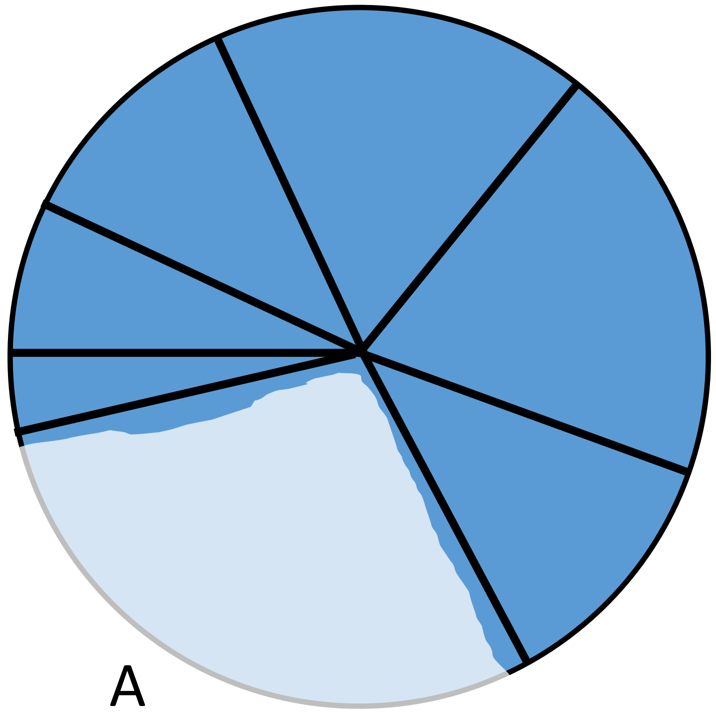

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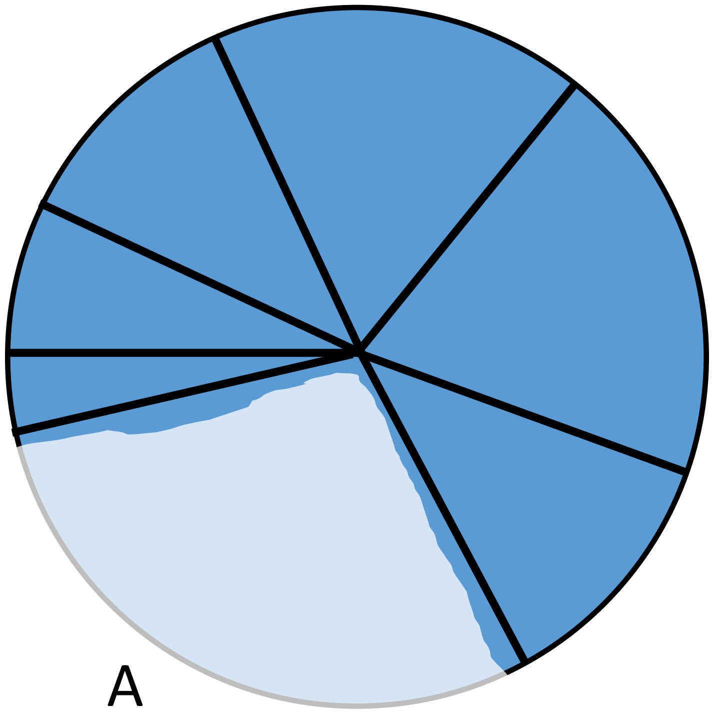

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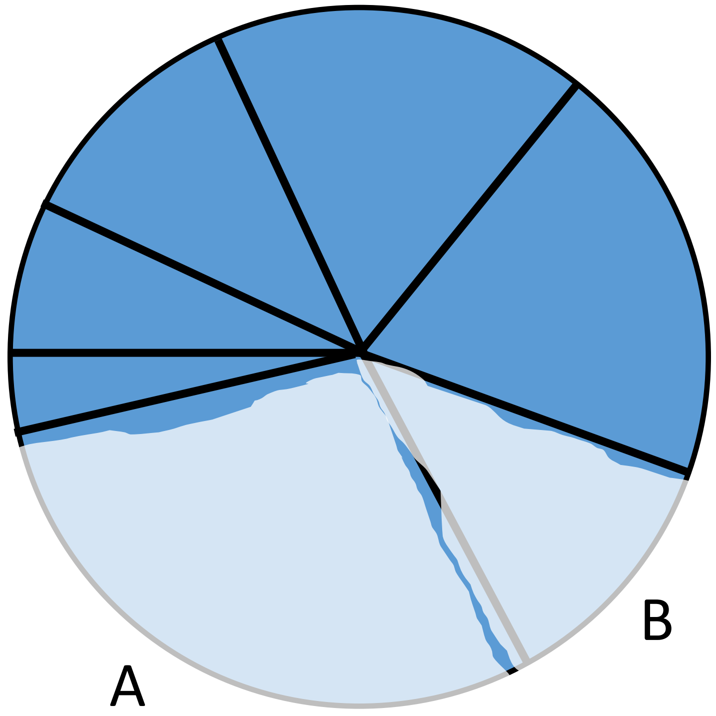

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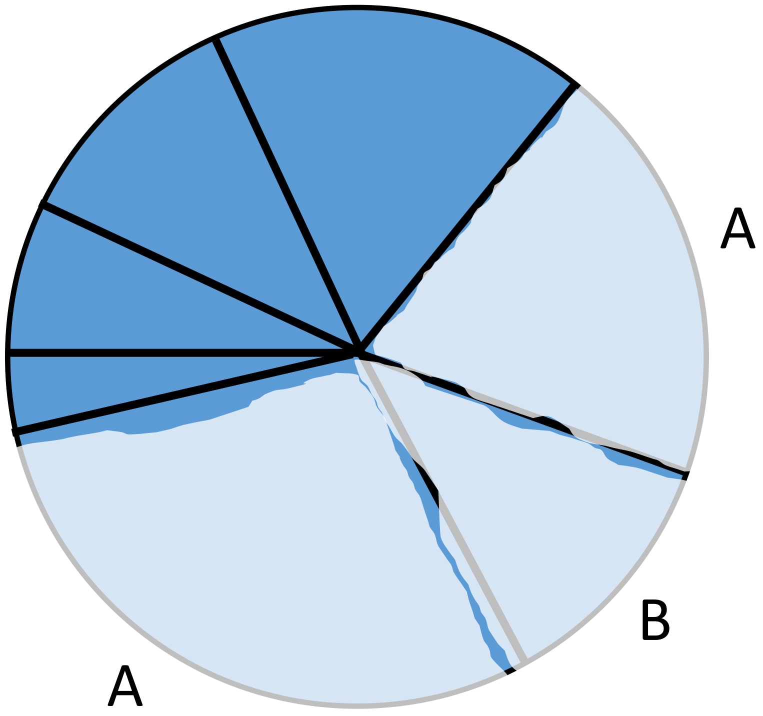

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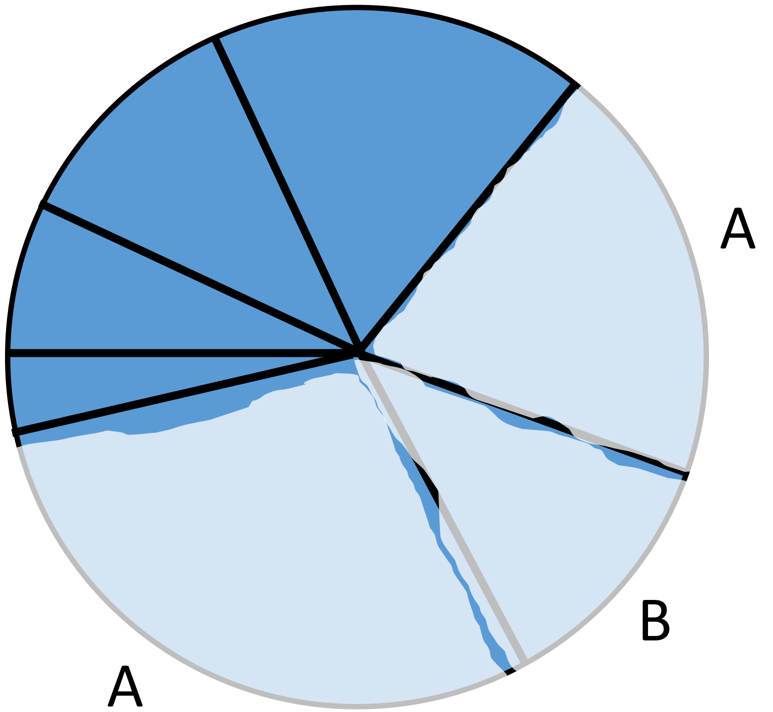

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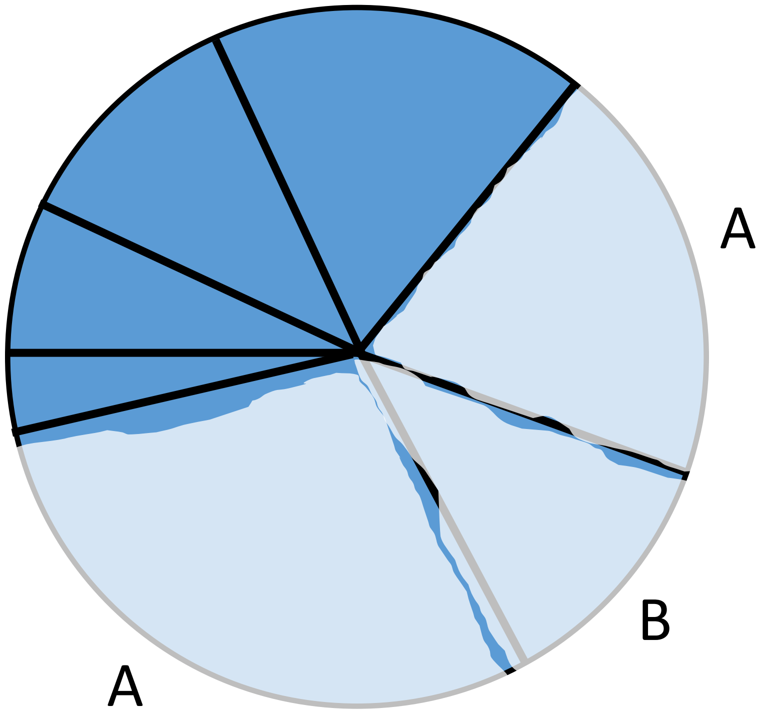

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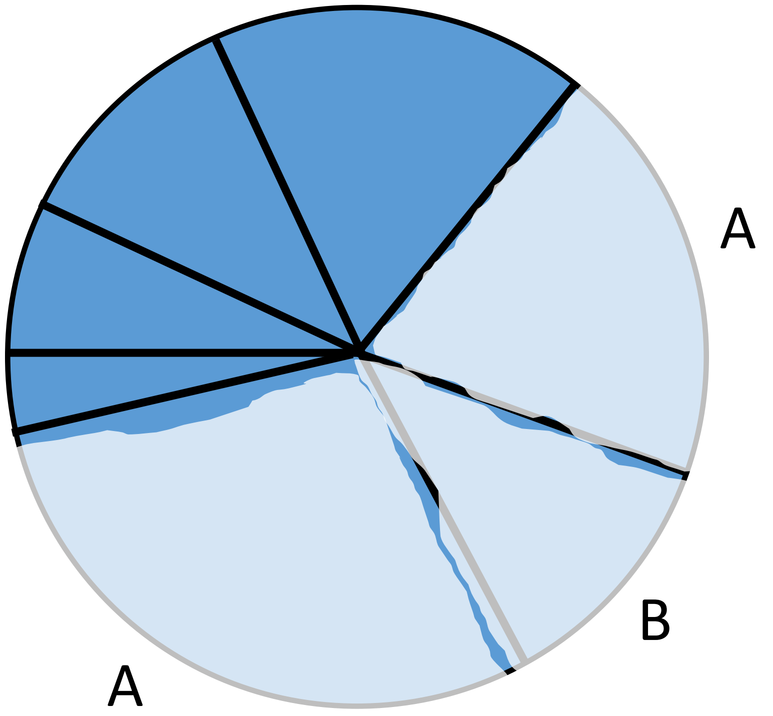

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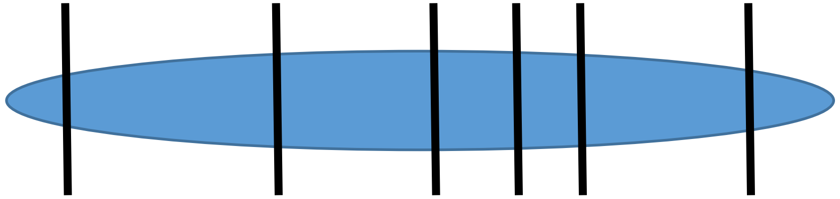
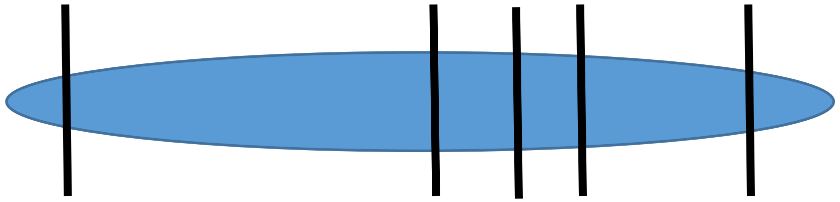
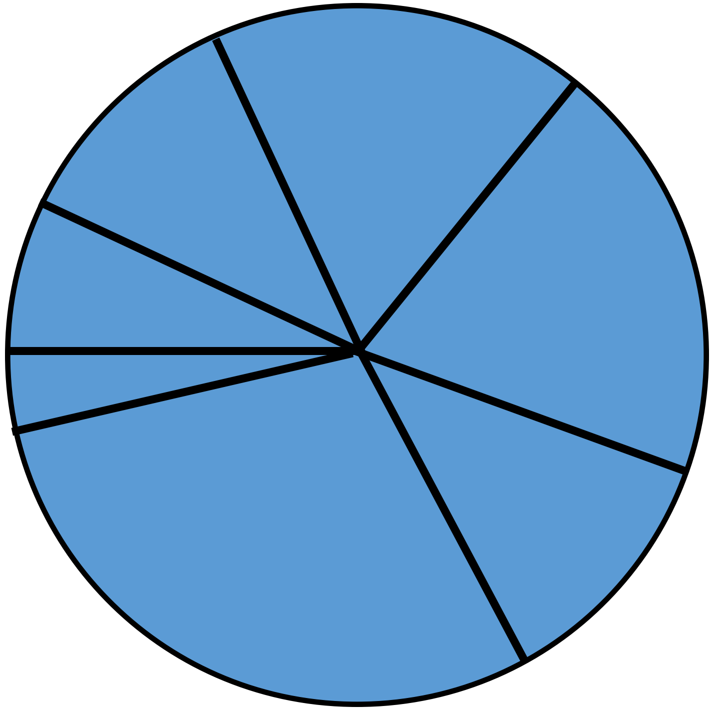
Version I: Submarine Sandwich



Version II: Submarine Sandwich (even number)



Version III: Pizza



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harine Sandwich

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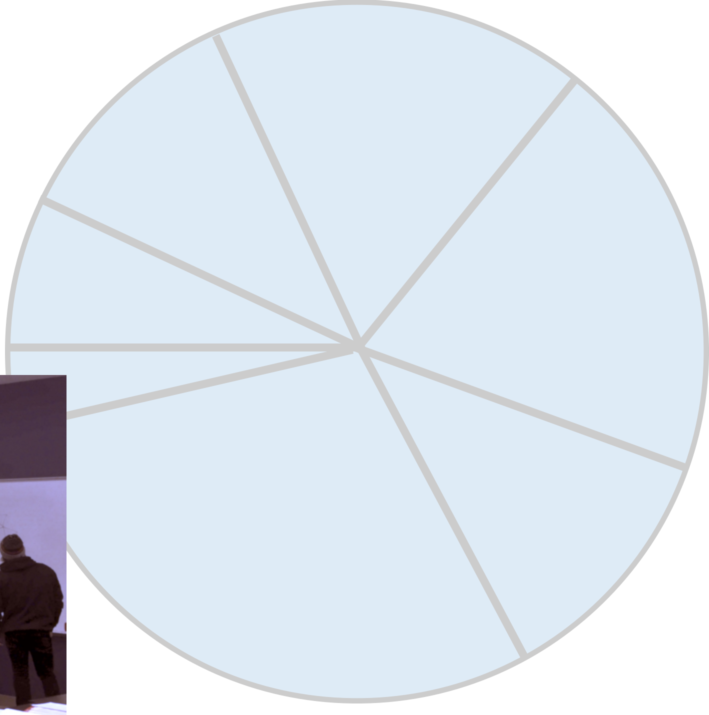

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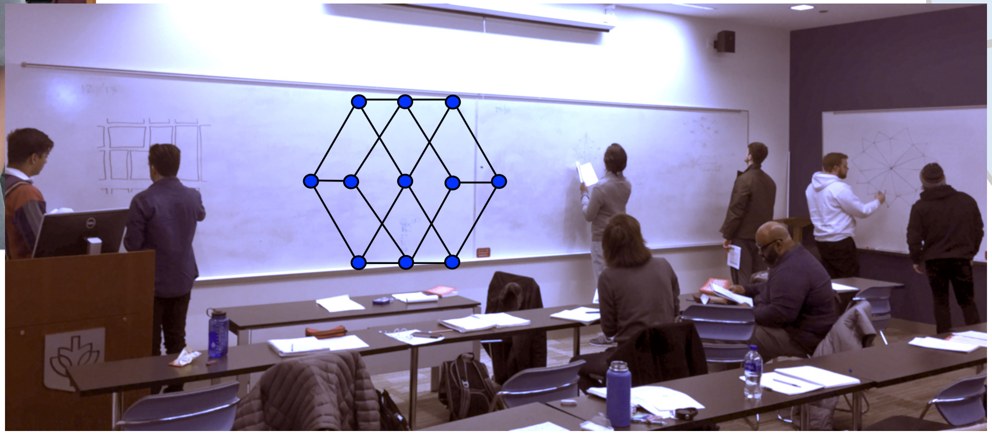

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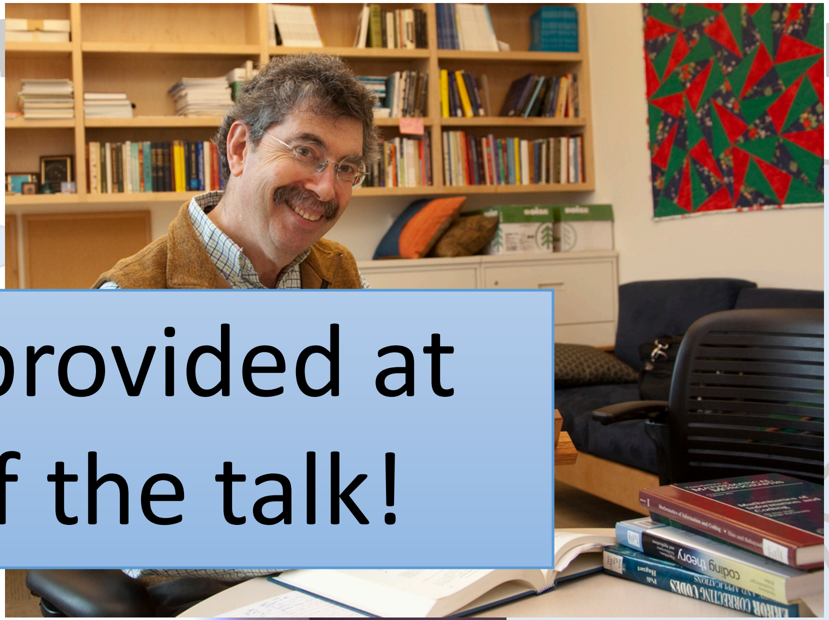

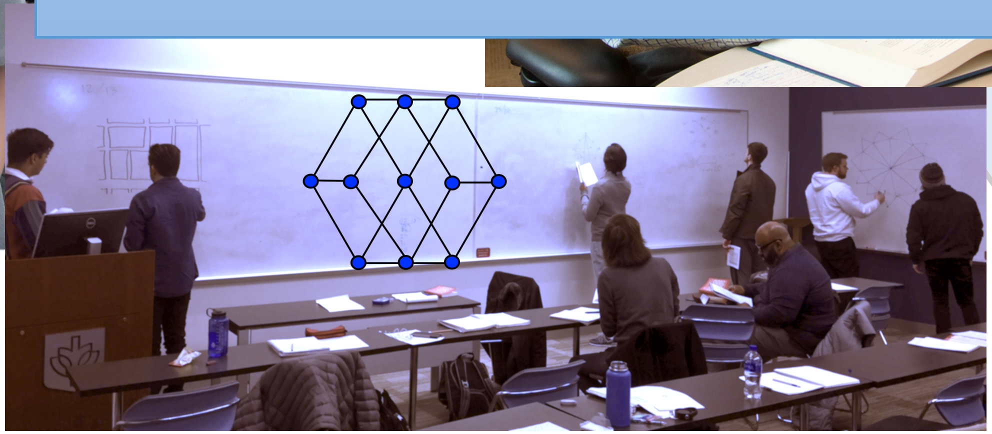

Answers provided at the end of the talk!

What is Mathematics Research?

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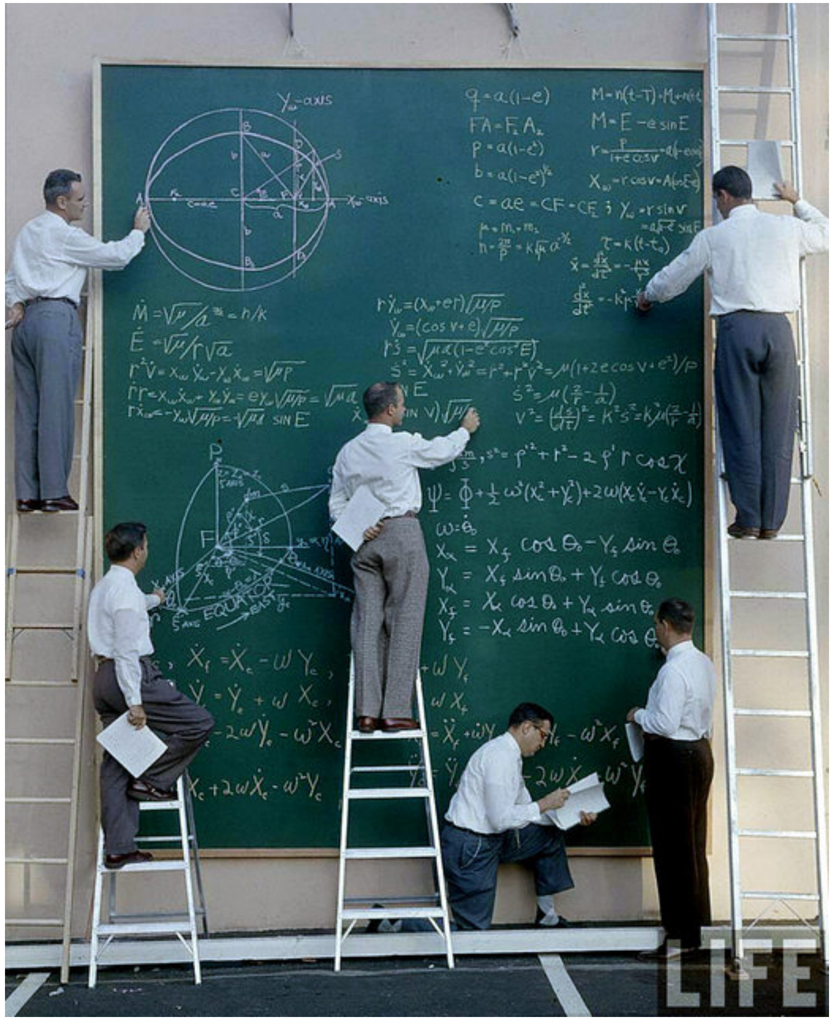

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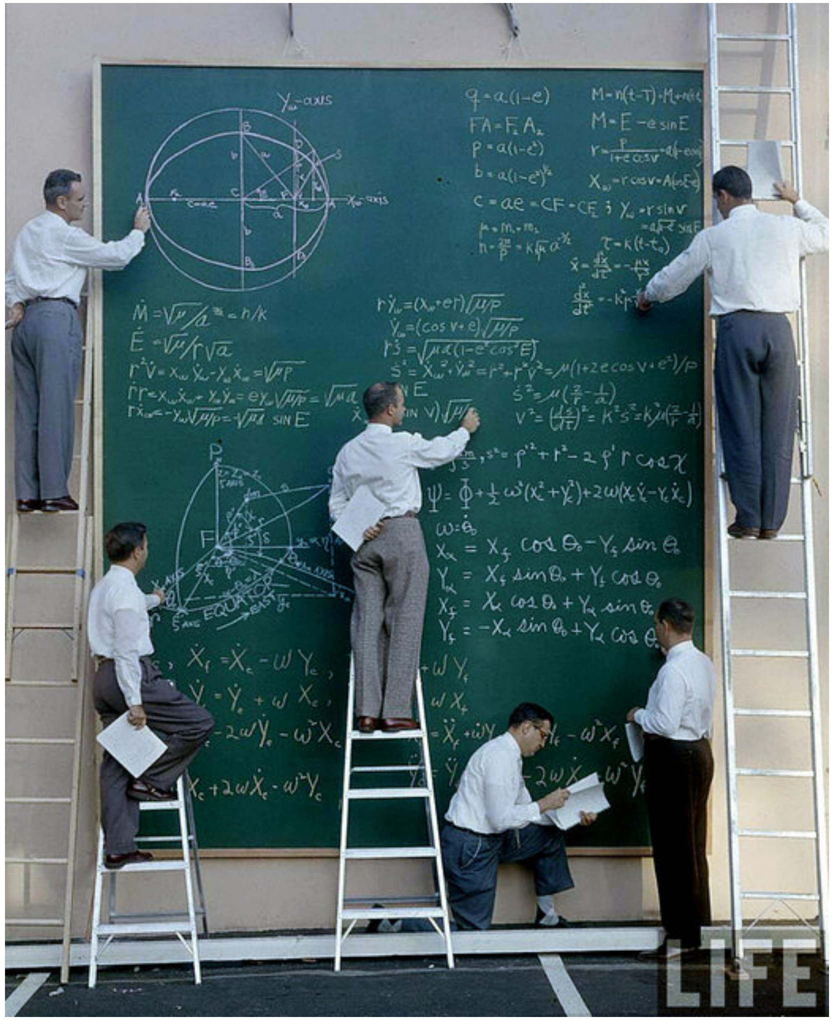

