

Alice and Bob have lunch

T. Kyle Petersen, PhD

Department of Mathematical Sciences

Alice and Bob, I

Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich



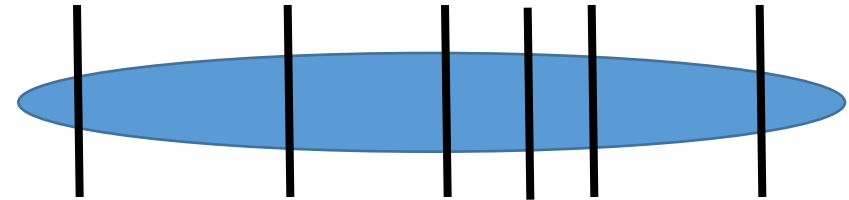
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces



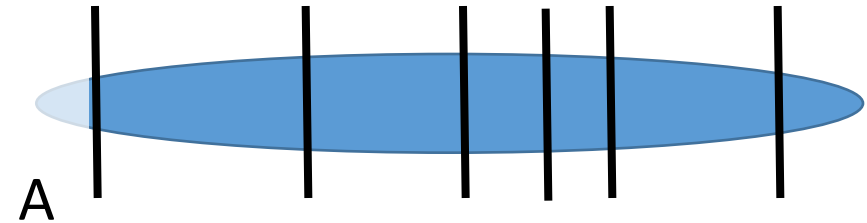
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces



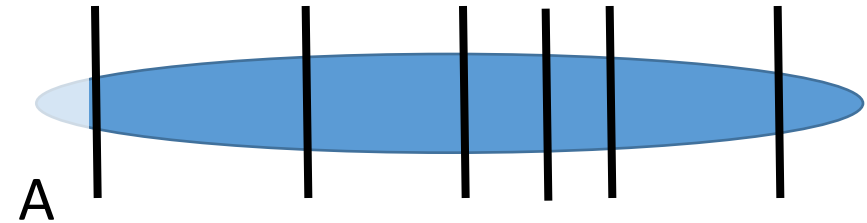
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end



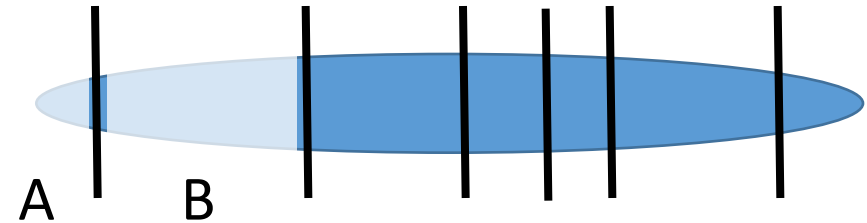
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end
- They alternate choosing end pieces



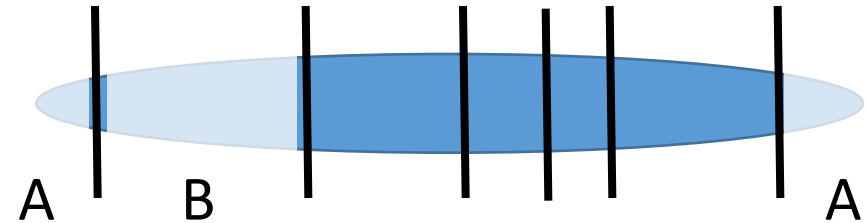
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end
- They alternate choosing end pieces



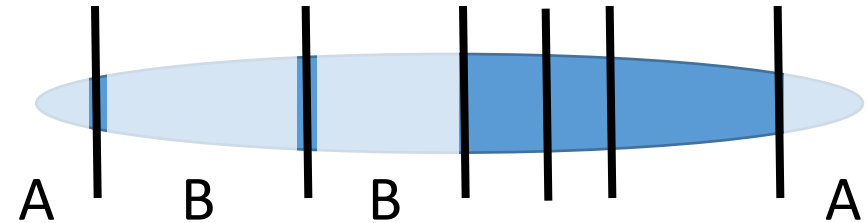
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end
- They alternate choosing end pieces



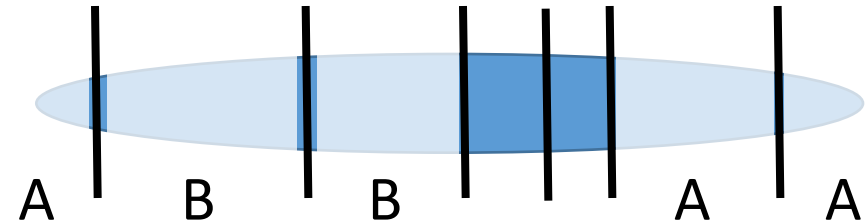
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end
- They alternate choosing end pieces