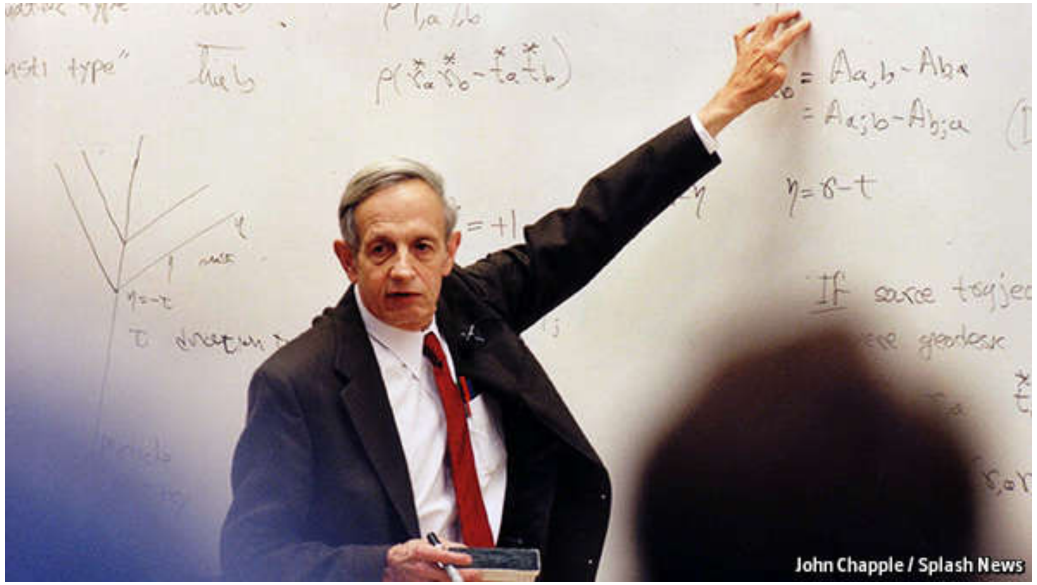

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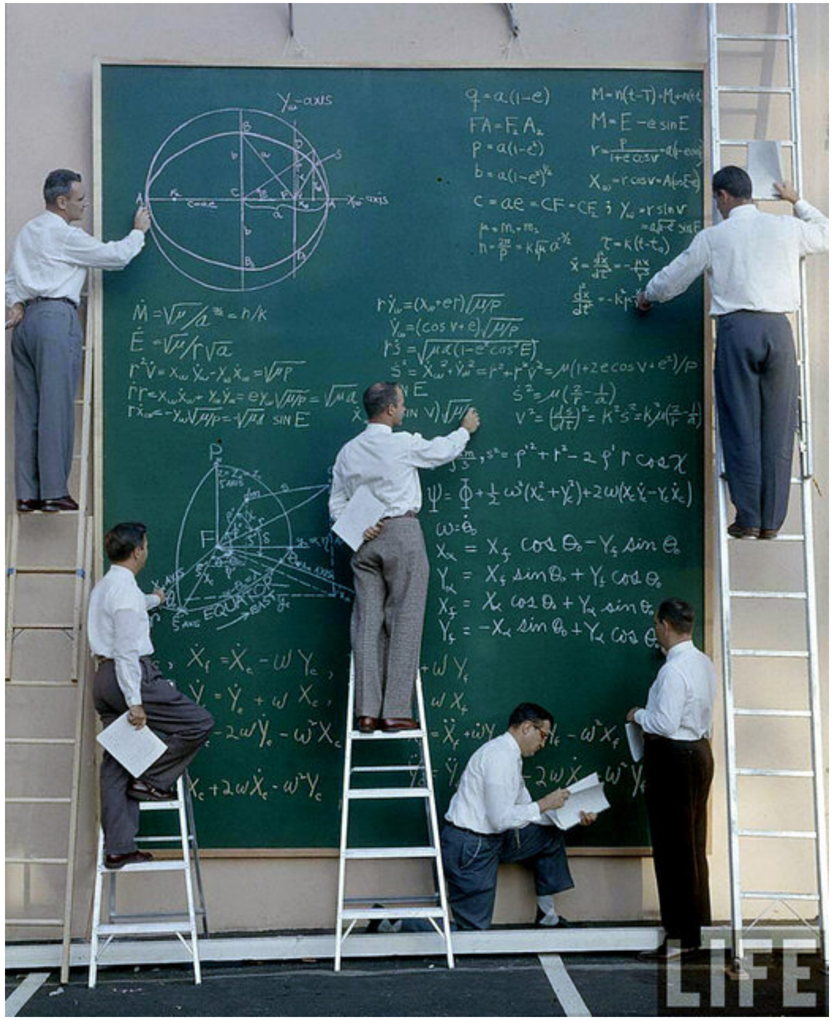

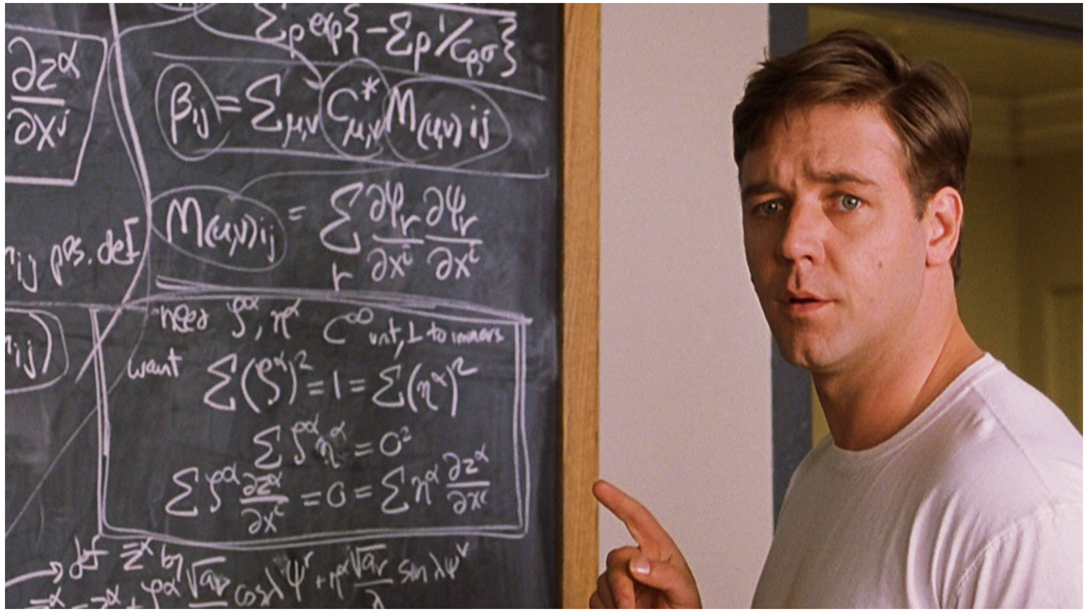

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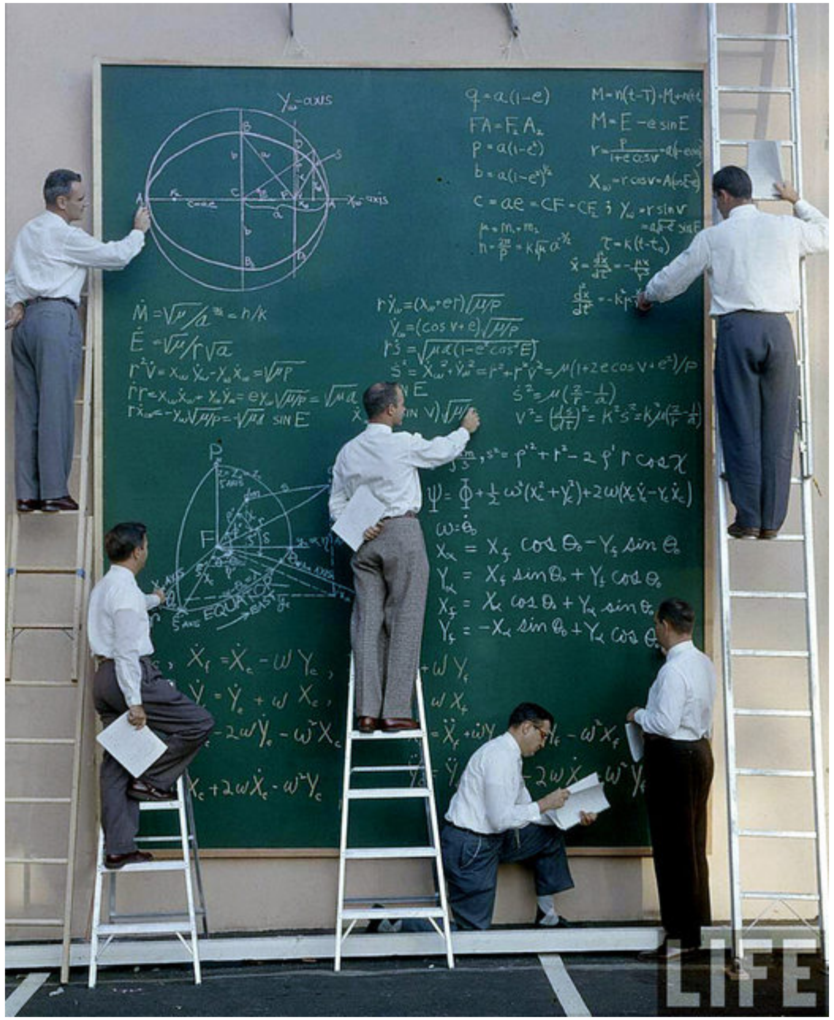
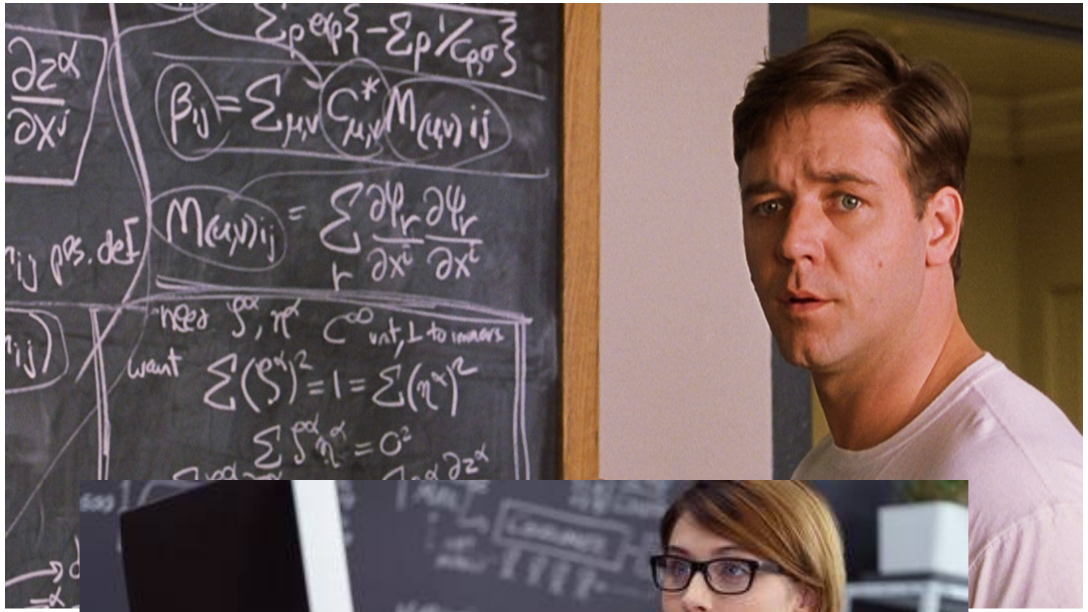
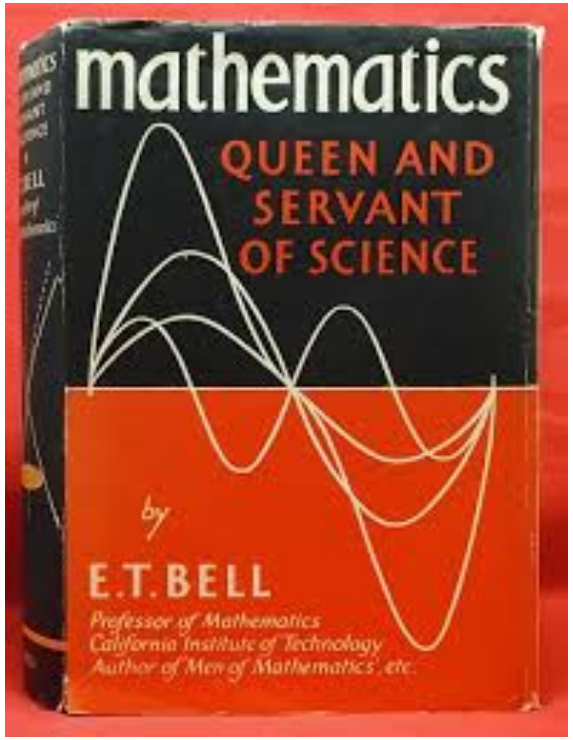




Mathematics, Queen and Servant of Science



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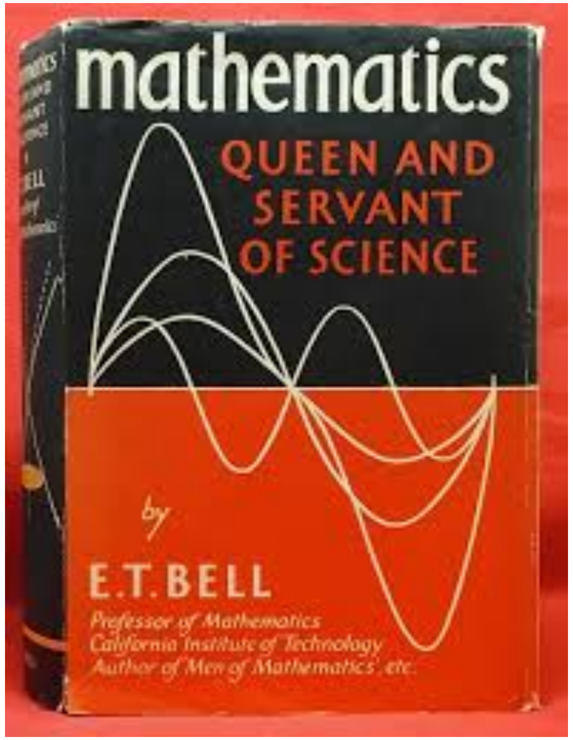

"Pure" versus "Applied" Math

Applied Math

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- starts from a real world problem, e.g., political gerrymandering

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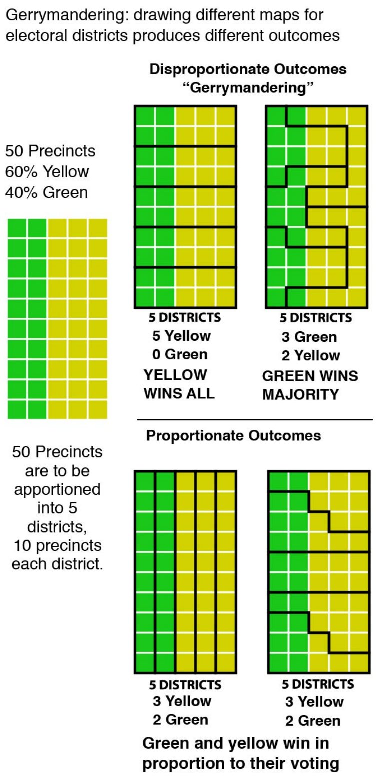
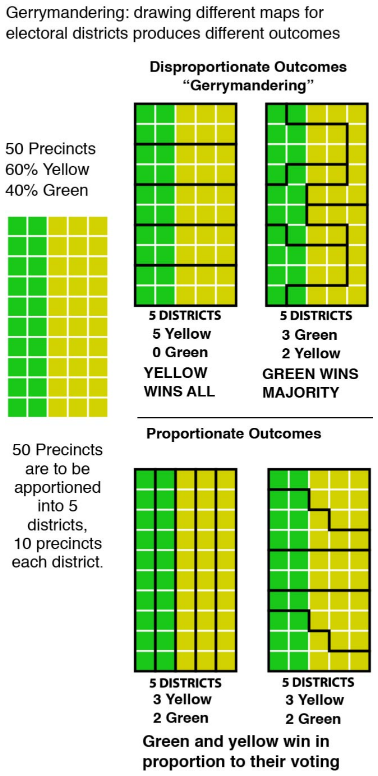


Applied Math

- starts from a real world problem, e.g., political gerrymandering
- constructs a mathematical model
- analyzes that model



$$PP(D) = \frac{4\pi A}{p^2}$$

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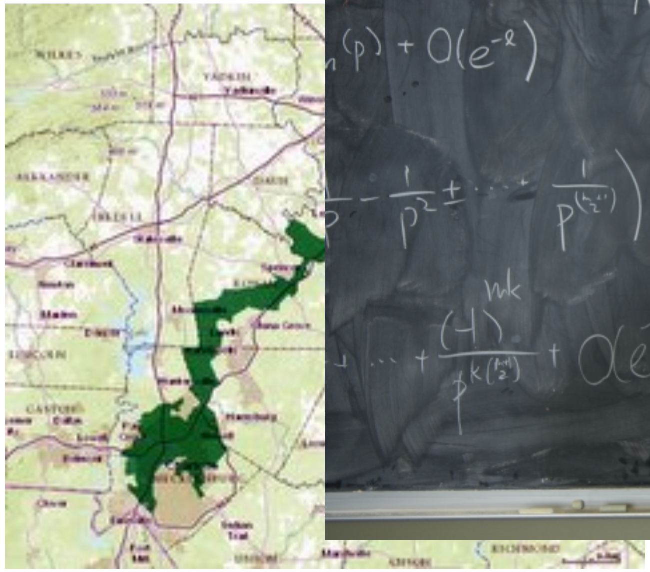

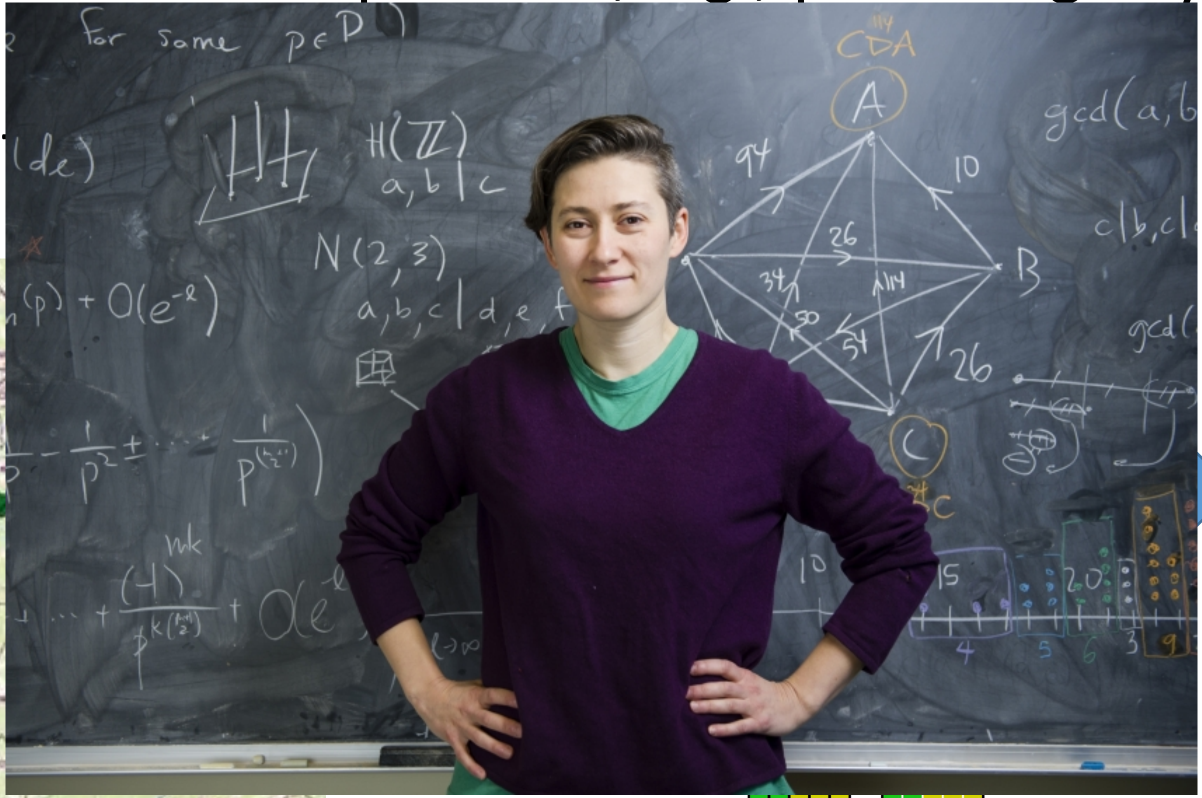

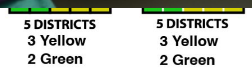

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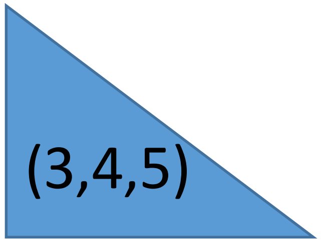

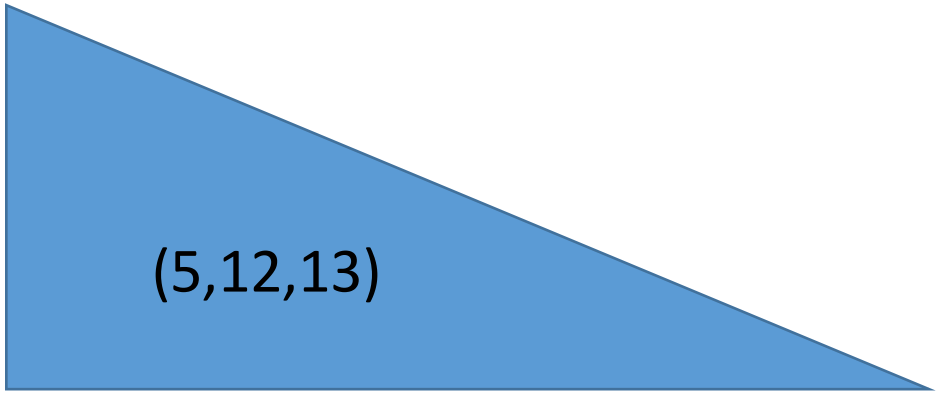

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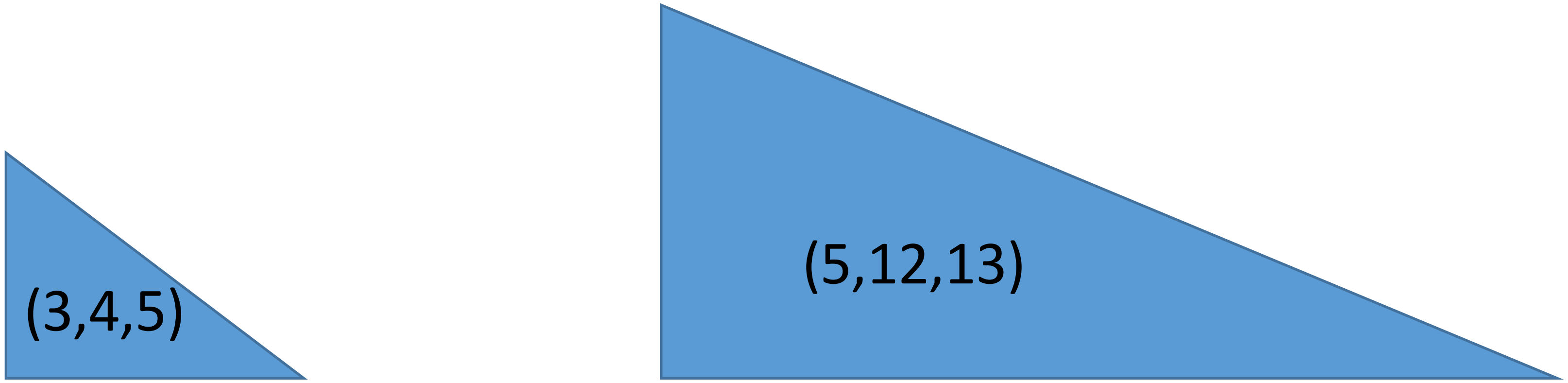
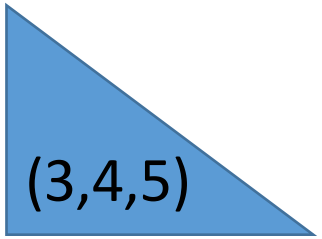
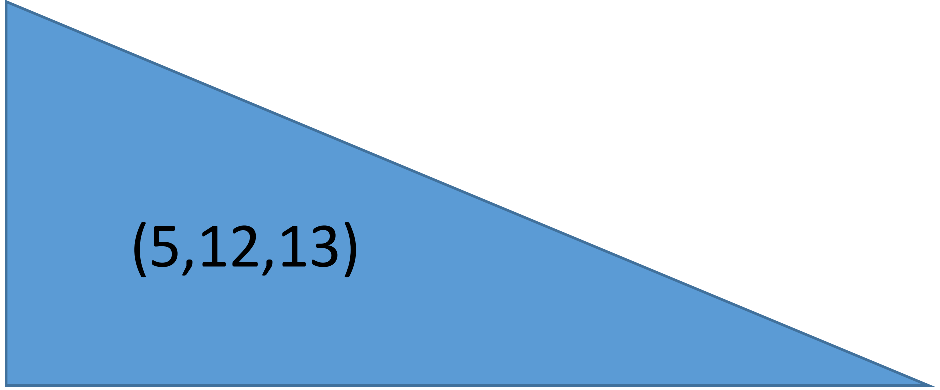


...but not (3,4,7)

Pure Math





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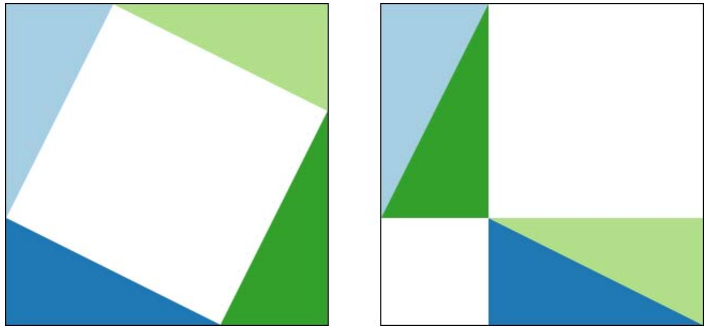

$$a^2 + b^2 = c^2$$

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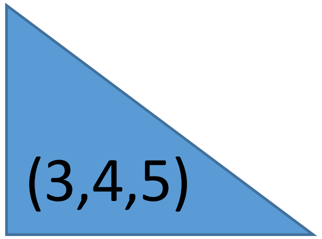

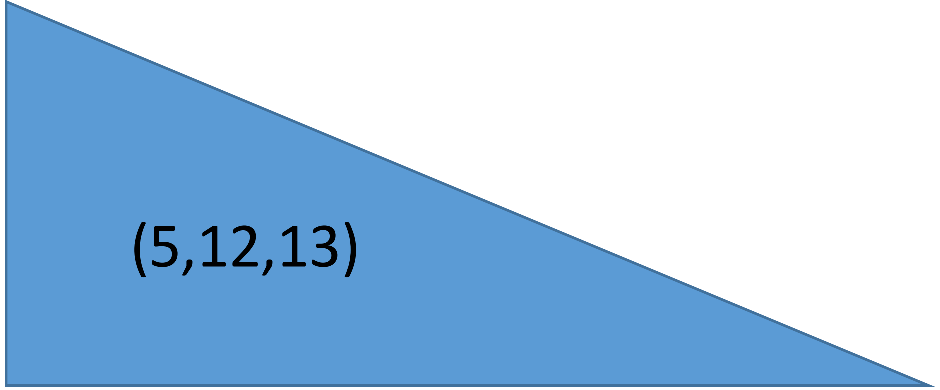

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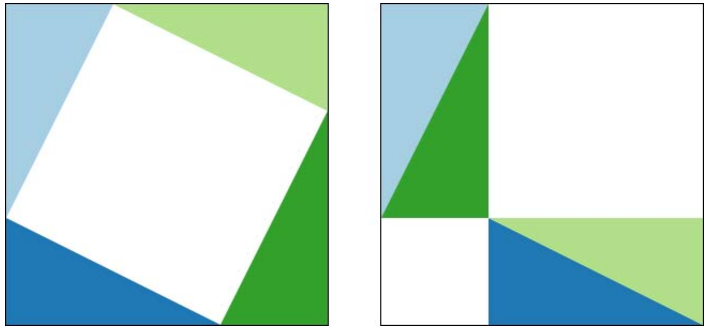
- a focus on justification (why is the answer correct?)



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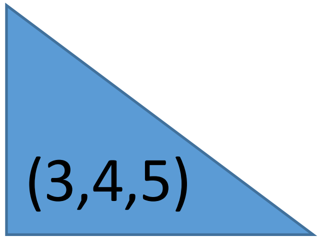

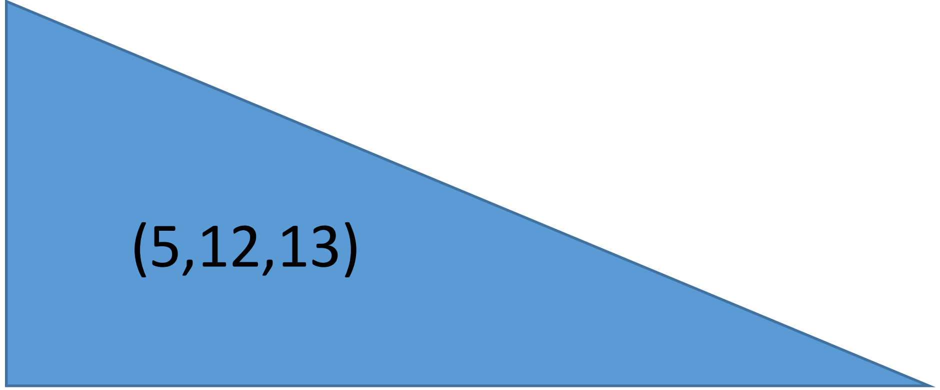

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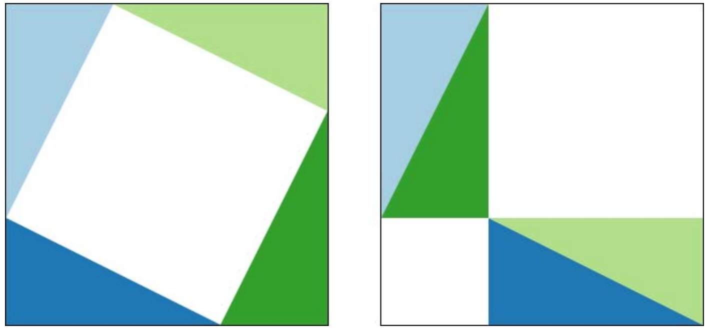
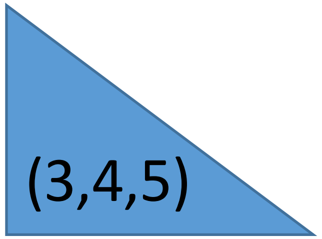
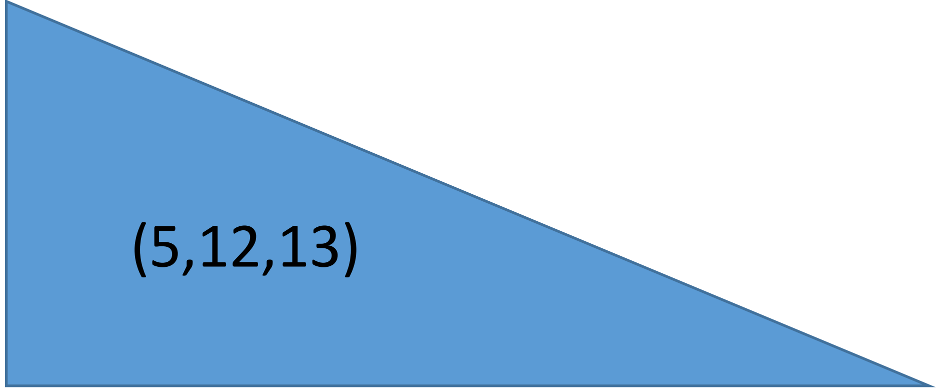
- a focus on justification (why is the answer correct?)
- style matters! Good questions and solutions breed more questions



$$a^2 + b^2 = c^2$$

Pure Math





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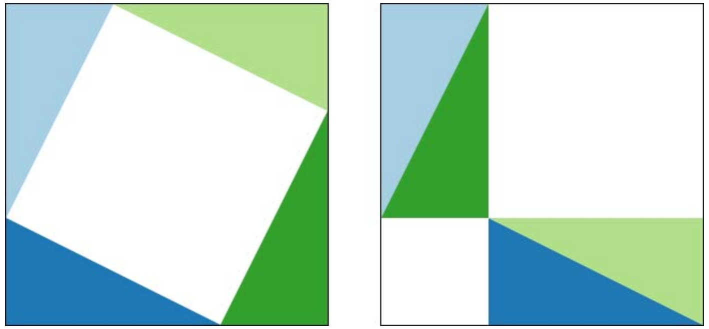
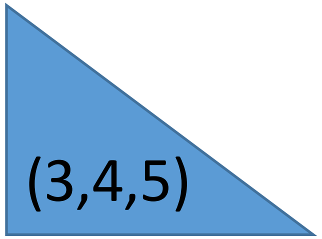


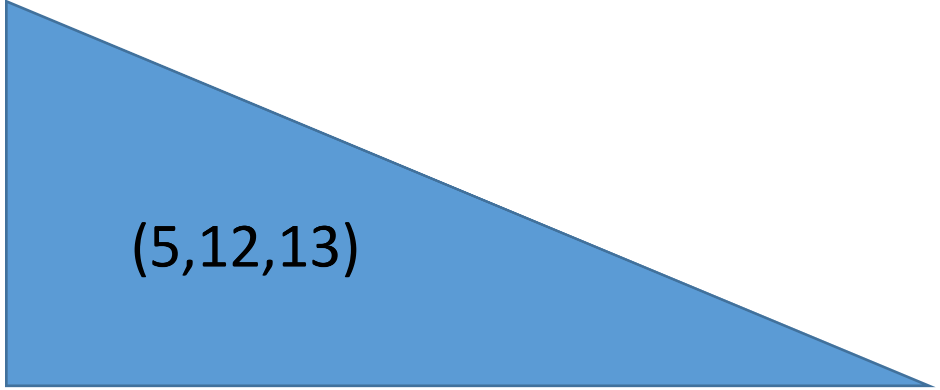
$$a^2 + b^2 = c^2$$

Do these other equations have any whole number solutions?

$$a^3 + b^3 = c^3$$
$$a^4 + b^4 = c^4$$
$$a^5 + b^5 = c^5$$
$$\vdots$$

Pure Math





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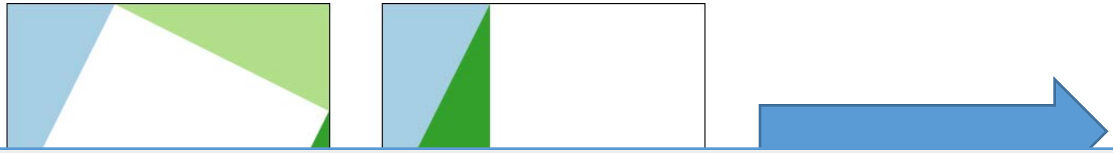

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$$a^3 + b^3 = c^3$$
$$a^4 + b^4 = c^4$$
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$$\vdots$$

Gauss: "Mathematics is queen of the sciences and ***Number Theory*** is the queen of mathematics"

Mathematics, Queen and Servant of Science

G. H. Hardy

Mathematics, Queen and Servant of Science

G. H. Hardy

"The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: **there is no permanent place in the world for ugly mathematics***." –A Mathematician's Apology*

Mathematics, Queen and Servant of Science

G. H. Hardy

“**I have never done anything `useful’.** *No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.” –A Mathematician’s Apology*

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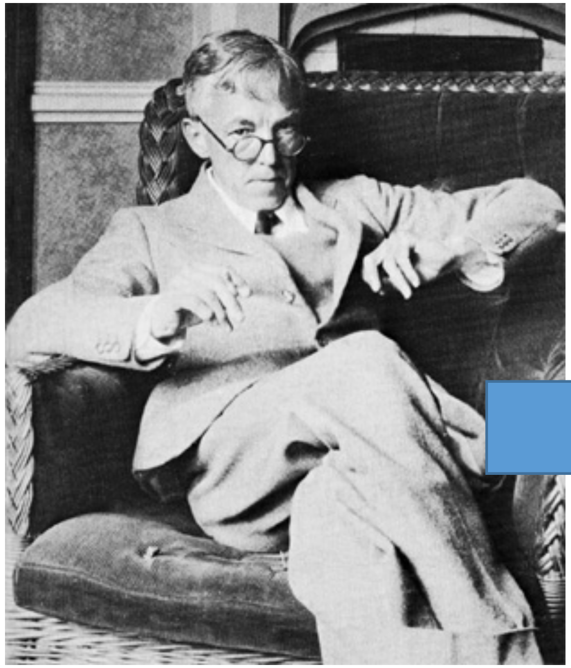

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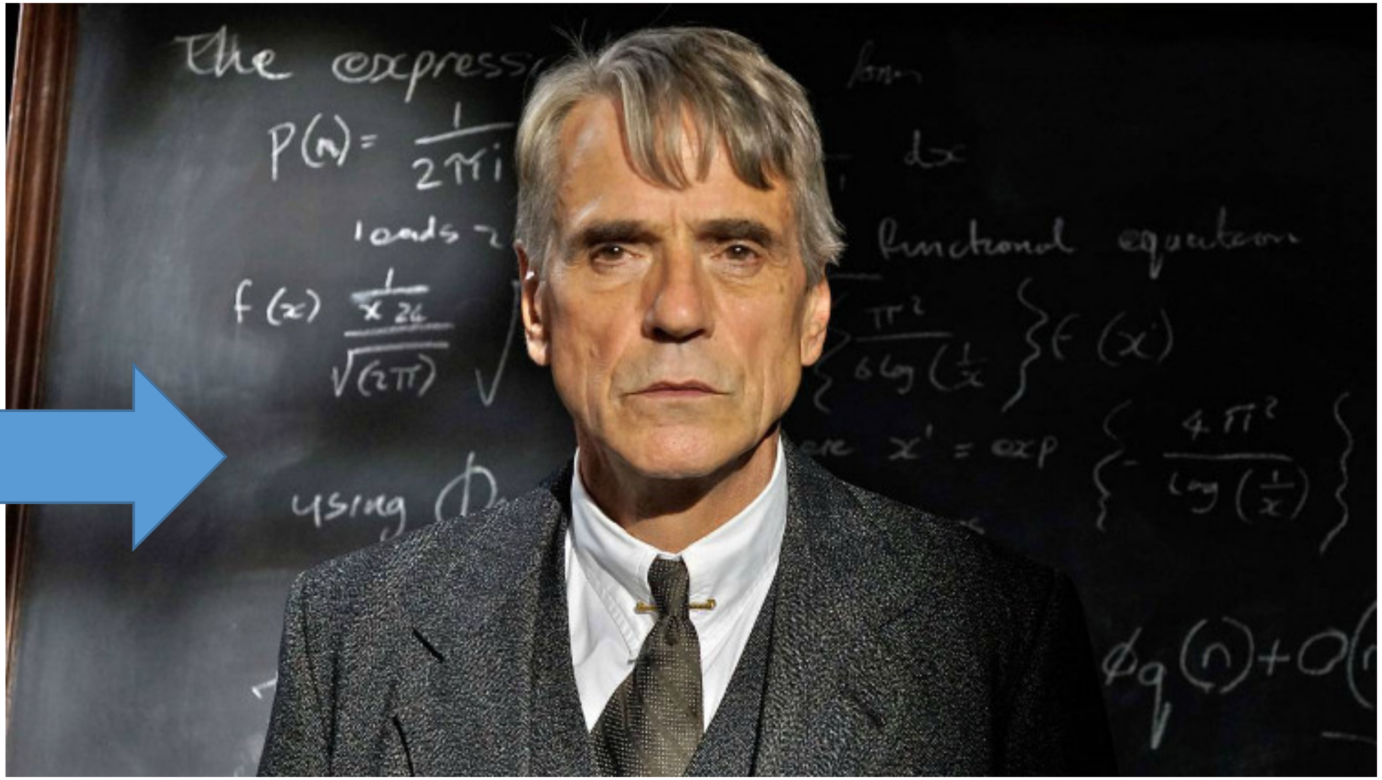


What do I do?

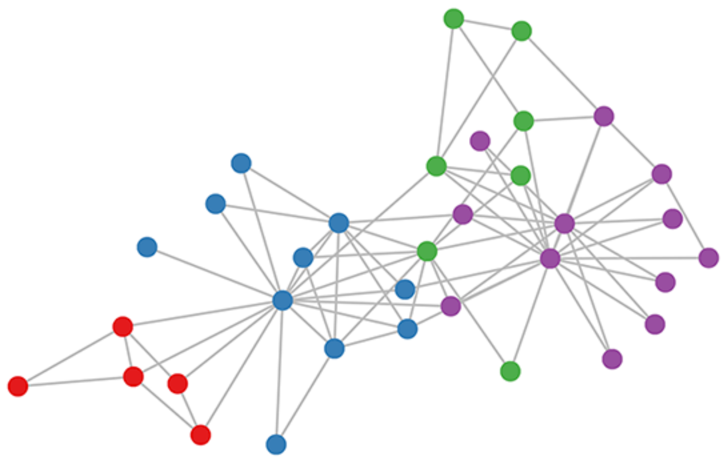
What do I do?

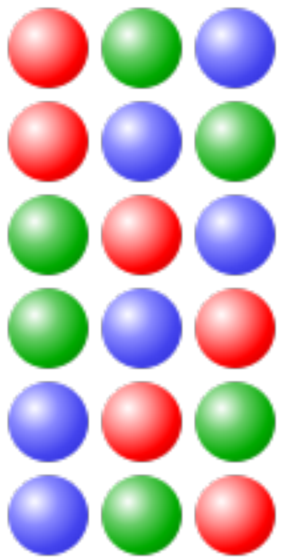
Combinatorics

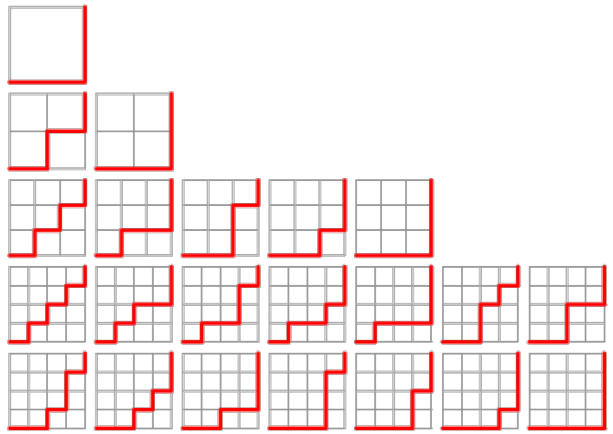
Combinatorics: art and science of counting

Combinatorics: art and science of counting

- Mostly deals in discrete mathematical models (originating in "real-world" applications, or in other parts of math)

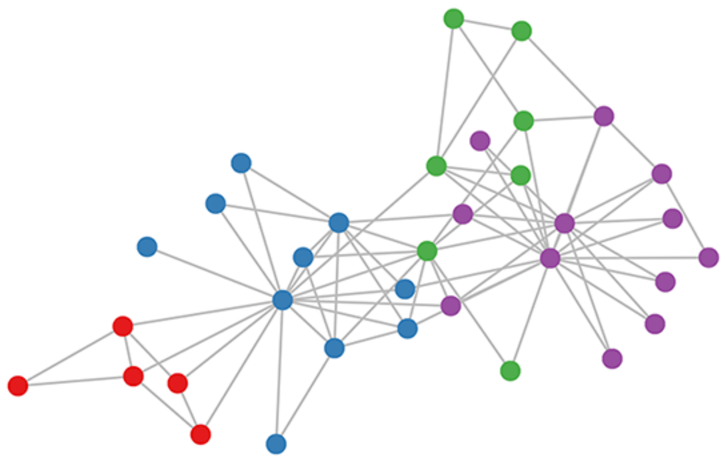


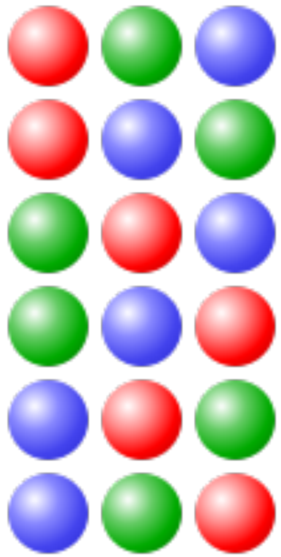


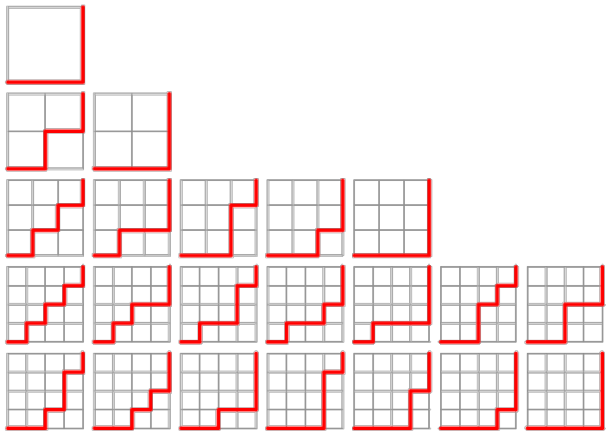


Combinatorics: art and science of counting

- Mostly deals in discrete mathematical models (originating in “real-world” applications, or in other parts of math)
- Search for patterns, count things, make pictures!







Combinatorics: art and science of counting

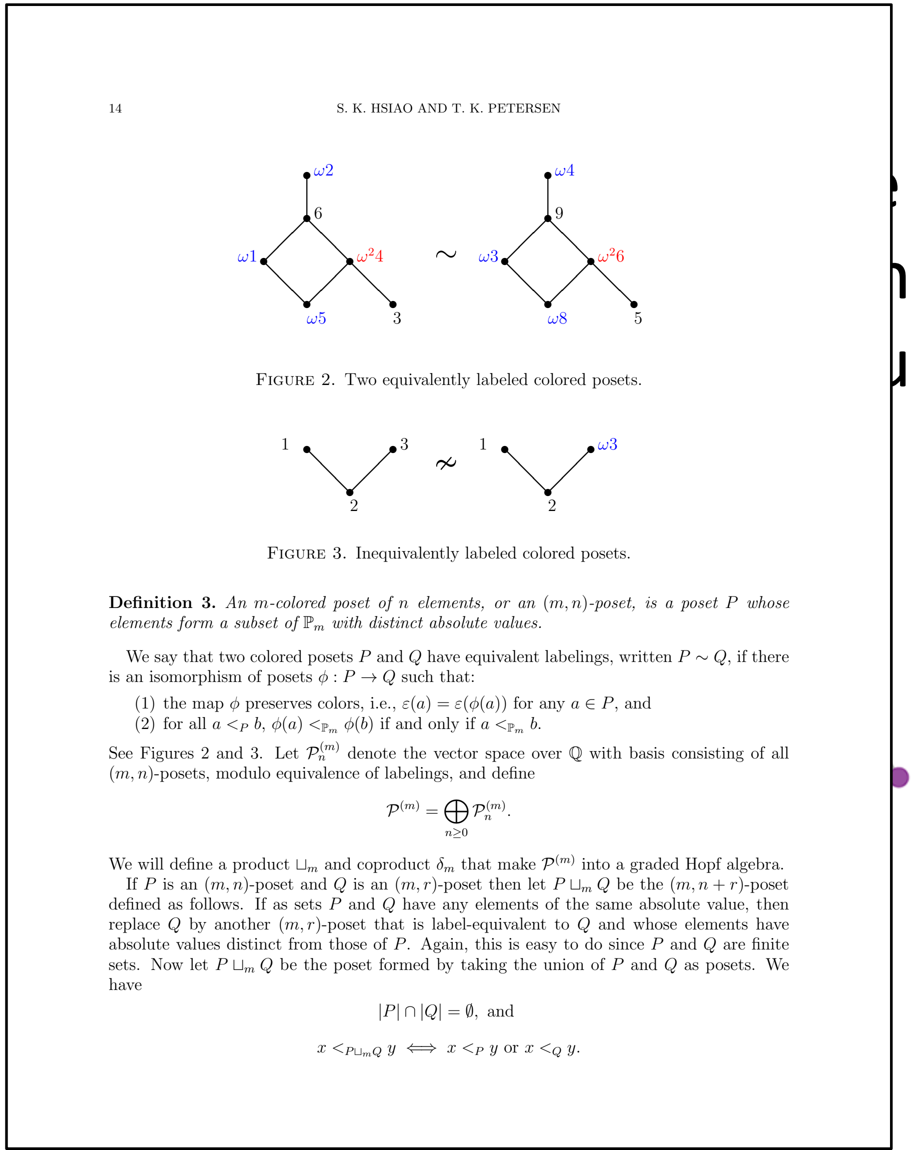


FIGURE 2. Two equivalently labeled colored posets.

FIGURE 3. Inequivalently labeled colored posets.

Definition 3. *An m-colored poset of n elements, or an (m,n)-poset, is a poset P whose elements form a subset of $\mathbb{P}_m$ with distinct absolute values.*

We say that two colored posets P and Q have equivalent labelings, written $P \sim Q$, if there is an isomorphism of posets $\phi : P \to Q$ such that:

(1) the map ϕ preserves colors, i.e., $\varepsilon(a) = \varepsilon(\phi(a))$ for any $a \in P$, and
(2) for all $a <_P b$, $\phi(a) <_{\mathbb{P}_m} \phi(b)$ if and only if $a <_{\mathbb{P}_m} b$.

See Figures 2 and 3. Let $\mathcal{P}_n^{(m)}$ denote the vector space over $\mathbb{Q}$ with basis consisting of all (m,n)-posets, modulo equivalence of labelings, and define

$$\mathcal{P}^{(m)} = \bigoplus_{n \geq 0} \mathcal{P}_n^{(m)}.$$

We will define a product $\sqcup_m$ and coproduct δ_m that make $\mathcal{P}^{(m)}$ into a graded Hopf algebra.

If P is an (m,n)-poset and Q is an (m,r)-poset then let $P \sqcup_m Q$ be the $(m, n+r)$-poset defined as follows. If as sets P and Q have any elements of the same absolute value, then replace Q by another (m,r)-poset that is label-equivalent to Q and whose elements have absolute values distinct from those of P. Again, this is easy to do since P and Q are finite sets. Now let $P \sqcup_m Q$ be the poset formed by taking the union of P and Q as posets. We have

$$|P| \cap |Q| = \emptyset, \text{ and}$$

$$x <_{P \sqcup_m Q} y \iff x <_P y \text{ or } x <_Q y.$$

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ns, or in other parts of math)
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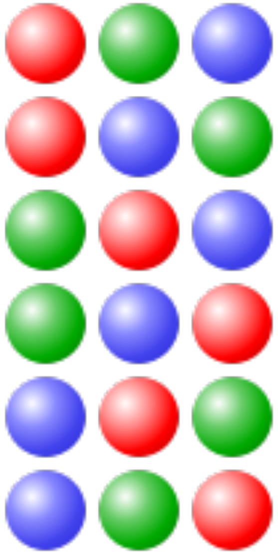
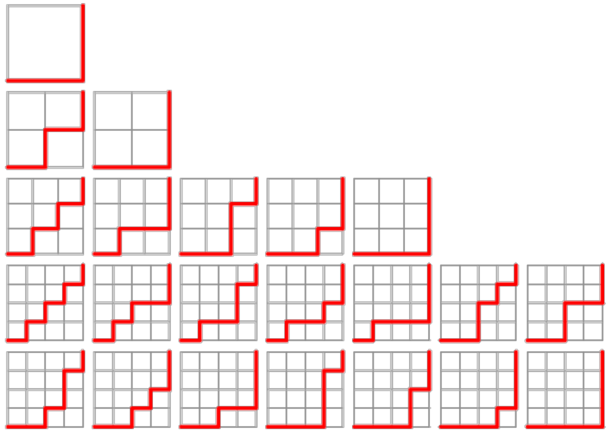
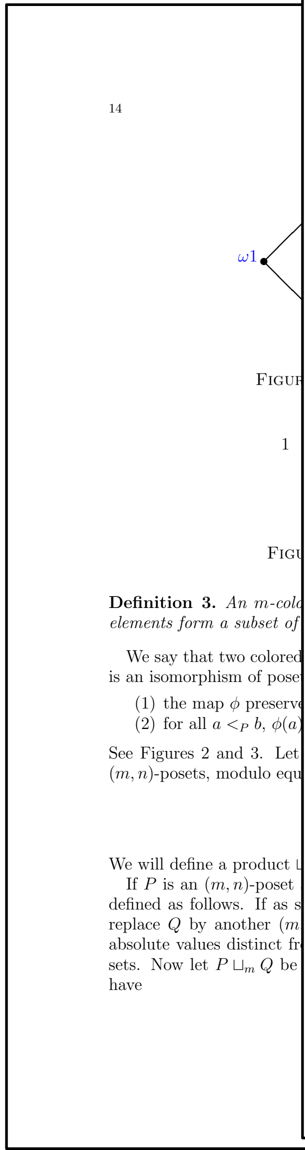




Combinatorics: art and science of counting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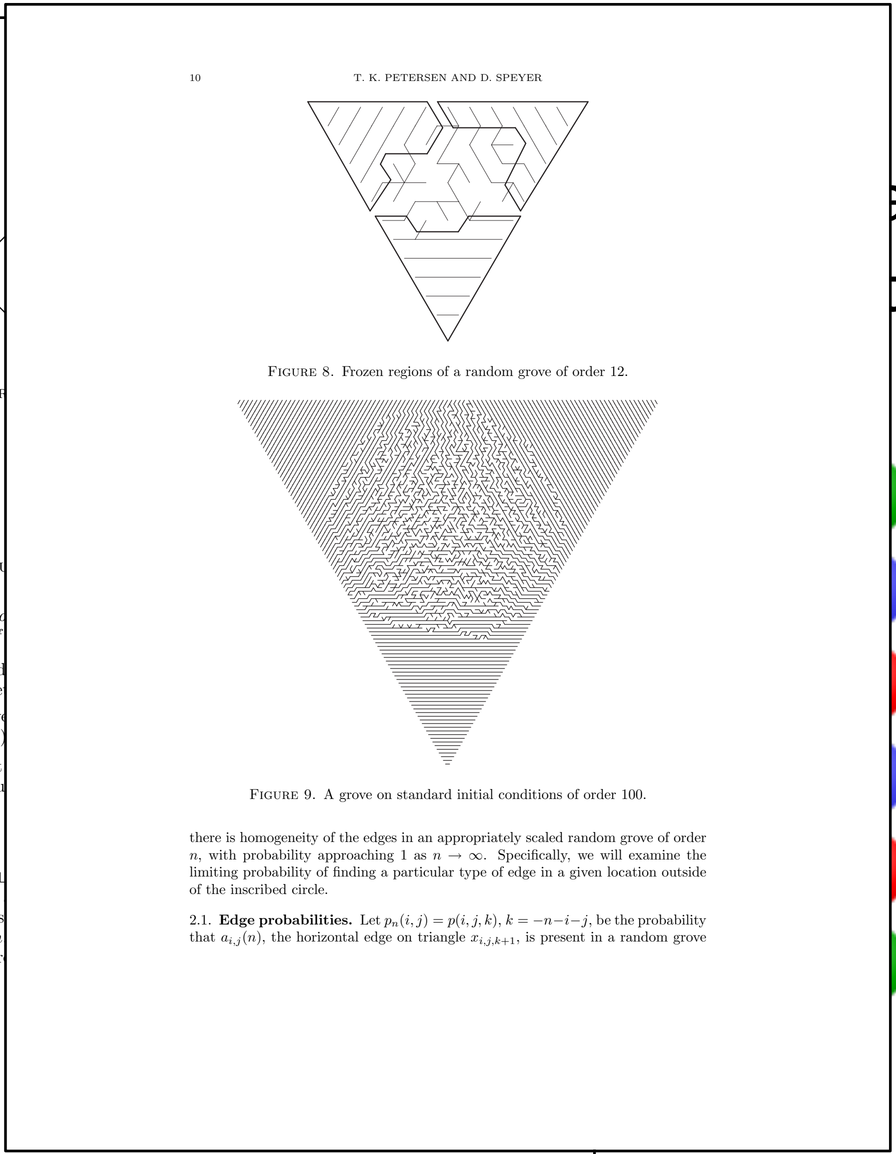

Figure 8. Frozen regions of a random grove of order 12.

Figure 9. A grove on standard initial conditions of order 100.

there is homogeneity of the edges in an appropriately scaled random grove of order n, with probability approaching 1 as $n \to \infty$. Specifically, we will examine the limiting probability of finding a particular type of edge in a given location outside of the inscribed circle.

2.1. **Edge probabilities.** Let $p_n(i, j) = p(i, j, k)$, $k = -n-i-j$, be the probability that $a_{i,j}(n)$, the horizontal edge on triangle $x_{i,j,k+1}$, is present in a random grove

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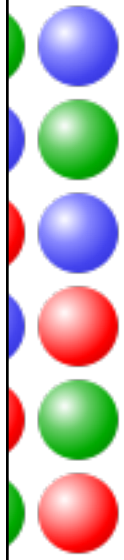
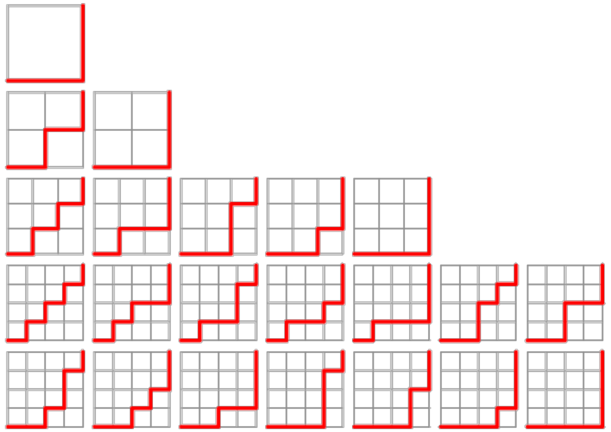
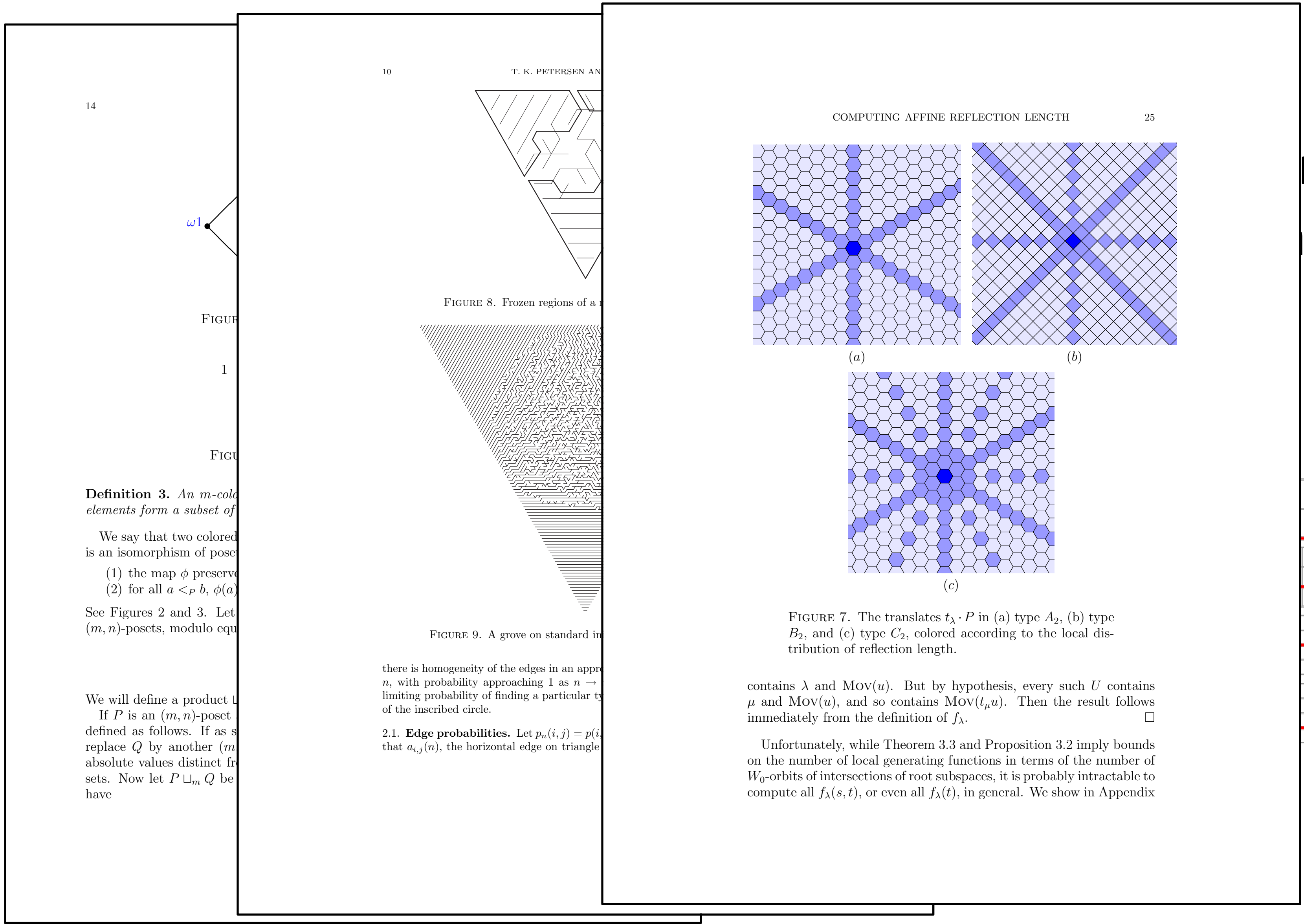
ther parts of math)
make pictures!





Combinatorics: art and science of counting


COMPUTING AFFINE REFLECTION LENGTH 25

(a) (b) (c)

FIGURE 7. The translates $t_\lambda \cdot P$ in (a) type A_2, (b) type B_2, and (c) type C_2, colored according to the local distribution of reflection length.

contains λ and $\mathrm{Mov}(u)$. But by hypothesis, every such U contains μ and $\mathrm{Mov}(u)$, and so contains $\mathrm{Mov}(t_\mu u)$. Then the result follows immediately from the definition of f_λ. □

Unfortunately, while Theorem 3.3 and Proposition 3.2 imply bounds on the number of local generating functions in terms of the number of W_0-orbits of intersections of root subspaces, it is probably intractable to compute all $f_\lambda(s,t)$, or even all $f_\lambda(t)$, in general. We show in Appendix

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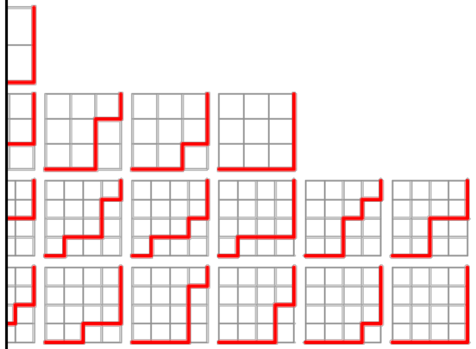
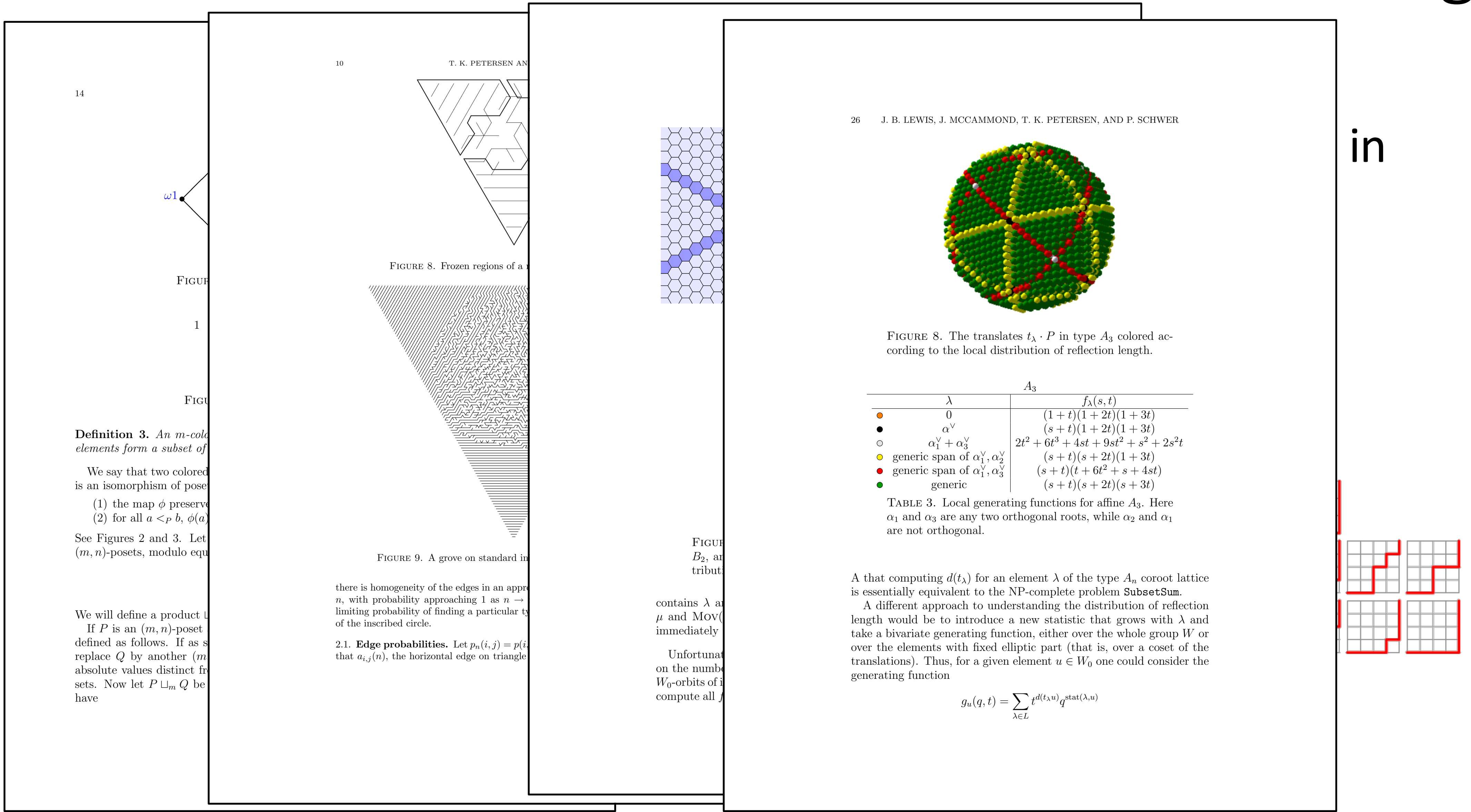


Combinatorics: art and science of counting



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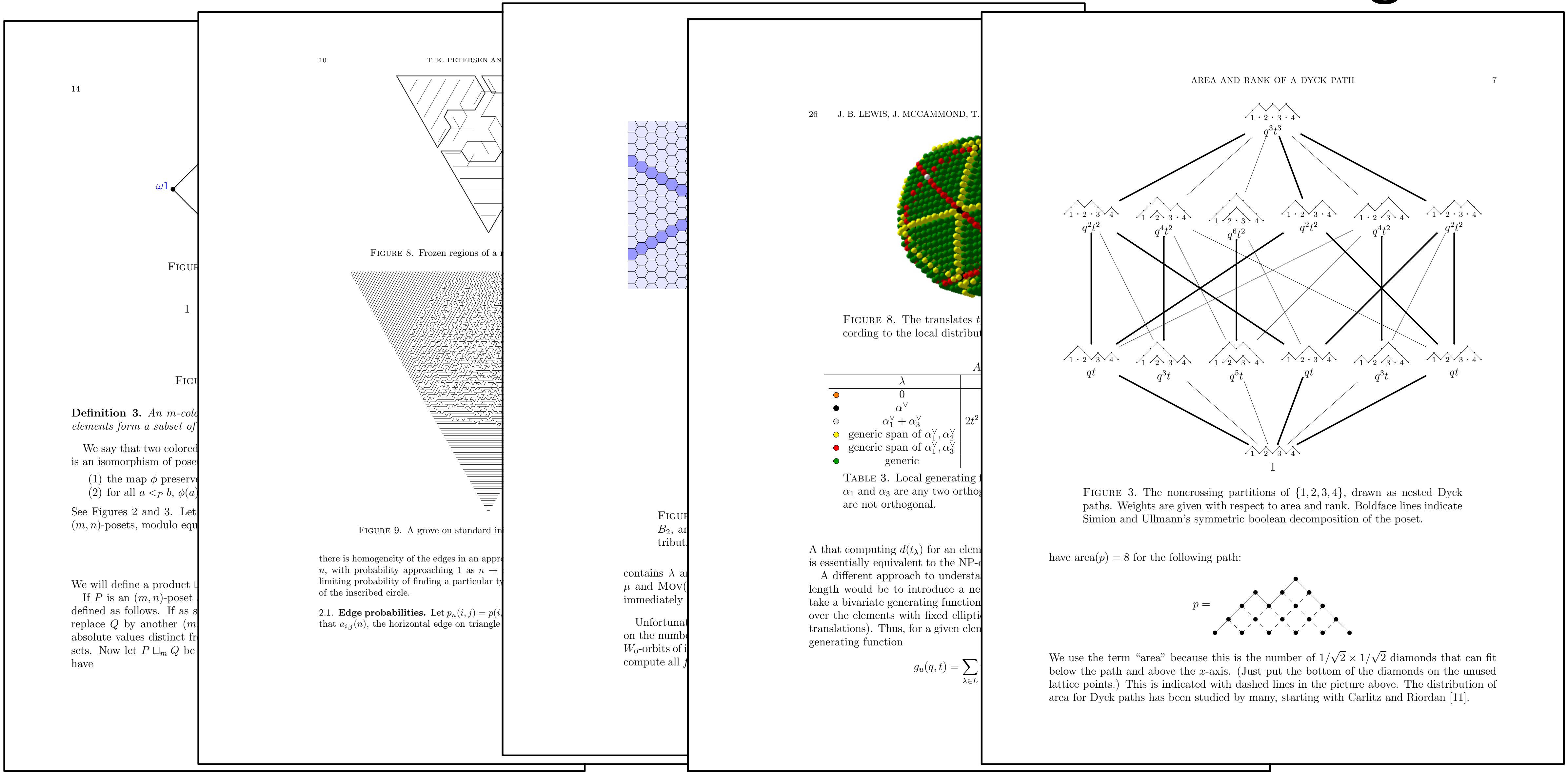
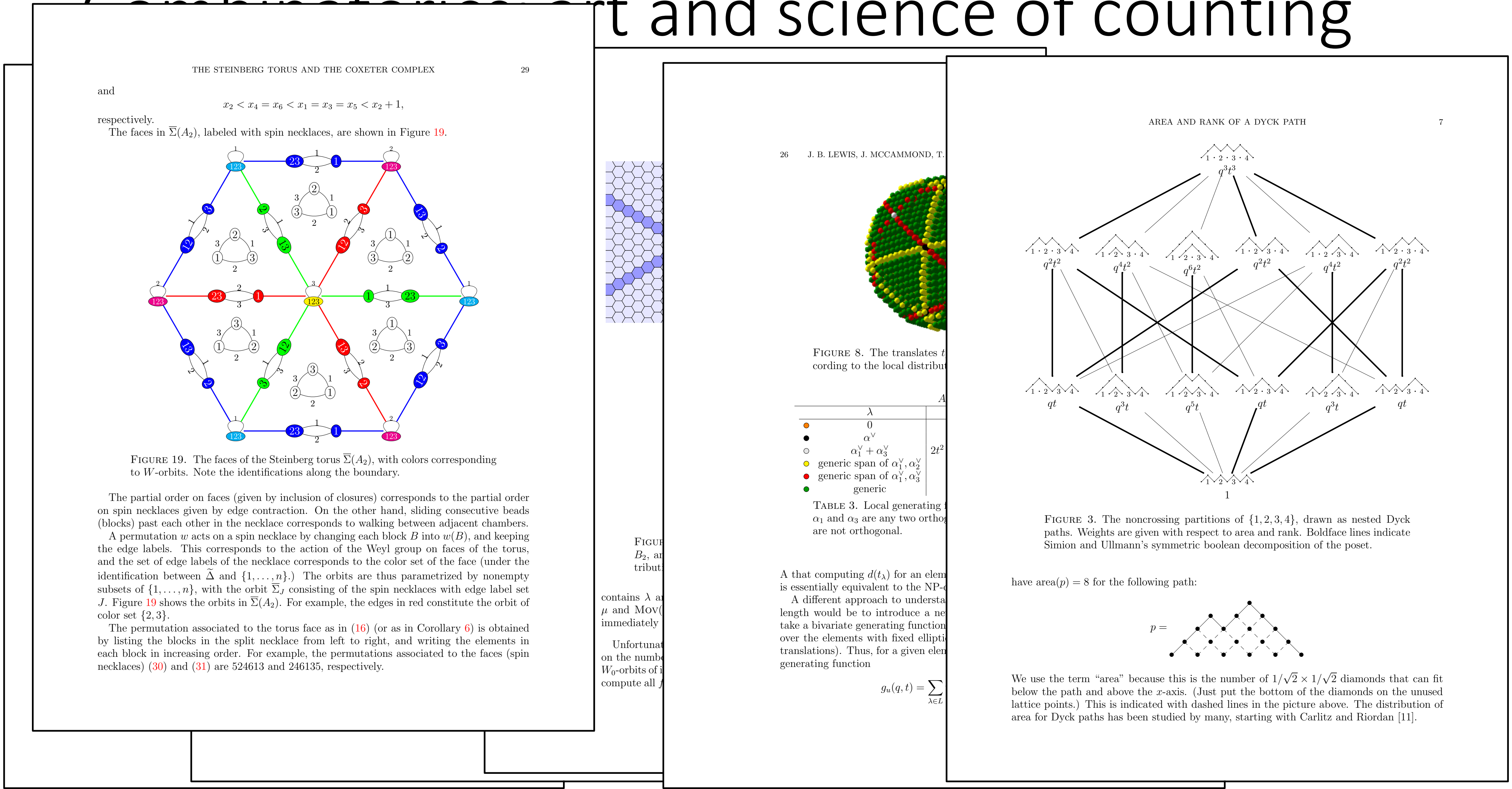
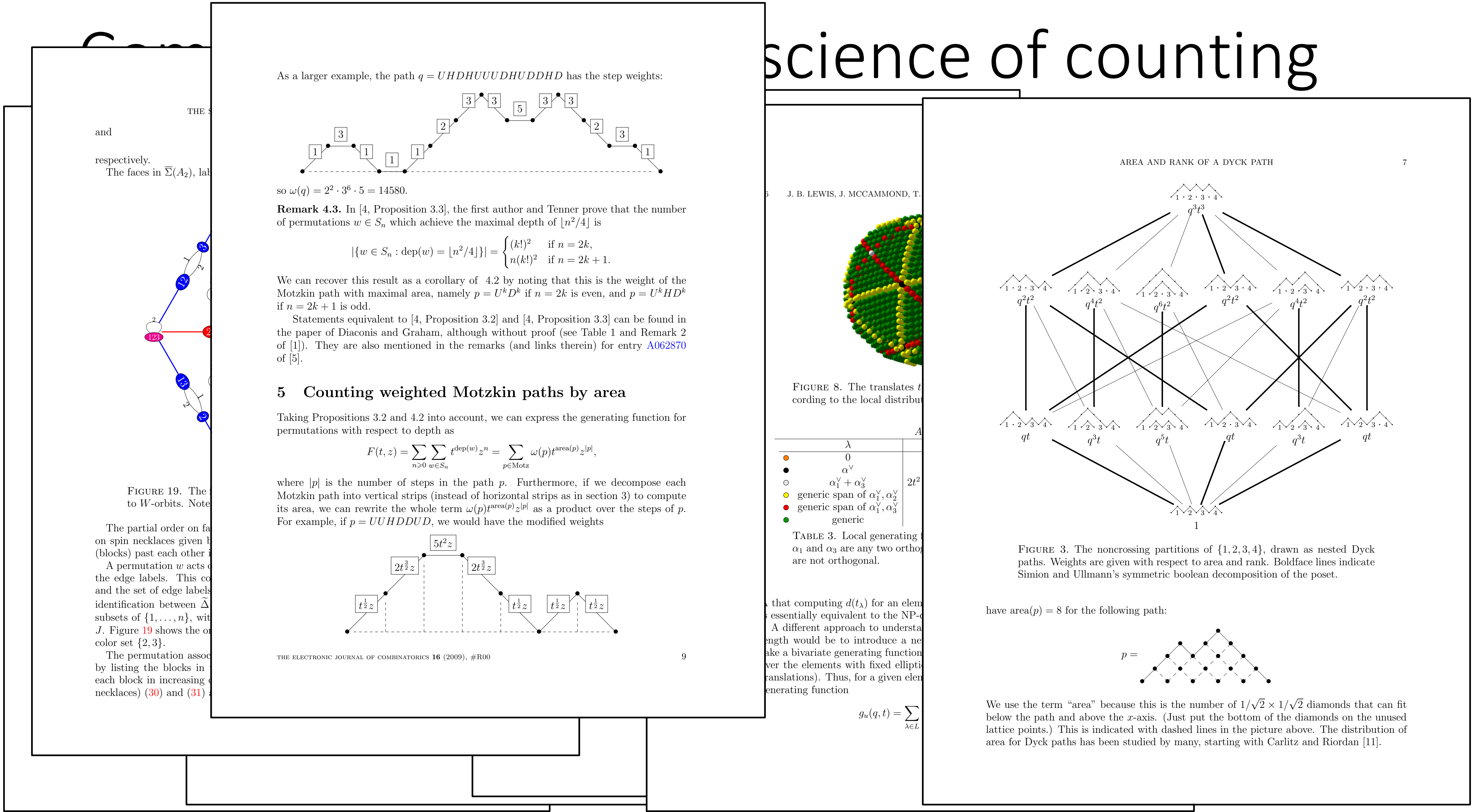


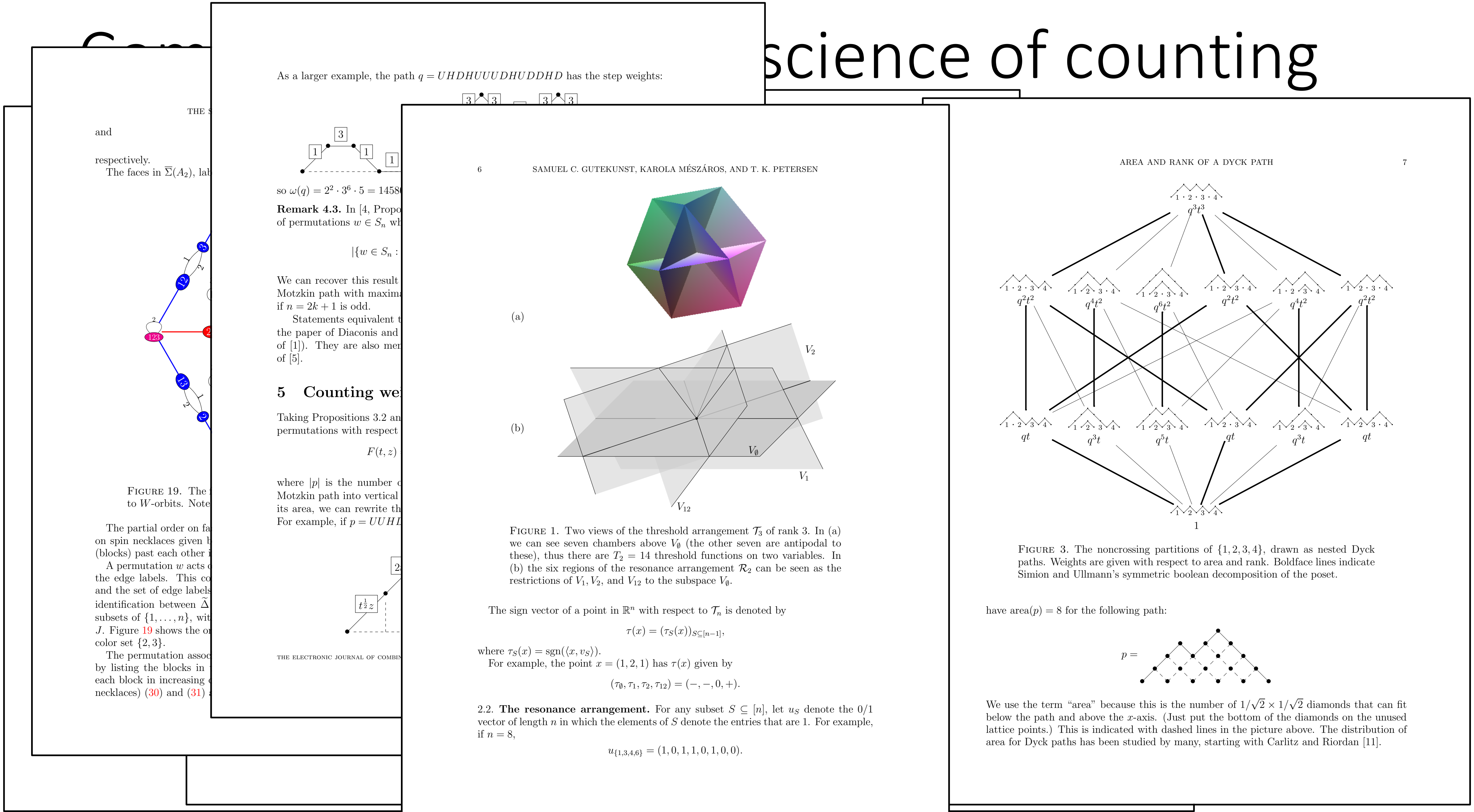
Combinatorics: art and science of counting

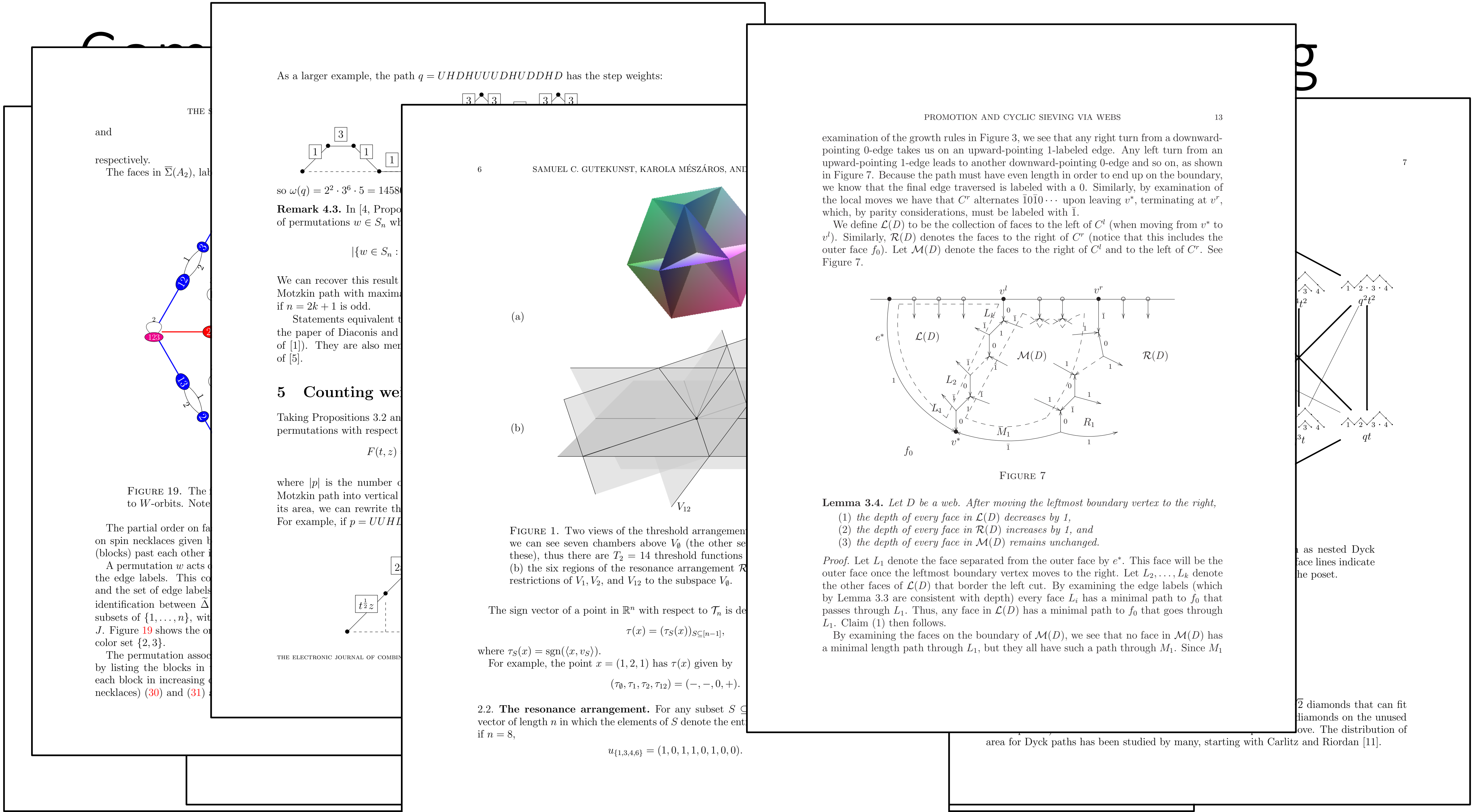


Com... science of counting



science of counting





examination of the growth rules in Figure 3, we see that any right turn from a downward-pointing 0-edge takes us on an upward-pointing 1-labeled edge. Any left turn from an upward-pointing 1-edge leads to another downward-pointing 0-edge and so on, as shown in Figure 7. Because the path must have even length in order to end up on the boundary, we know that the final edge traversed is labeled with a 0. Similarly, by examination of the local moves we have that C^r alternates $\bar{1}0\bar{1}0\cdots$ upon leaving v^*, terminating at v^r, which, by parity considerations, must be labeled with $\bar{1}$.

We define $\mathcal{L}(D)$ to be the collection of faces to the left of C^l (when moving from v^* to v^l). Similarly, $\mathcal{R}(D)$ denotes the faces to the right of C^r (notice that this includes the outer face f_0). Let $\mathcal{M}(D)$ denote the faces to the right of C^l and to the left of C^r. See Figure 7.

FIGURE 7

Lemma 3.4. *Let D be a web. After moving the leftmost boundary vertex to the right,*

(1) *the depth of every face in $\mathcal{L}(D)$ decreases by 1,*
(2) *the depth of every face in $\mathcal{R}(D)$ increases by 1, and*
(3) *the depth of every face in $\mathcal{M}(D)$ remains unchanged.*

Proof. Let L_1 denote the face separated from the outer face by e^*. This face will be the outer face once the leftmost boundary vertex moves to the right. Let $L_2, \ldots, L_k$ denote the other faces of $\mathcal{L}(D)$ that border the left cut. By examining the edge labels (which by Lemma 3.3 are consistent with depth) every face L_i has a minimal path to f_0 that passes through L_1. Thus, any face in $\mathcal{L}(D)$ has a minimal path to f_0 that goes through L_1. Claim (1) then follows.

By examining the faces on the boundary of $\mathcal{M}(D)$, we see that no face in $\mathcal{M}(D)$ has a minimal length path through L_1, but they all have such a path through M_1. Since M_1

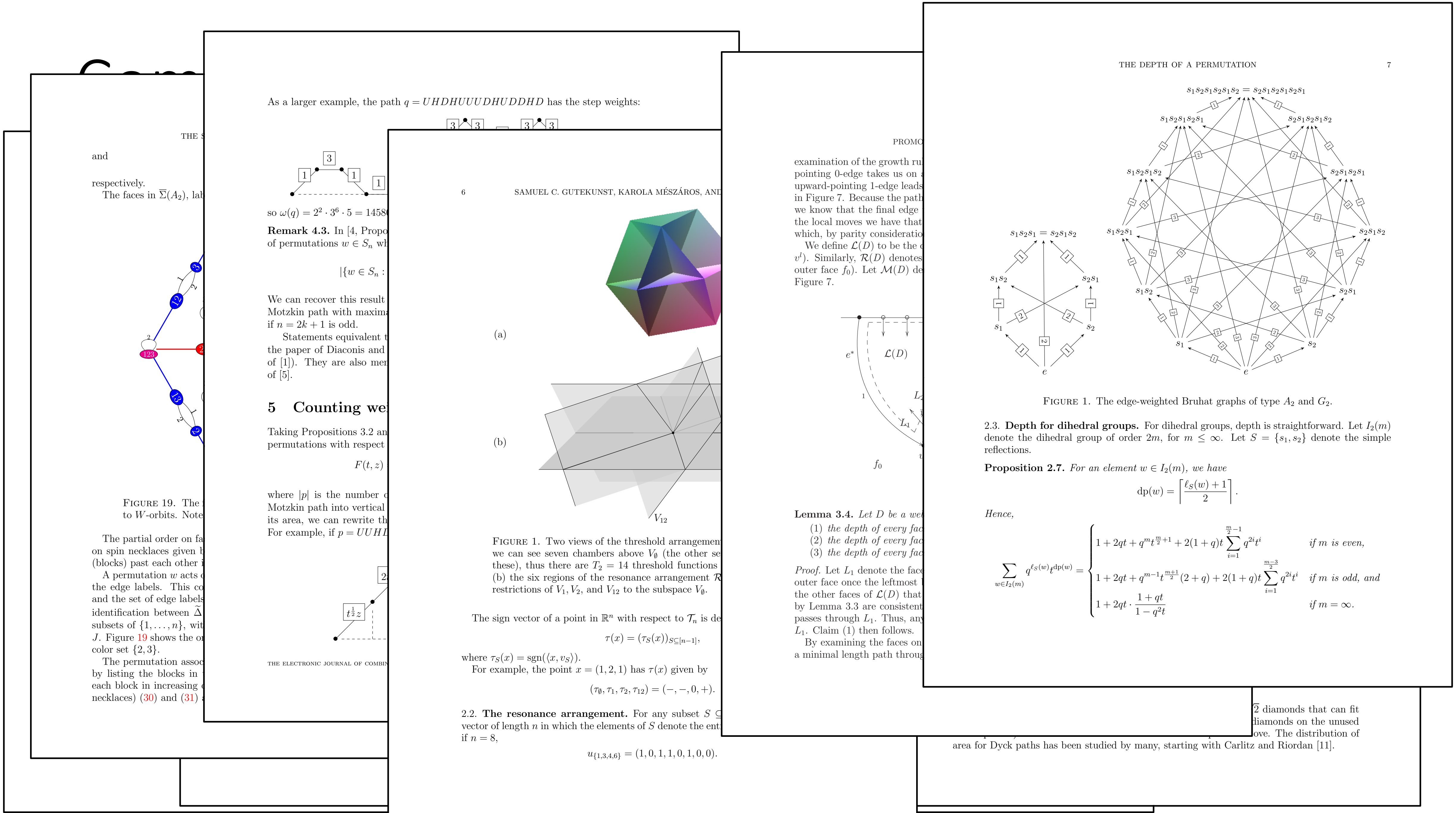


FIGURE 1. The edge-weighted Bruhat graphs of type A_2 and G_2.

2.3. **Depth for dihedral groups.** For dihedral groups, depth is straightforward. Let $I_2(m)$ denote the dihedral group of order $2m$, for $m \leq \infty$. Let $S = \{s_1, s_2\}$ denote the simple reflections.

Proposition 2.7. *For an element $w \in I_2(m)$, we have*

$$\mathrm{dp}(w) = \left\lceil \frac{\ell_S(w)+1}{2} \right\rceil.$$

Hence,

$$\sum_{w\in I_2(m)} q^{\ell_S(w)} t^{\mathrm{dp}(w)} = \begin{cases} 1+2qt+q^m t^{\frac{m}{2}+1} + 2(1+q)t \sum_{i=1}^{\frac{m}{2}-1} q^{2i}t^i & \text{if } m \text{ is even,} \\ 1+2qt+q^{m-1}t^{\frac{m+1}{2}}(2+q) + 2(1+q)t \sum_{i=1}^{\frac{m-3}{2}} q^{2i}t^i & \text{if } m \text{ is odd, and} \\ 1+2qt\cdot \dfrac{1+qt}{1-q^2t} & \text{if } m=\infty. \end{cases}$$

Example 8.11. Let us take $m = 5$ and take the shape L with sinks $(1,4)$ and $(3,2)$. Then the associated complex $\mathbf{m}_L^{(nc)^*}$ and a realization of the dual polytope are shown

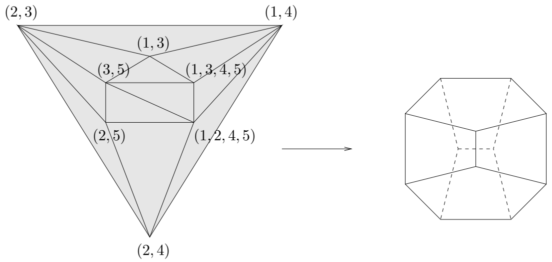


FIGURE 7

on Figure 7. Here the outer triangle $\{(2,3),(1,4),(2,4)\}$ should also be understood as a face.

Example 8.12. If L is a $2 \times n$ (or an $n \times 2$) rectangle, $\mathbf{m}_L^{(nc)^*}$ is the type *A cluster complex* of [FZ]. It is known to be polytopal, and the dual polytope is known as the *associahedron*. However, Γ_L is usually not simplicial. The first counter-example is when $n = 4$ (so $m = 6$). Here Γ_L has a square face whose vertices correspond to 14, 15, 25 and 24. In $\mathbf{m}_L^{(nc)^*}$, this square is subdivided into two triangles, along the diagonal joining $(1,5)$ and $(2,4)$.

Example 8.13. Let L be a 3×3 rectangle (so $m = 6$). In this example, we will explore the difference between $\mathbf{m}_L^{(nc)}$ and $\mathbf{m}_L^{(ws)}$. There are 6 solid paths and $N(L) = 9$, so $\mathbf{m}_L^{(nc)^*}$ is a 3-sphere. We write $\mathbf{m}_L^{(ws)^*}$ for the subcomplex of $\mathbf{m}_L^{(nc)^*}$ corresponding to weakly separated paths. There are two pairs of 3-element subsets of [6] which are non-crossing but not weakly separated, namely the pairs (145, 236) and (124, 356). (The first pair of paths crosses twice; the second pair has an hourglass.) Each of these pairs corresponds to an edge in $\mathbf{m}_L^{(nc)^*}$. Each of these edges is surrounded by four tetrahedra and these tetrahedra fit together to form an octahedron subdivided around a central axis. These two octahedra are disjoint from one another. In $\mathbf{m}_L^{(ws)^*}$, these two octahedra are removed, leaving behind a complex homeomorphic to $S^2 \times [0,1]$. The endpoints of this product are a pair of 2-spheres, each triangulated as the boundary of the octahedron. The simplicial complex $\mathbf{m}_L^{(ws)^*}$ is a subcomplex of the D_4-cluster complex, which is again a 3-sphere. In the D_4 cluster complex, two new vertices are added. One of these vertices is compatible

he step weights:

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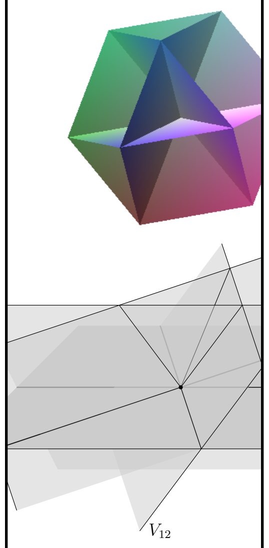

views of the threshold arrangement
chambers above $V_\emptyset$ (the other se
are $T_2 = 14$ threshold functions
s of the resonance arrangement $\mathcal{R}$
V_2, and V_{12} to the subspace $V_\emptyset$.

oint in $\mathbb{R}^n$ with respect to $\mathcal{T}_n$ is de

$$\tau(x) = (\tau_S(x))_{S \subseteq [n-1]},$$

)).
t $x = (1,2,1)$ has $\tau(x)$ given by

$$(\tau_\emptyset, \tau_1, \tau_2, \tau_{12}) = (-,-,0,+).$$

2.2. **The resonance arrangement.** For any subset $S \subseteq$
vector of length n in which the elements of S denote the ent
if $n = 8$,

$$u_{\{1,3,4,6\}} = (1,0,1,1,0,1,0,0).$$

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examination of the growth ru
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upward-pointing 1-edge leads
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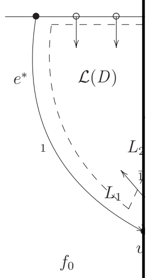

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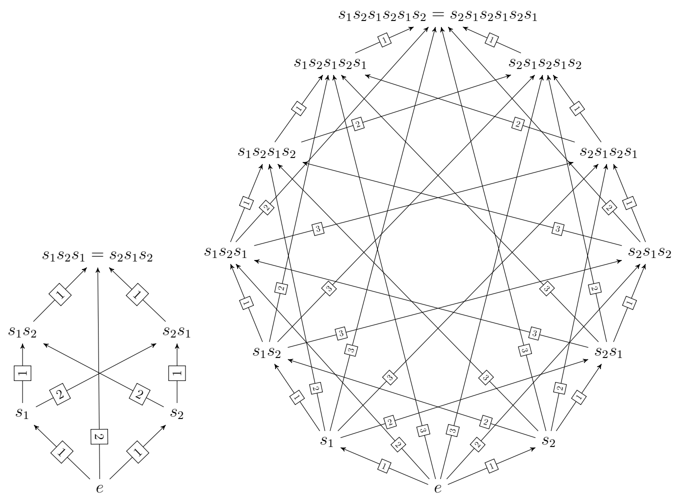
diamonds on the unused
ove. The distribution of
area for Dyck paths has been studied by many, starting with Carlitz and Riordan [11].



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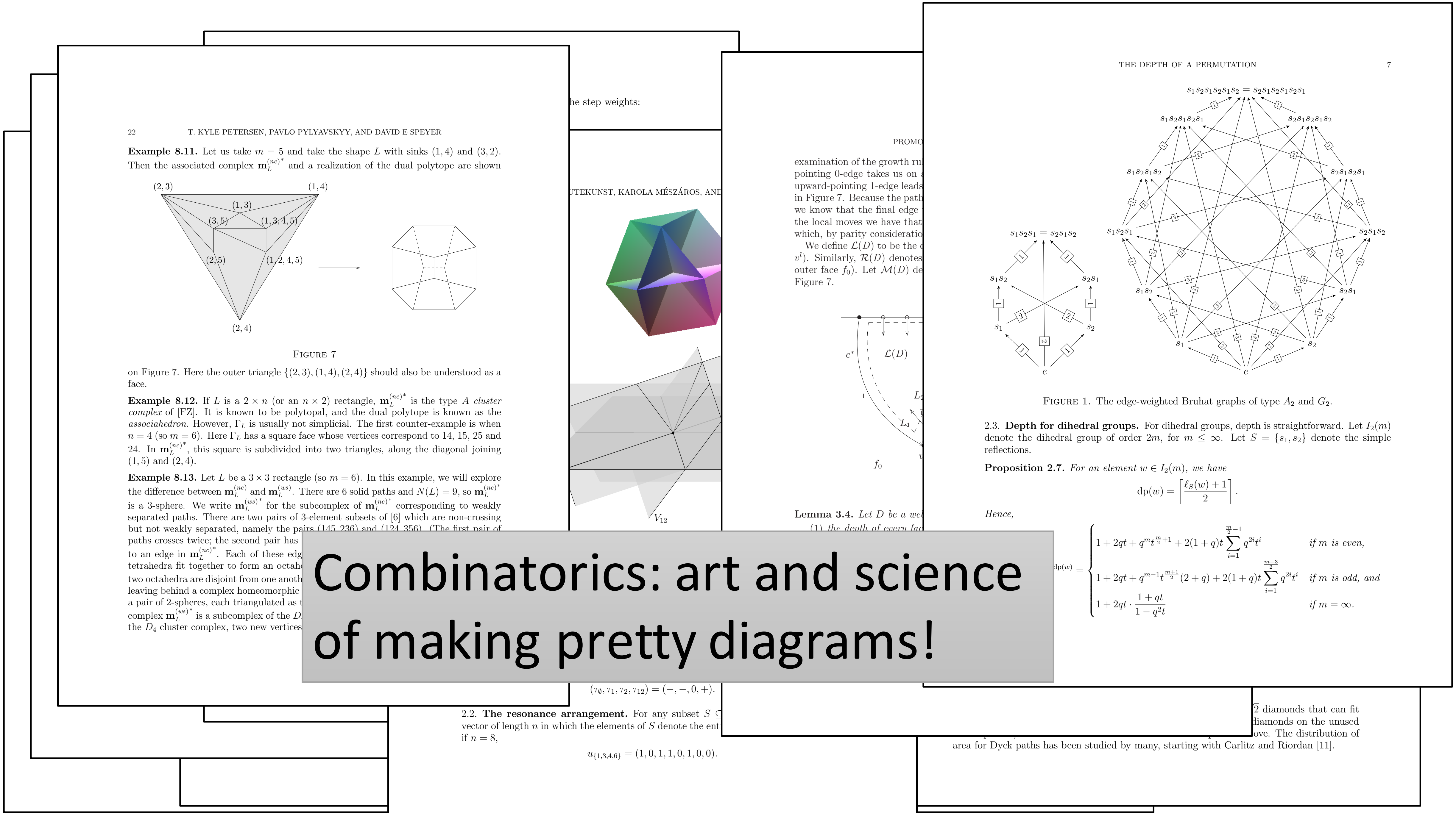
$$\sum_{w \in I_2(m)} q^{\ell_S(w)} t^{\mathrm{dp}(w)} = \begin{cases} 1 + 2qt + q^m t^{\frac{m}{2}+1} + 2(1+q)t \sum_{i=1}^{\frac{m}{2}-1} q^{2i} t^i & \text{if } m \text{ is even,} \\ 1 + 2qt + q^{m-1} t^{\frac{m+1}{2}} (2+q) + 2(1+q)t \sum_{i=1}^{\frac{m-3}{2}} q^{2i} t^i & \text{if } m \text{ is odd, and} \\ 1 + 2qt \cdot \dfrac{1+qt}{1-q^2 t} & \text{if } m = \infty. \end{cases}$$

Combinatorics: art and science of making pretty diagrams!

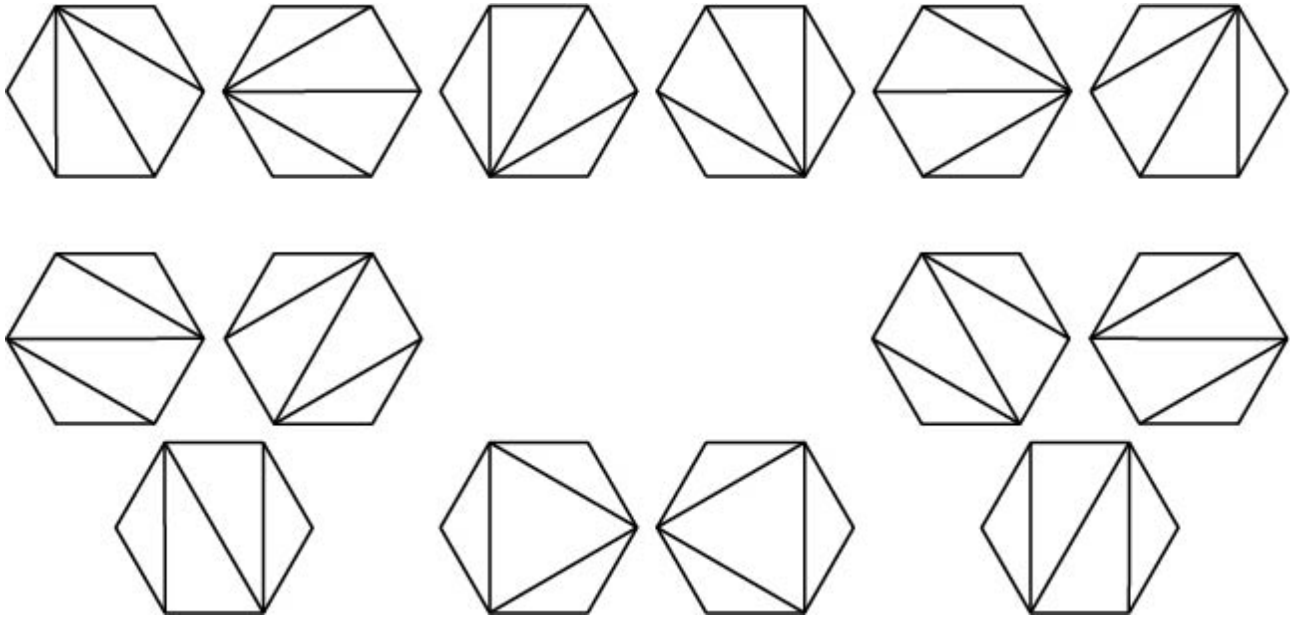


Combinatorics: art and science of counting

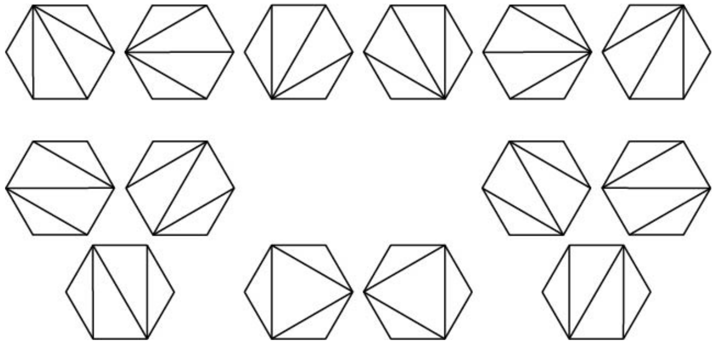
Leonhard Euler ca. 1750 asked: How many ways to triangulate a polygon?

Combinatorics: art and science of counting

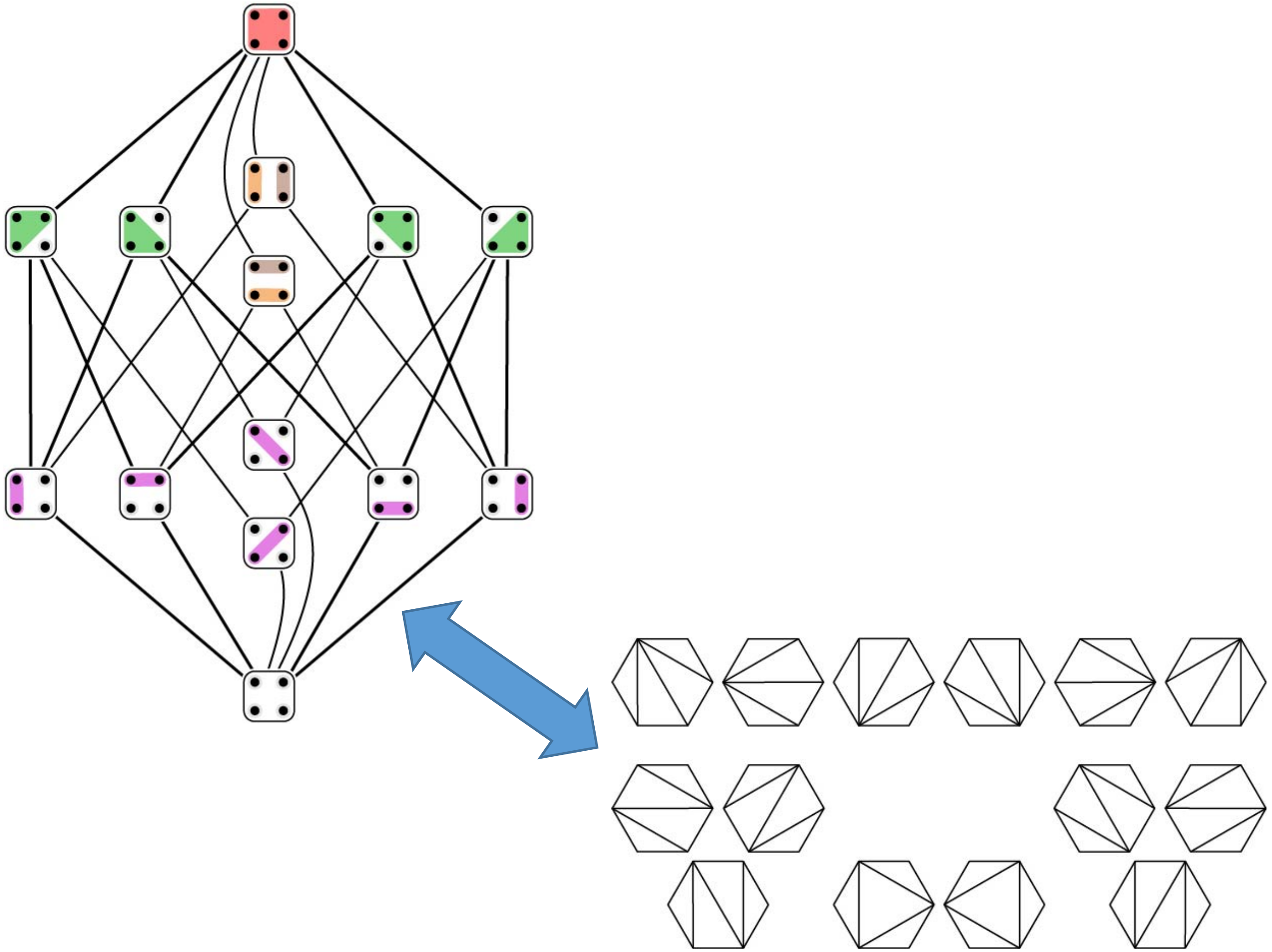
Leonhard Euler ca. 1750 asked: How many ways to triangulate a polygon?



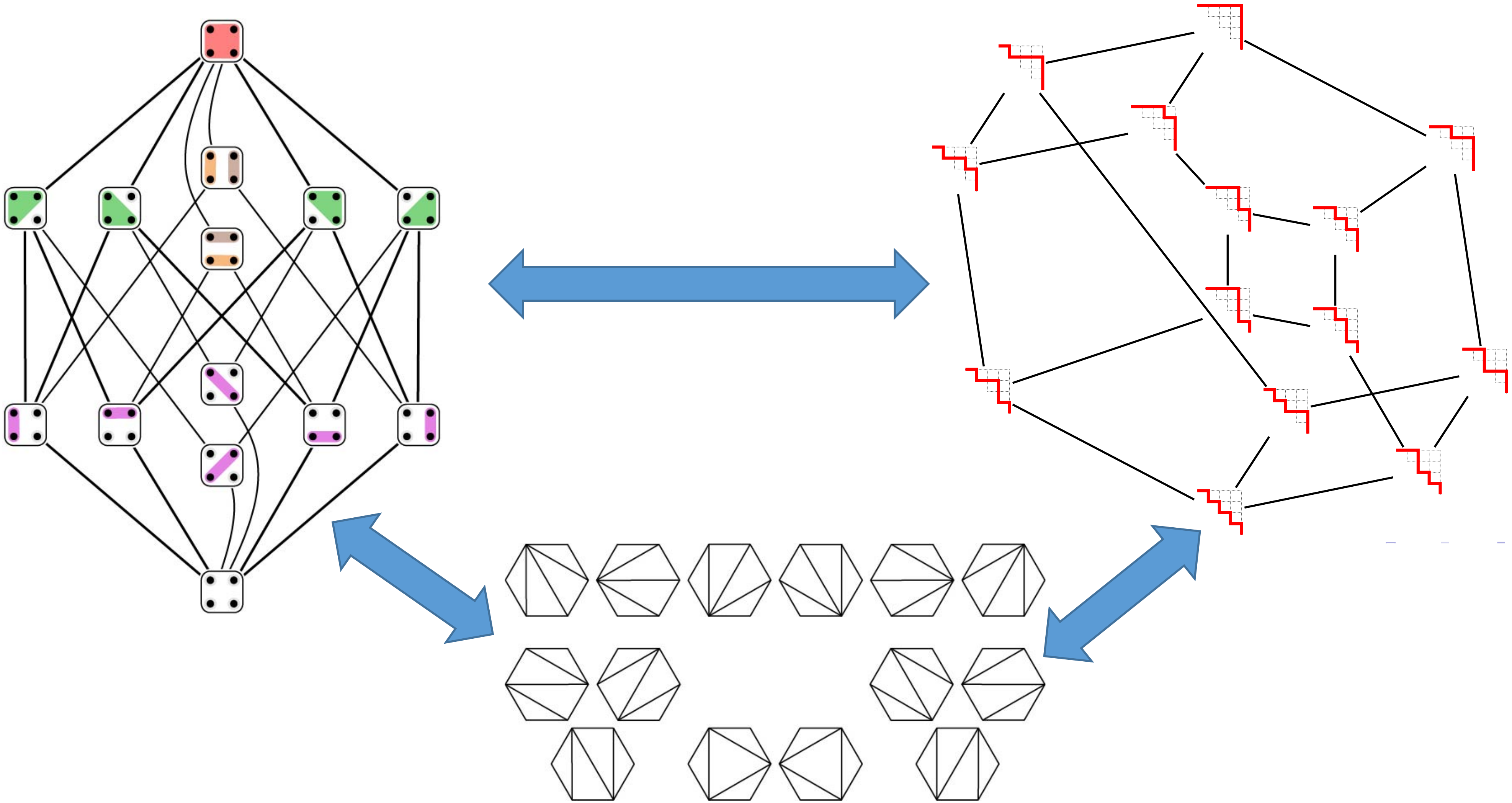
Combinatorics: art and science of counting



Combinatorics: art and science of counting

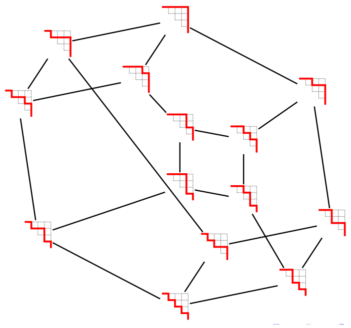


Combinatorics: art and science of counting



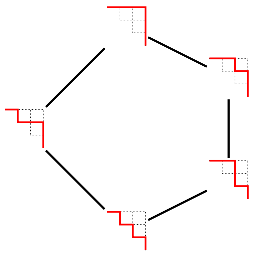
Combinatorics: art and science of counting

n=4

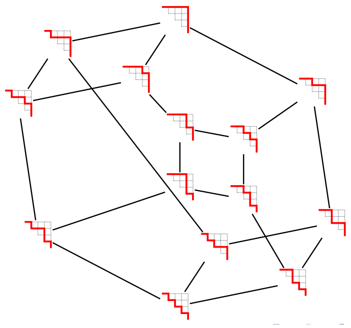


Combinatorics: art and science of counting

n=3



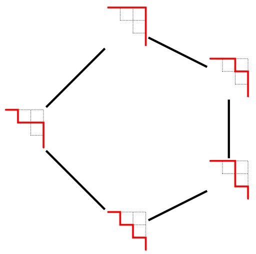
n=4



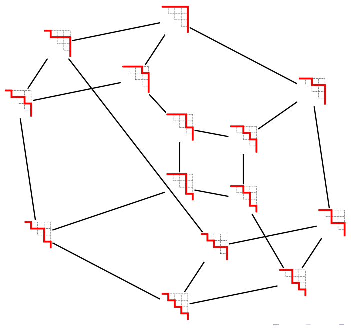
Combinatorics: art and science of counting

n=2

n=3



n=4



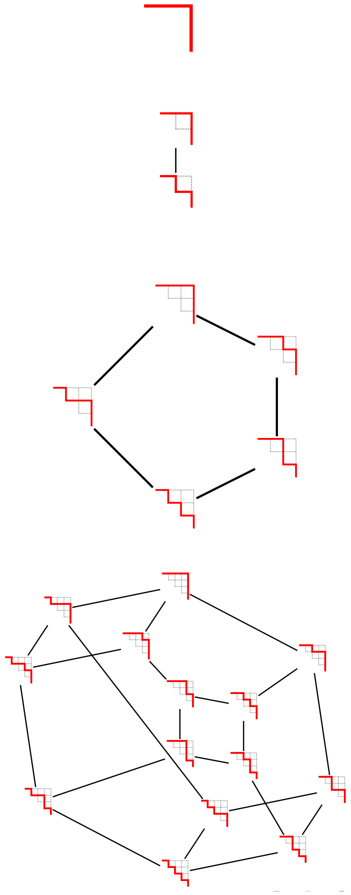
Combinatorics: art and science of counting

n=1

n=2

n=3

n=4



Combinatorics: art and science of counting

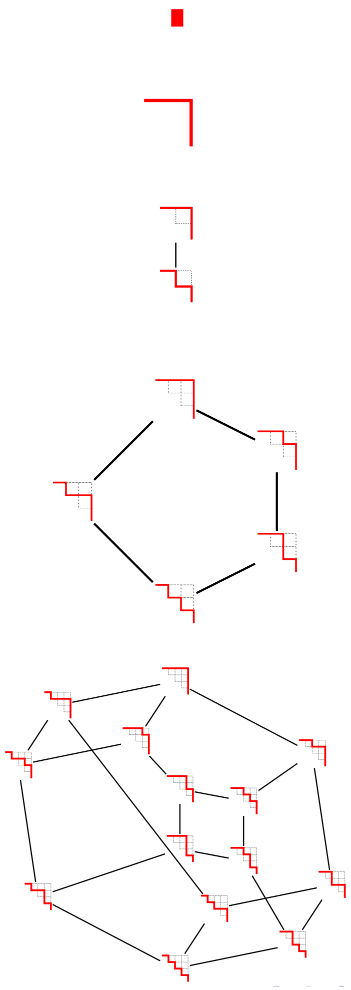
n=0

n=1

n=2

n=3

n=4



Combinatorics: art and science of counting

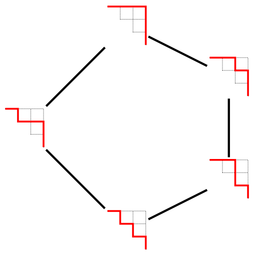
n=0

$C_0=1$

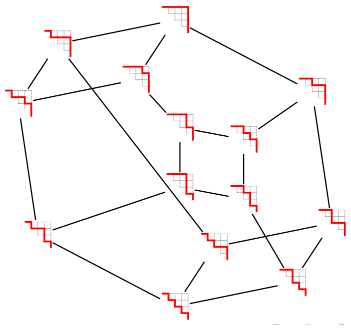
n=1

n=2

n=3



n=4

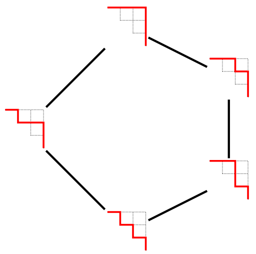


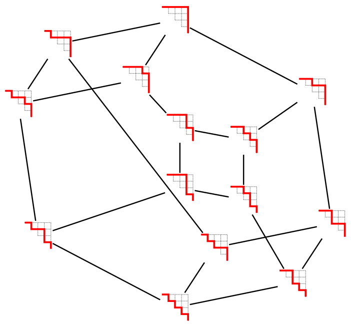
Combinatorics: art and science of counting

n=0 $C_0=1$

n=1  $C_1=1$

n=2 

n=3 

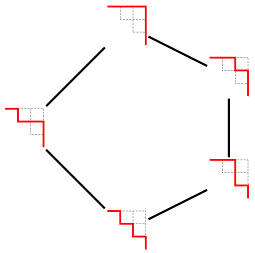
n=4 

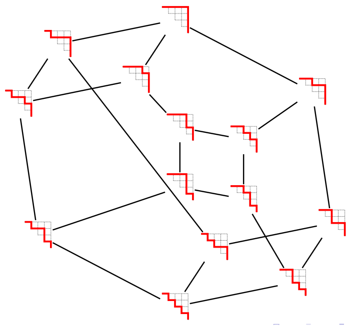
Combinatorics: art and science of counting

n=0 $C_0=1$

n=1  $C_1=1$

n=2  $C_2=2$

n=3 

n=4 

Combinatorics: art and science of counting

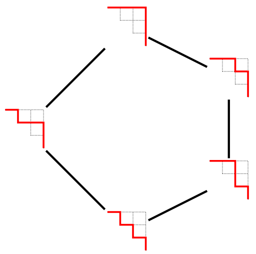
n=0
$C_0=1$

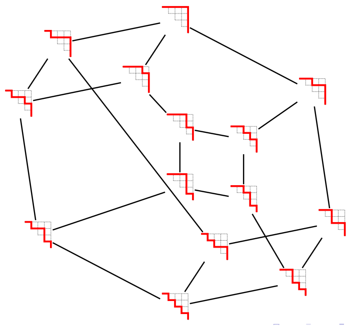
n=1

$C_1=1$

n=2

$C_2=2$

n=3

$C_3=5$

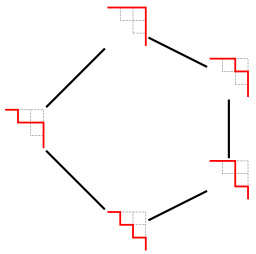
n=4


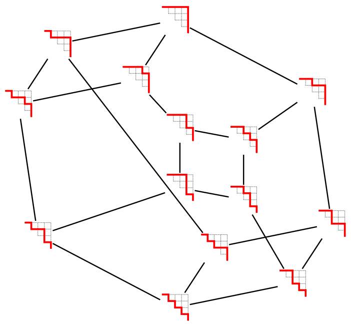
Combinatorics: art and science of counting

n=0 $C_0=1$

n=1  $C_1=1$

n=2  $C_2=2$

n=3  $C_3=5$

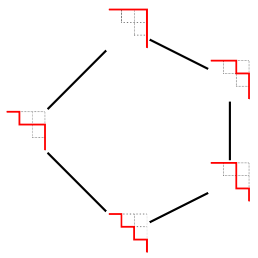
n=4  $C_4=14$

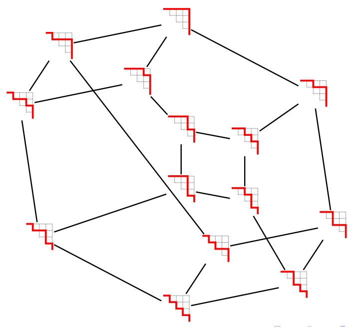
Combinatorics: art and science of counting

n=0 $C_0=1$

n=1  $C_1=1$

n=2  $C_2=2$

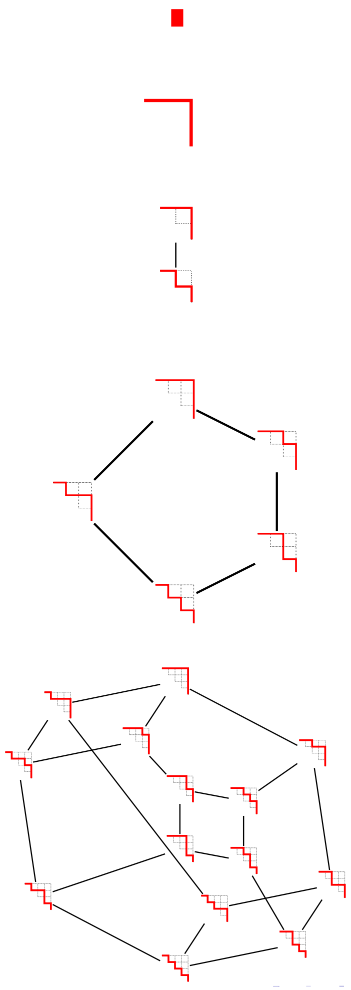
n=3  $C_3=5$

n=4  $C_4=14$

Formula:

$$C_n = \frac{(2n)!}{n!\,(n+1)!}$$

Combinatorics: art and science of counting



n=0 — $C_0=1$

n=1 — $C_1=1$

n=2 — $C_2=2$

n=3 — $C_3=5$

n=4 — $C_4=14$

Formula:

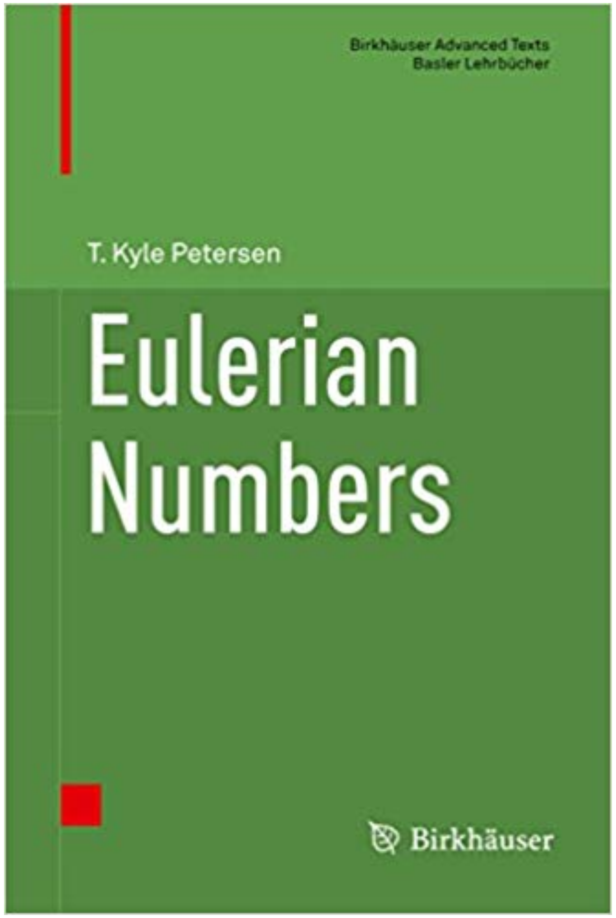
$$C_n = \frac{(2n)!}{n!\,(n+1)!}$$

Generating Function:

$$\frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots$$

Number Triangles and Geometry

Number Triangles and Geometry



Pascal’s Triangle

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

Pascal's Triangle

Sums

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

Pascal's Triangle

					Sums
1					**1**
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	

Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			
1	3	3	1		
1	4	6	4	1	

Pascal's Triangle

						Sums
1						**1**
1	1					**2**
1	2	1				**4**
1	3	3	1			
1	4	6	4	1		

Pascal's Triangle

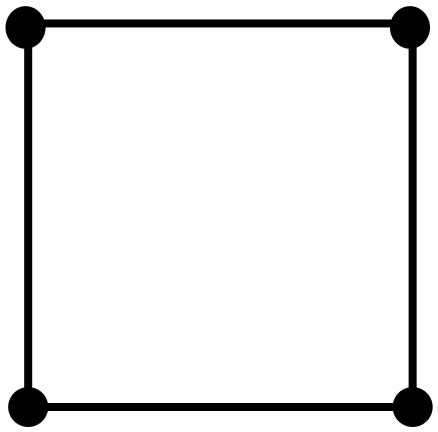
					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	

Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**

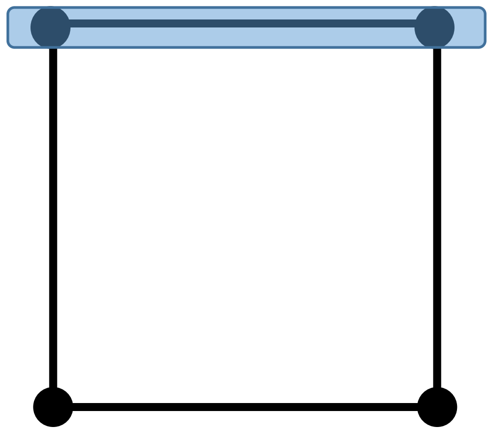
Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**



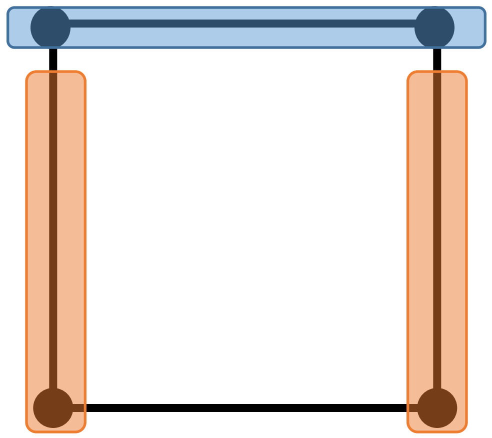
Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**



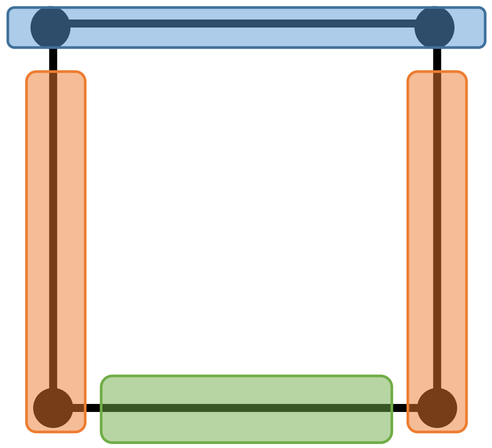
Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**



Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**



Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
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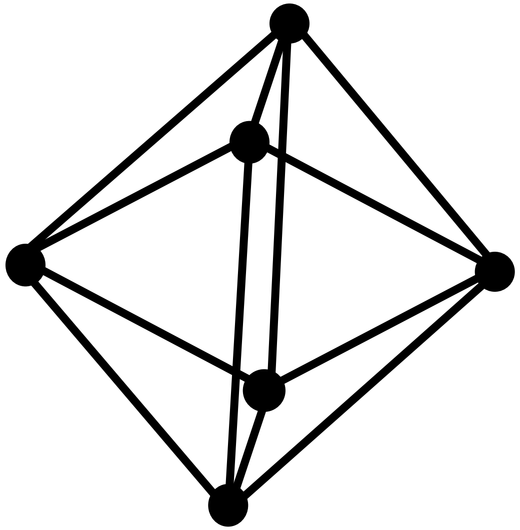
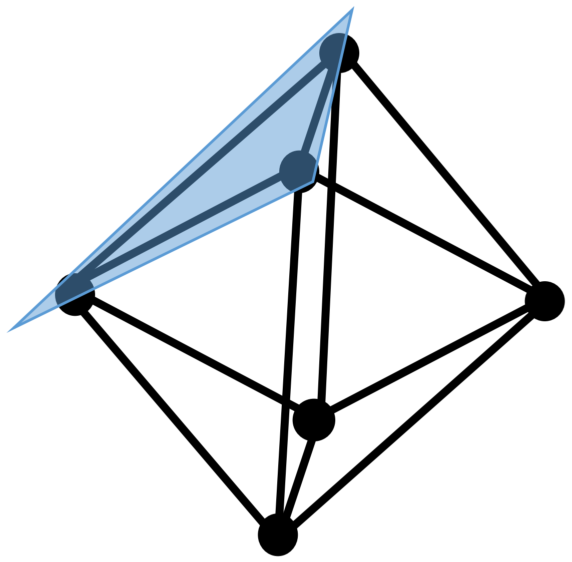


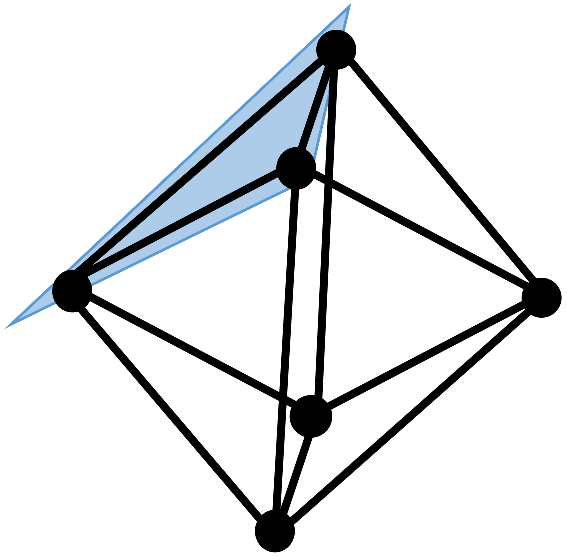
Pascal's Triangle

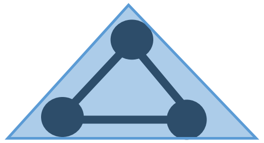
					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**



Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**





Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
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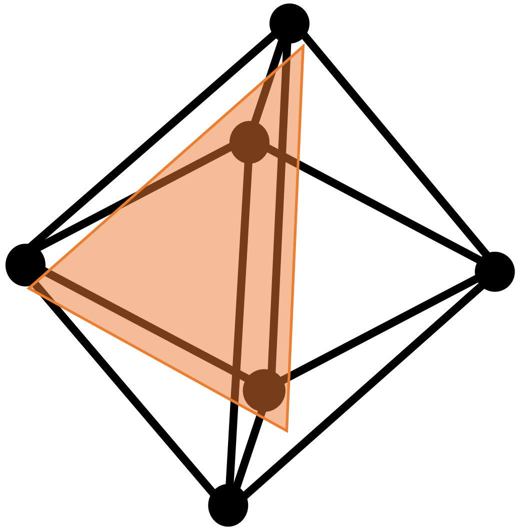
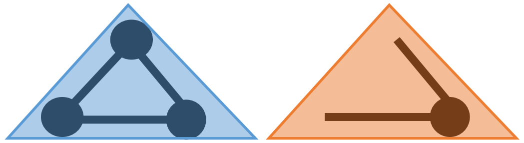
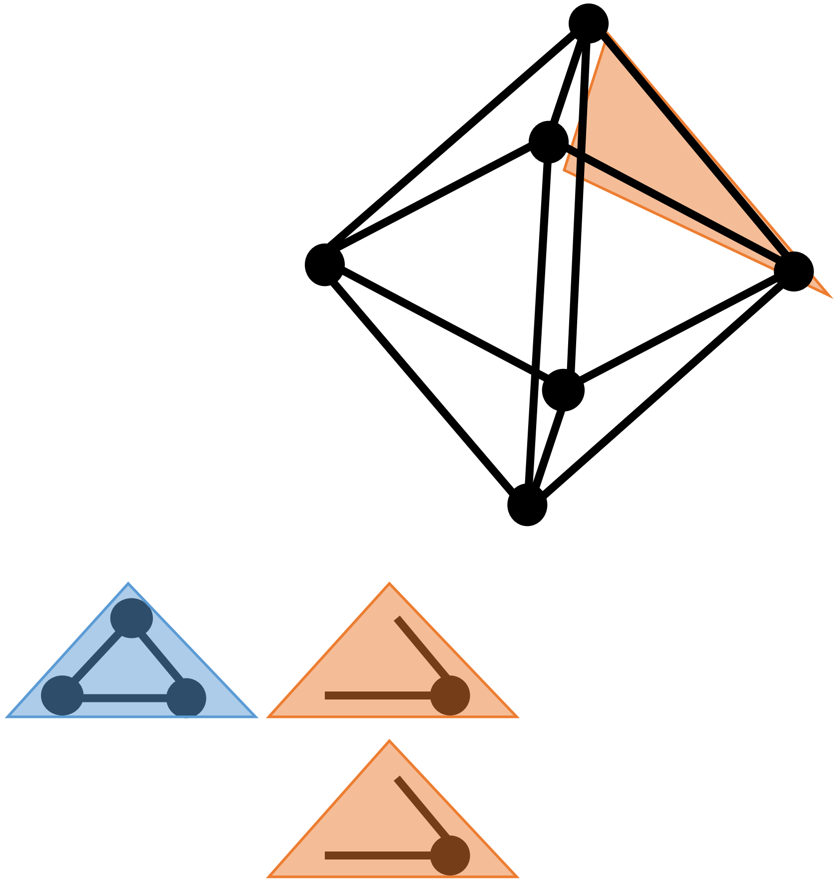




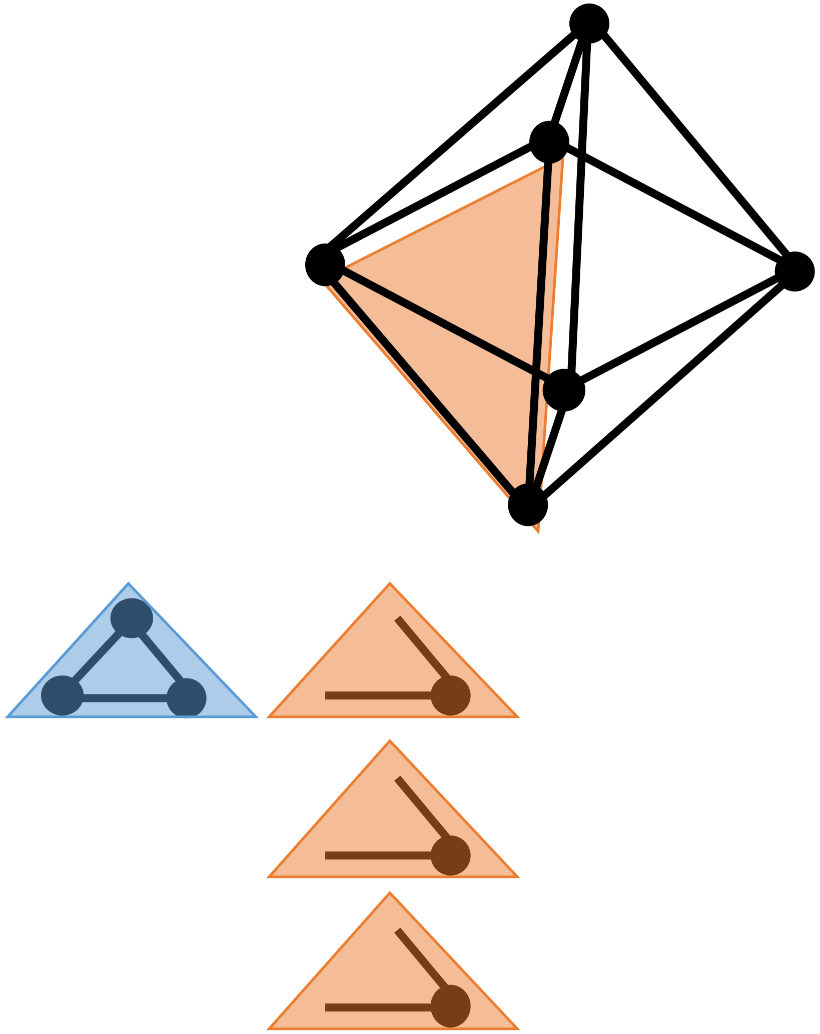
Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**



Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**



Pascal’s Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
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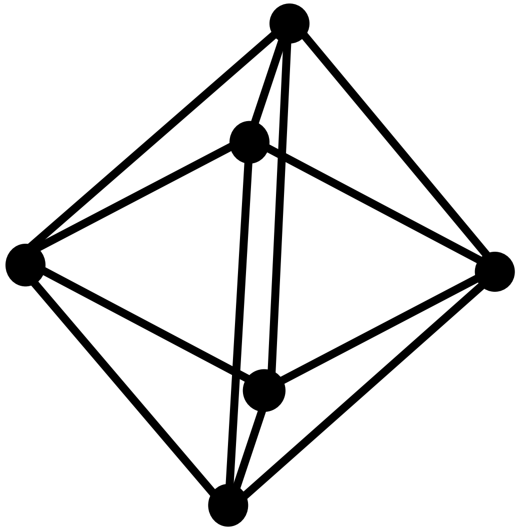
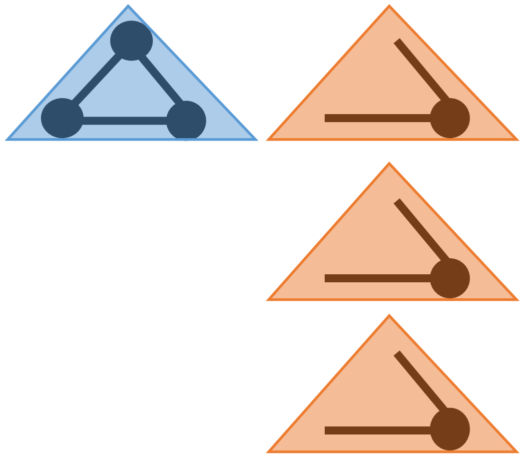
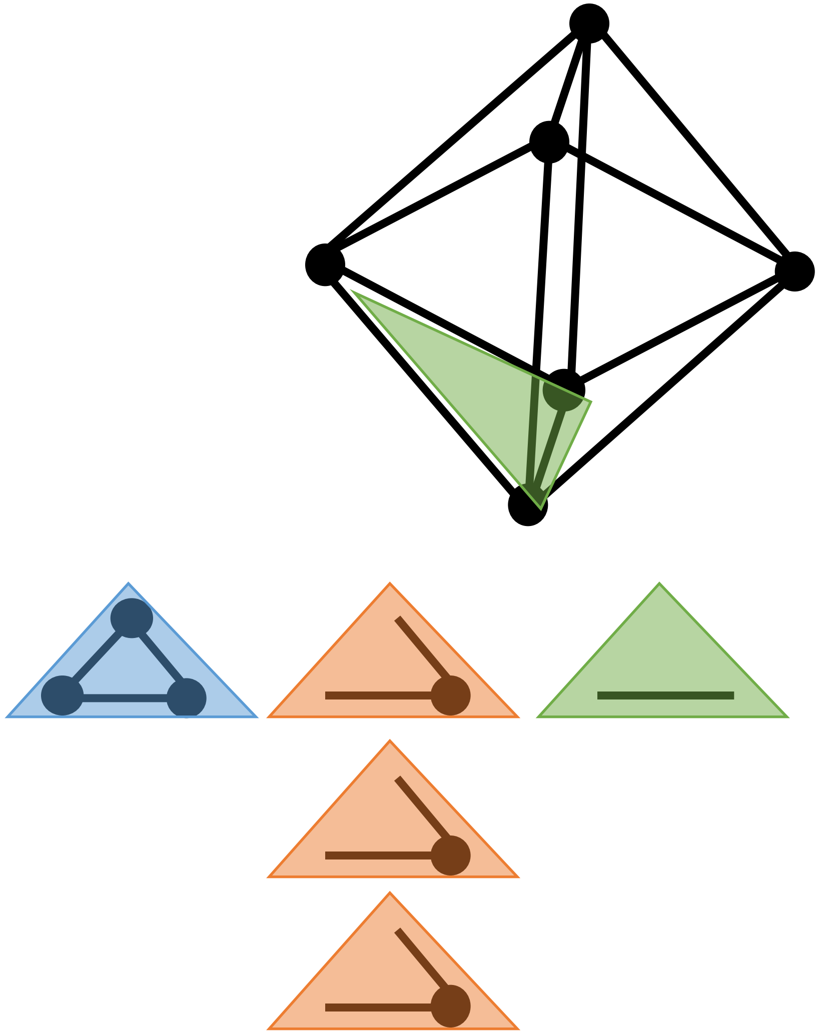




Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**



Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
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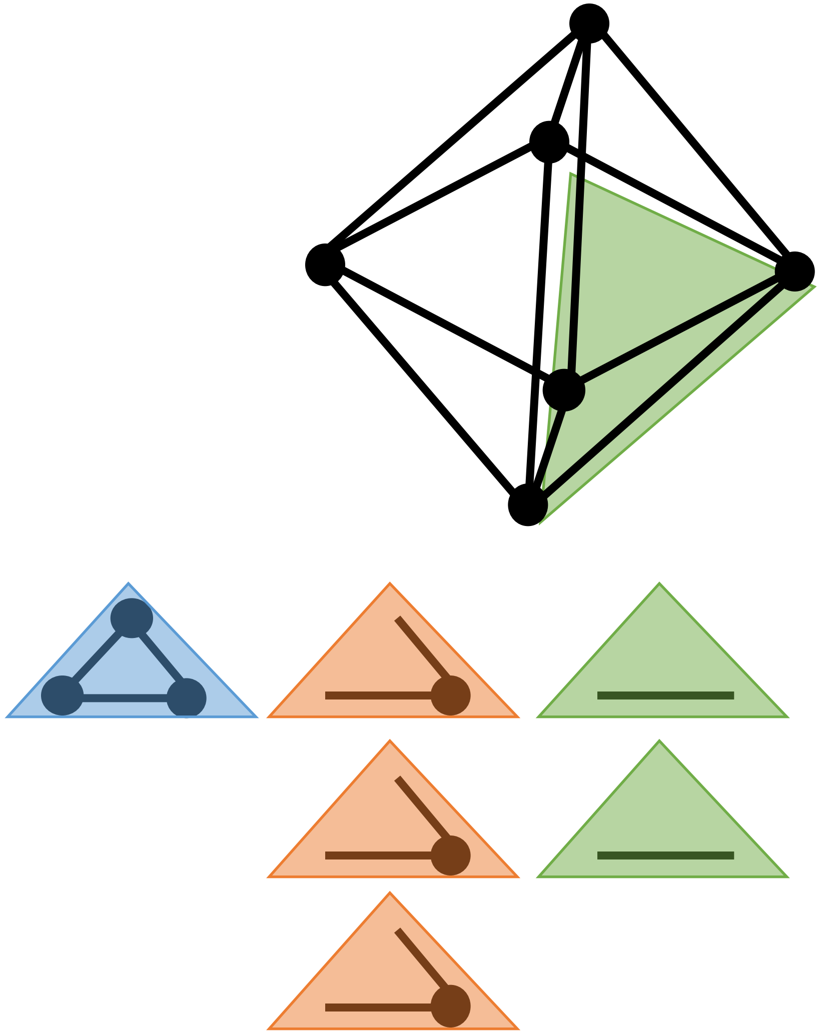
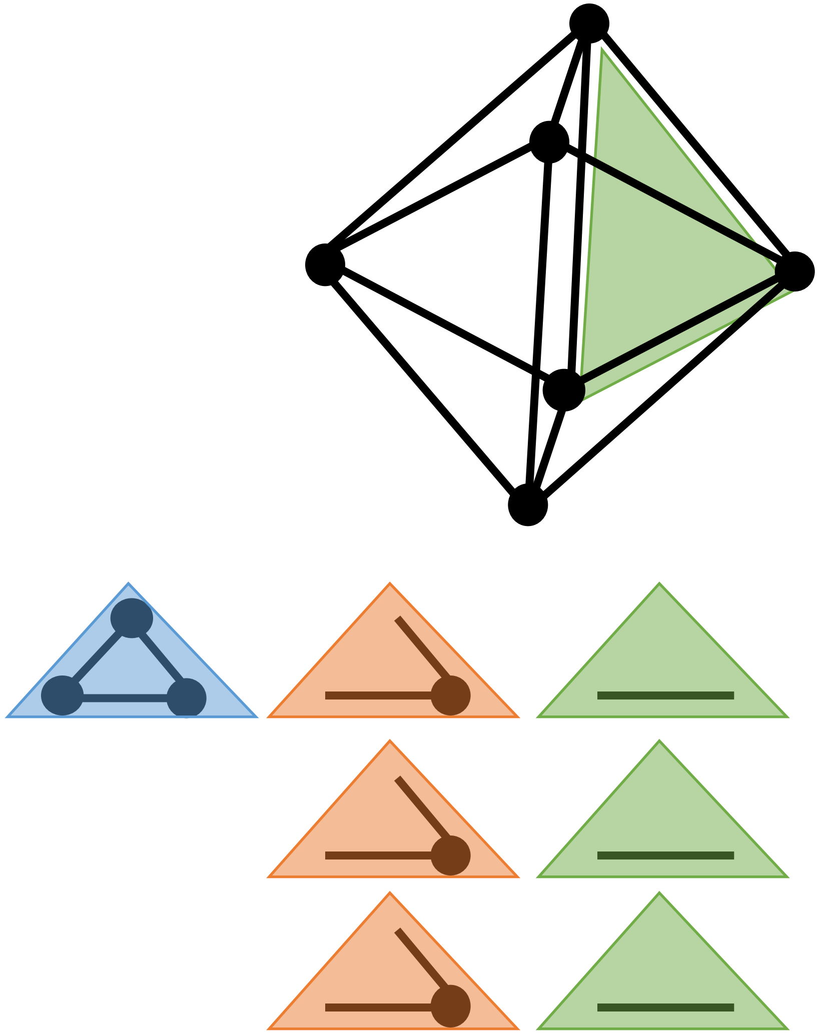


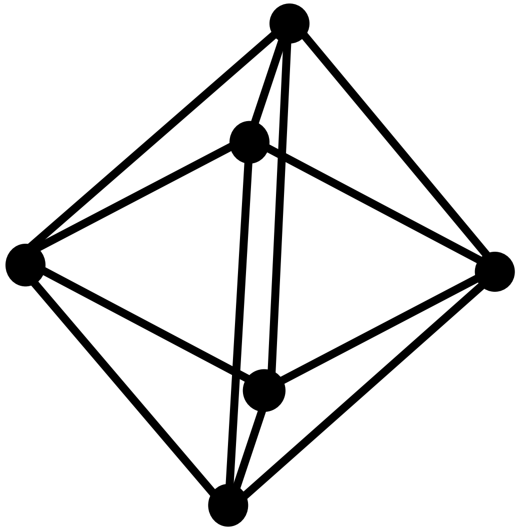
Pascal's Triangle

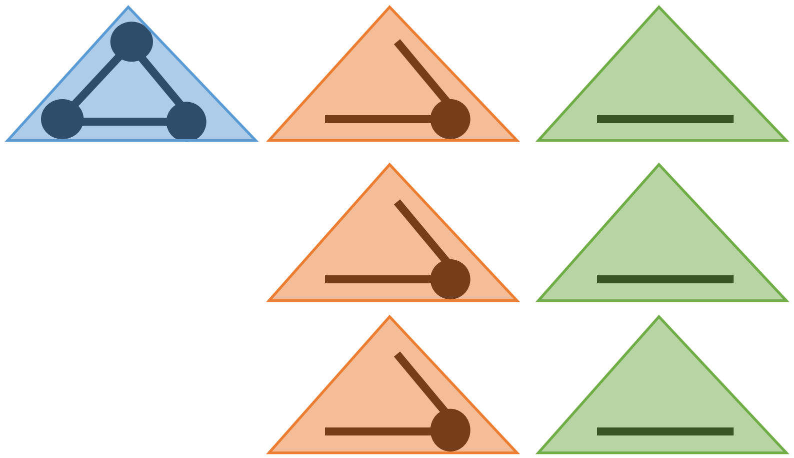
					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**



Pascal’s Triangle

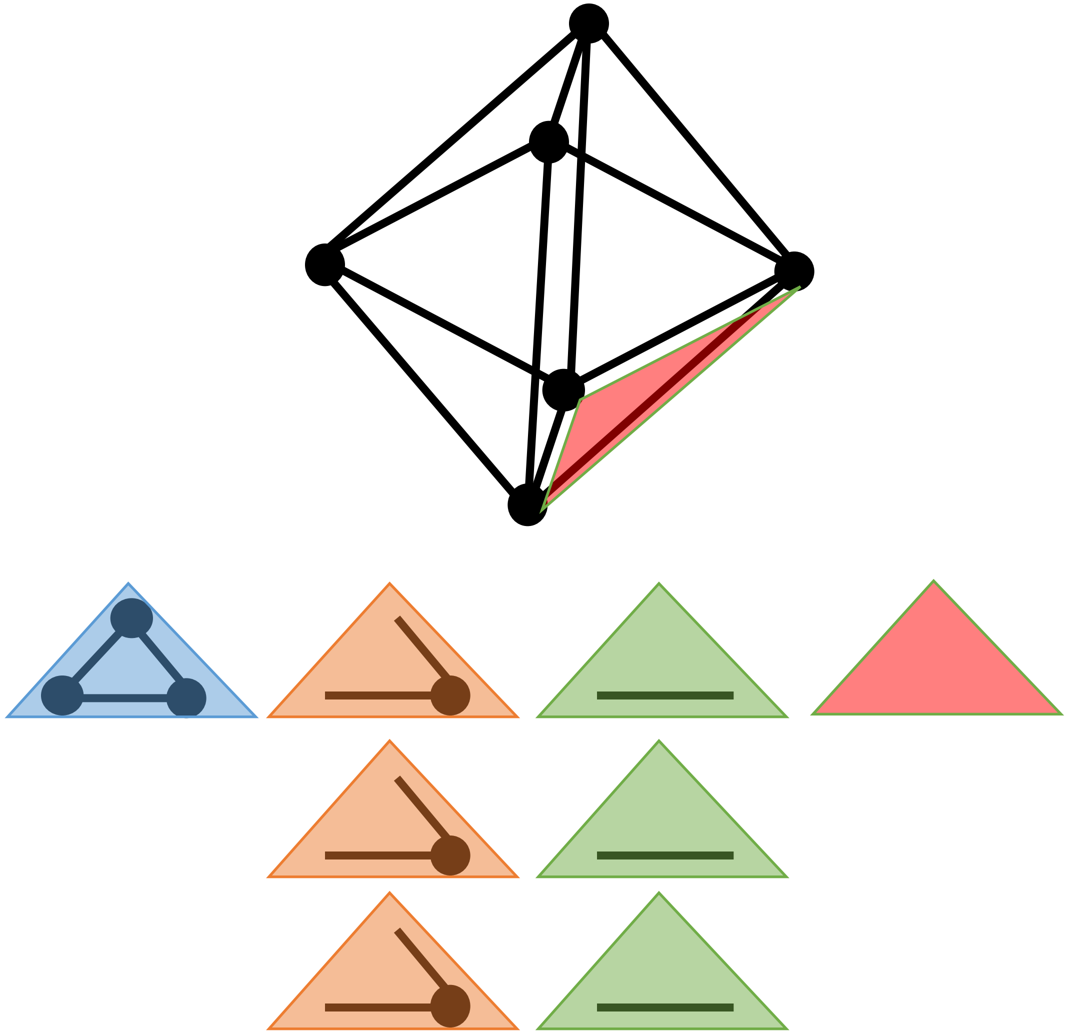
					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**





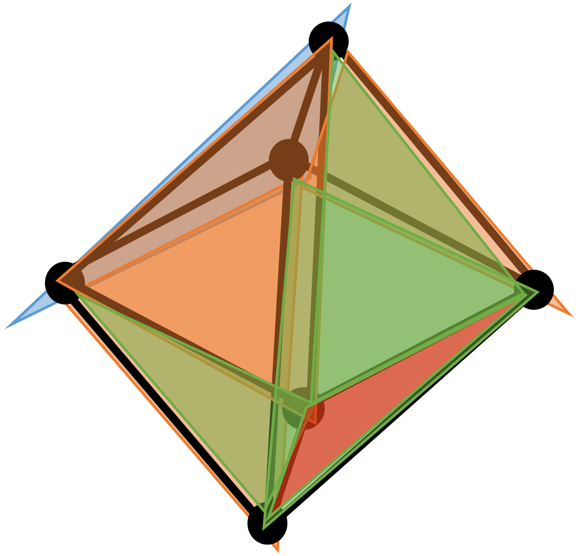
Pascal's Triangle

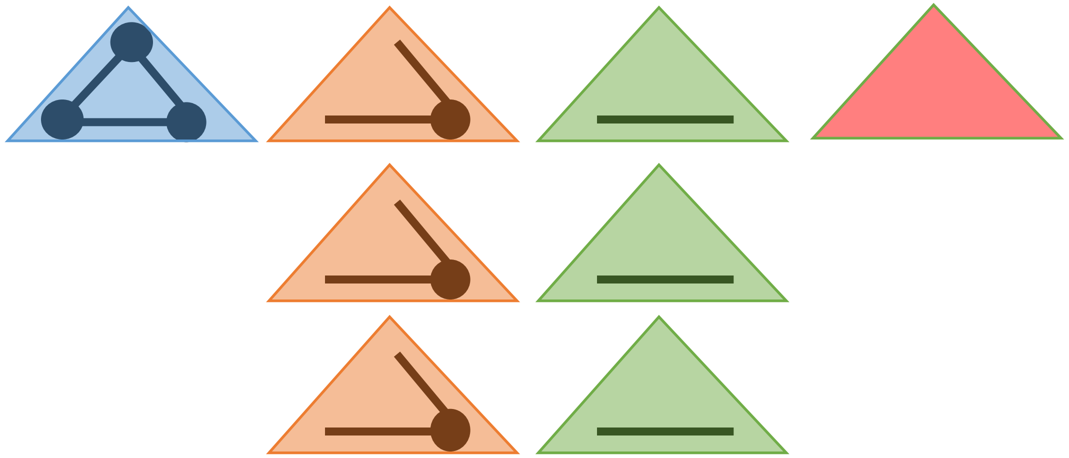
					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**



Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2	1			**4**
1	3	3	1		**8**
1	4	6	4	1	**16**

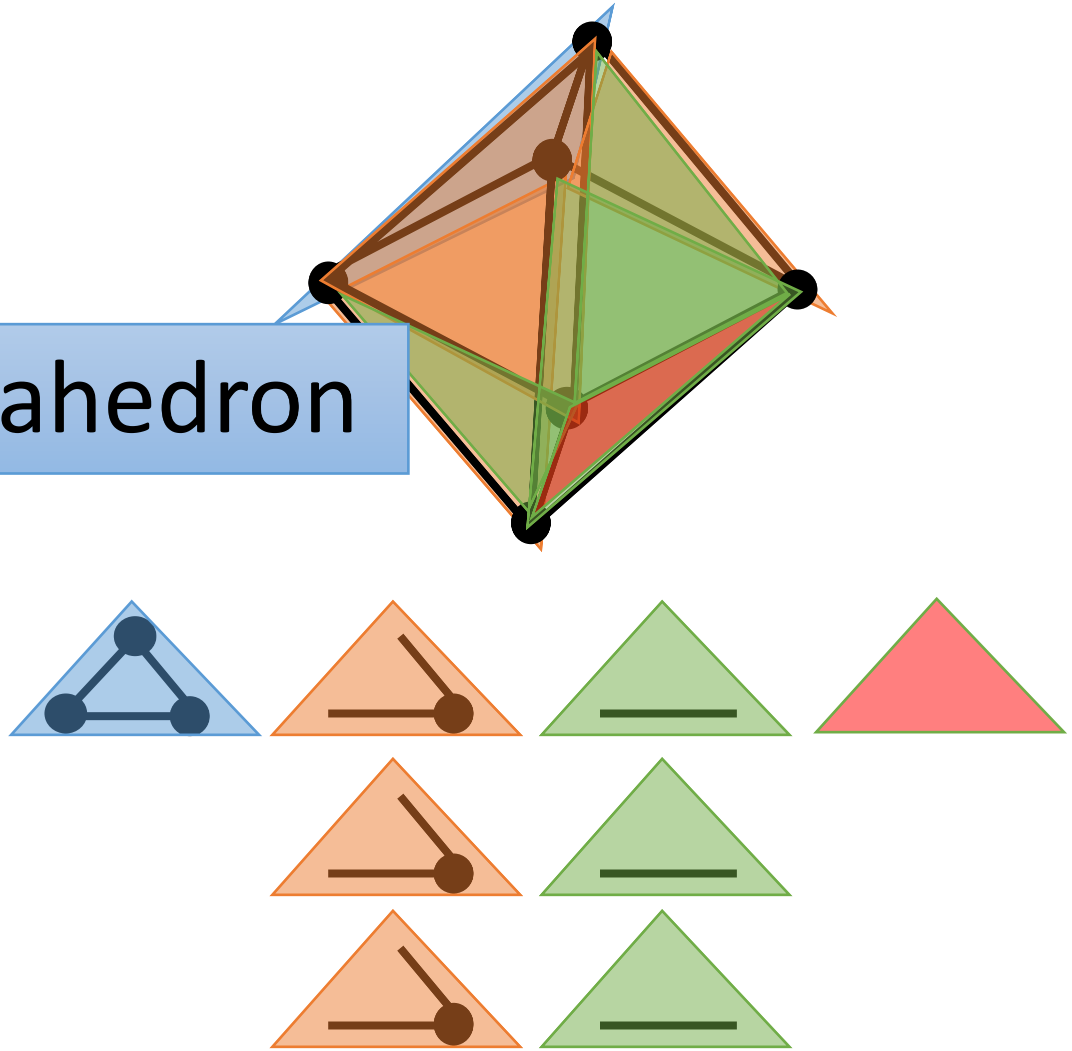




Pascal's Triangle

					Sums
1					**1**
1	1				**2**
1	2				
1	3	3	1		**8**
1	4	6	4	1	**16**

"Shelling" the octahedron



Euler's Triangle

1				
1	1			
1	4	1		
1	11	11	1	
1	26	66	26	1

Euler's Triangle

Sums

1				
1	1			
1	4	1		
1	11	11	1	
1	26	66	26	1

Euler’s Triangle

					Sums
1					**1**
1	1				
1	4	1			
1	11	11	1		
1	26	66	26	1	

Euler's Triangle

					Sums
1					**1**
1	1				**2**
1	4	1			
1	11	11	1		
1	26	66	26	1	

Euler's Triangle

					Sums
1					**1**
1	1				**2**
1	4	1			**6**
1	11	11	1		
1	26	66	26	1	

Euler's Triangle

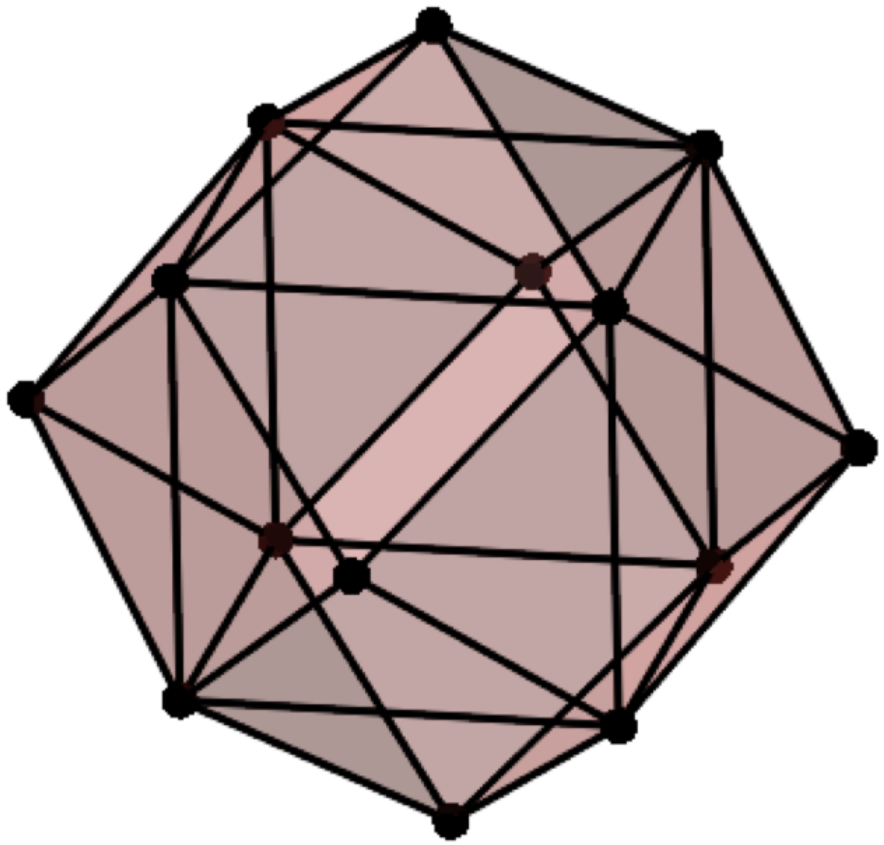
					Sums
1					**1**
1	1				**2**
1	4	1			**6**
1	11	11	1		**24**
1	26	66	26	1	

Euler’s Triangle

					Sums
1					**1**
1	1				**2**
1	4	1			**6**
1	11	11	1		**24**
1	26	66	26	1	**120**

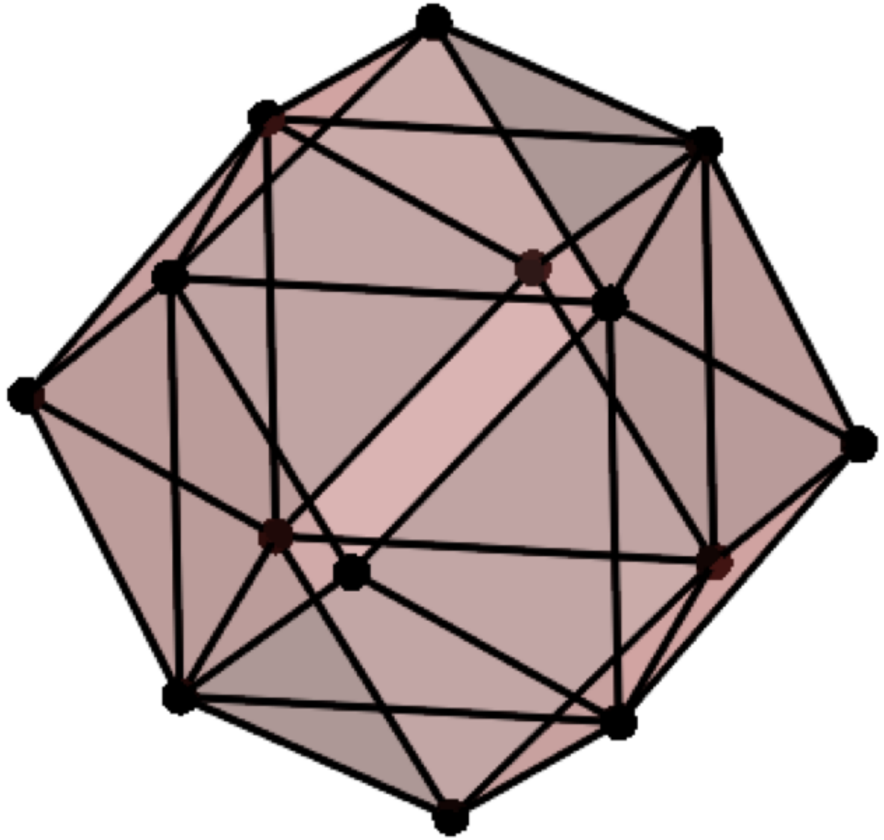
Euler's Triangle

					Sums
1					**1**
1	1				**2**
1	4	1			**6**
1	11	11	1		**24**
1	26	66	26	1	**120**



Euler's Triangle

					Sums
1					**1**
1	1				**2**
1	4	1			**6**
1	11	11	1		**24**
1	26	66	26	1	**120**



"Shelling" the Coxeter complex

Narayana’s Triangle

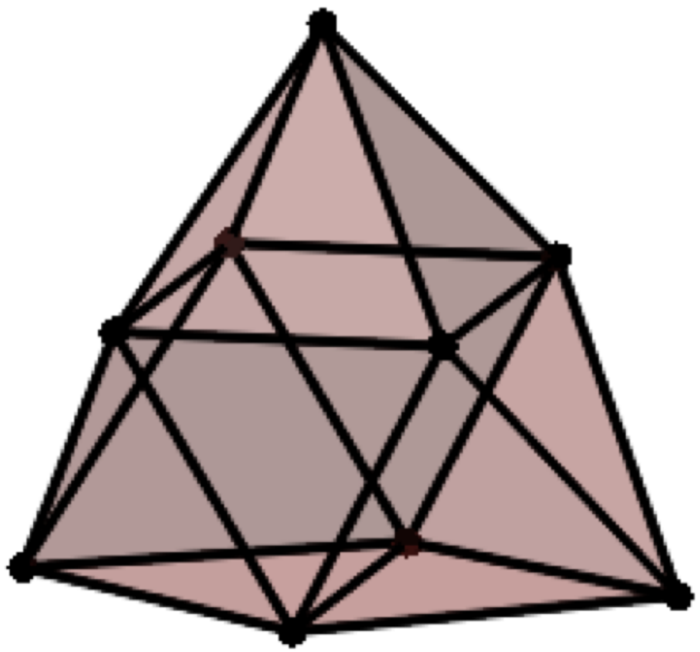
1				
1	1			
1	3	1		
1	6	6	1	
1	10	20	10	1

Narayana’s Triangle

1					**1**
1	1				**2**
1	3	1			**5**
1	6	6	1		**14**
1	10	20	10	1	**42**

Narayana's Triangle

1					**1**
1	1				**2**
1	3	1			**5**
1	6	6	1		**14**
1	10	20	10	1	**42**



"Shelling" the noncrossing complex

Combinatorial Topology

1

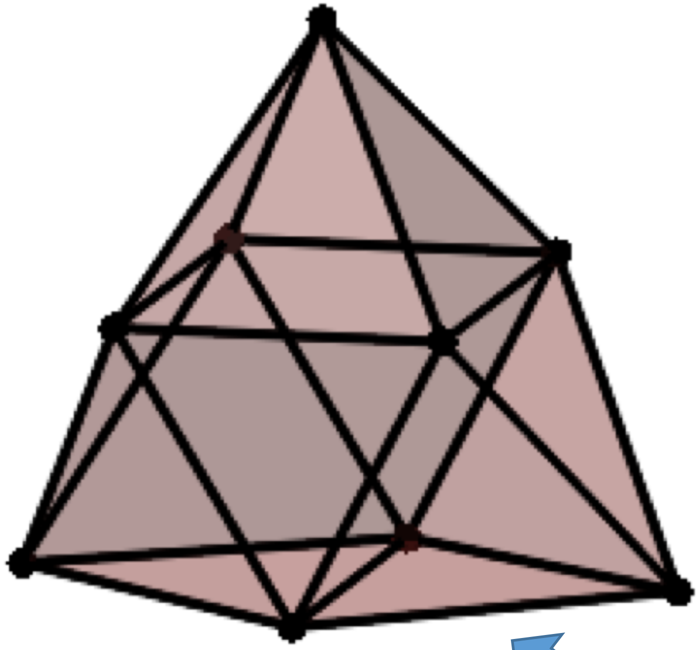
1 1

1 3 1

1 6 6 1

1 10 20 10 1

“h-vector”



“triangulated sphere”

Combinatorial Topology

1

1 1

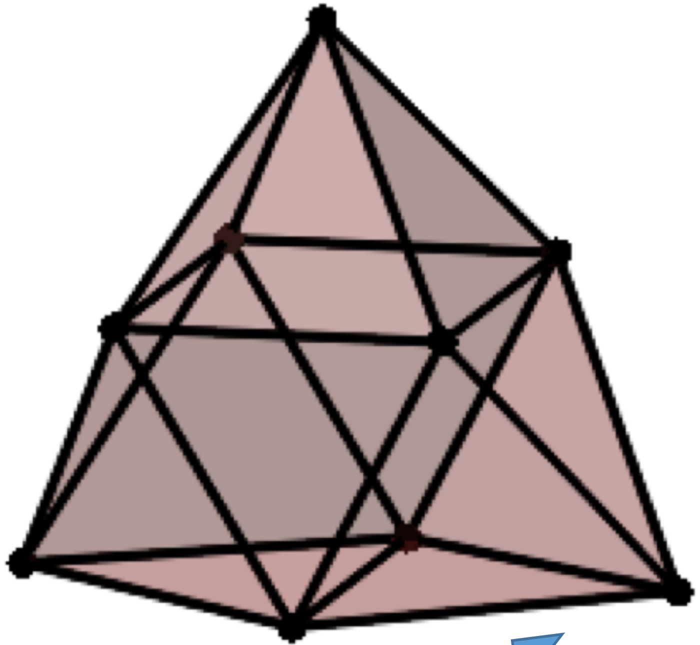
1 3 1

1 6 6 1

1 10 20 10 1

"h-vector"

Big fish: Hopf conjecture, Charney-Davis conjecture, g-conjecture…



"triangulated sphere"

Back to Alice and Bob…

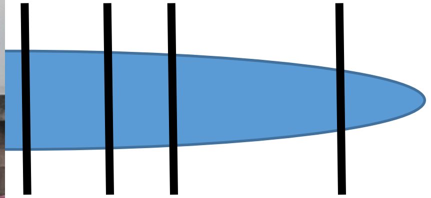
Version I: Submarine Sandwich

Version II: Submarine Sandwich (even number)

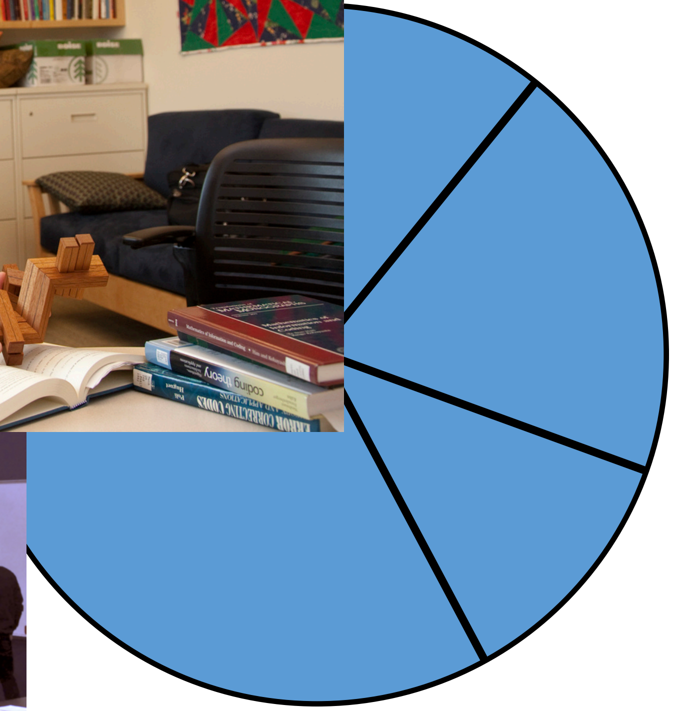
Version III: Pizza

Alice and Bob have lunch

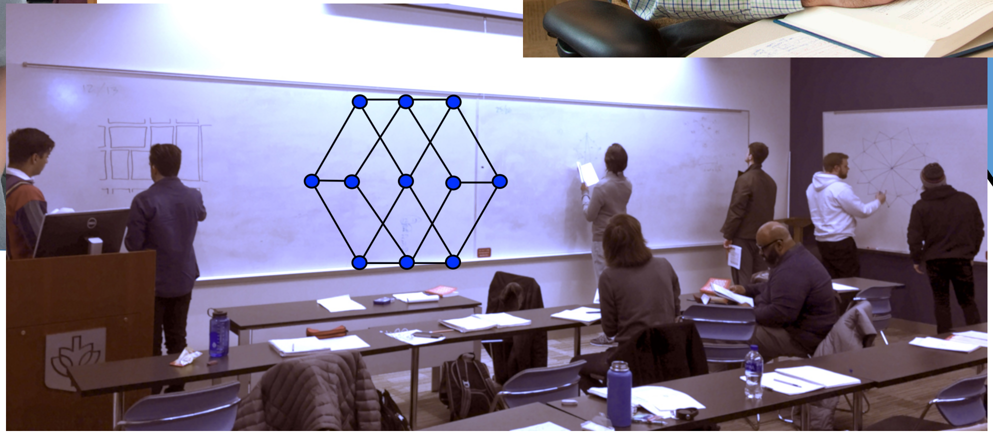
marine Sandv



l: Pizza

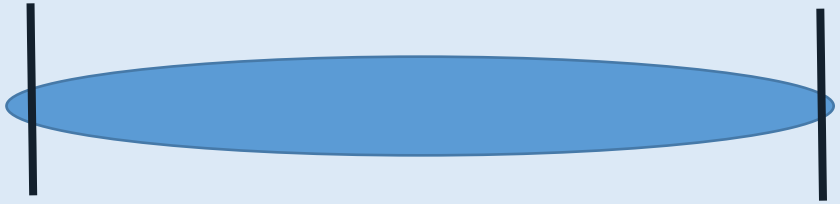


marine Sand

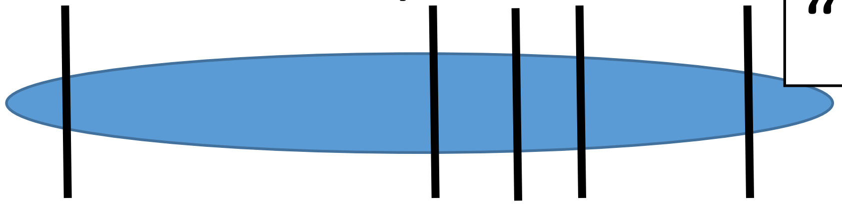


Alice and Bob have lunch

Version I: Submarine Sandwich



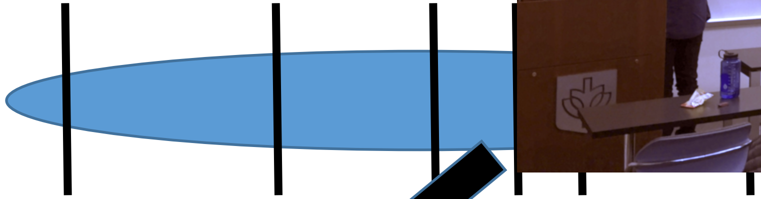
Version II: Submarine Sandwich (even number)

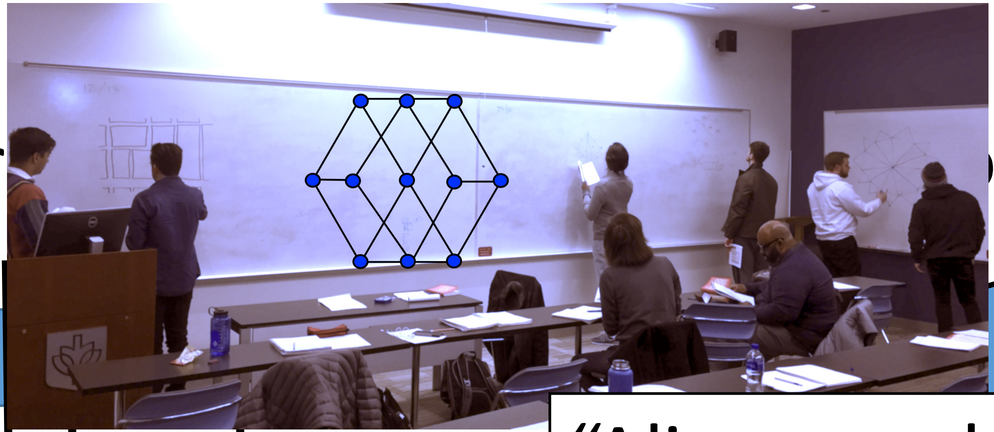


"Bob can get as much as he wants!"

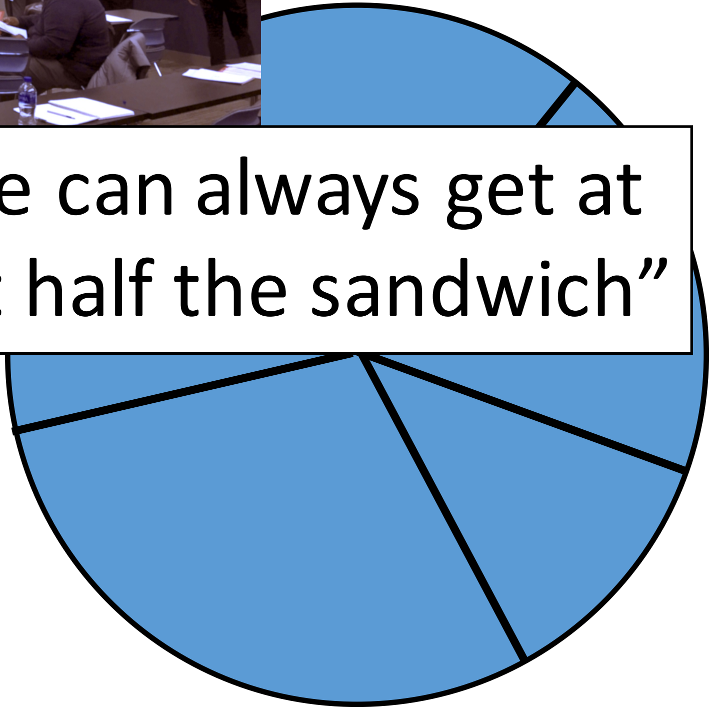
Alice and Bob have lunch

Version I: Submar



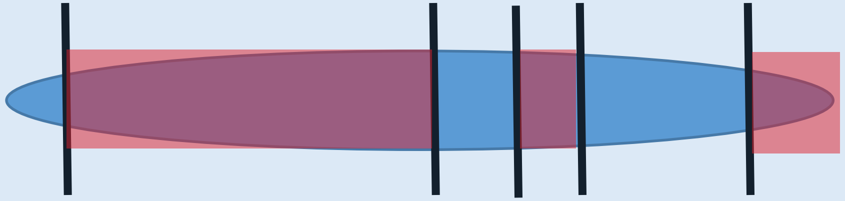


n III: Pizza



"Alice can always get at least half the sandwich"

Version II: Submarine Sandwich (even number)

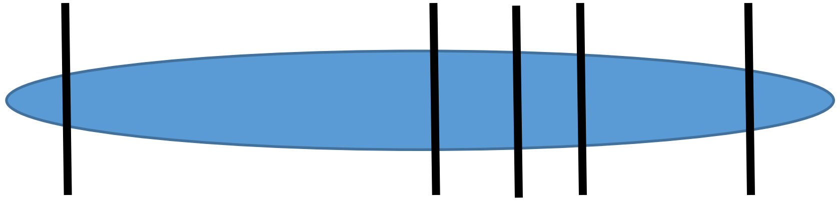


Alice and Bob have lunch

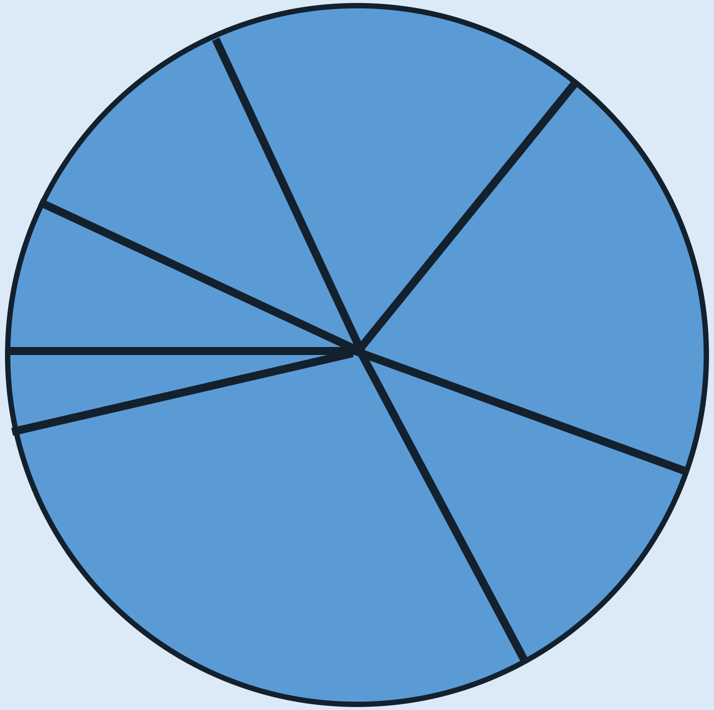
Ver wich

Ver dwich

(eve)



Version III: Pizza



Alice and Bob have lunch

Ver wich

Ver dwich

“Bob can get 5/9 of the pizza”
(…??!!?)

Version III: Pizza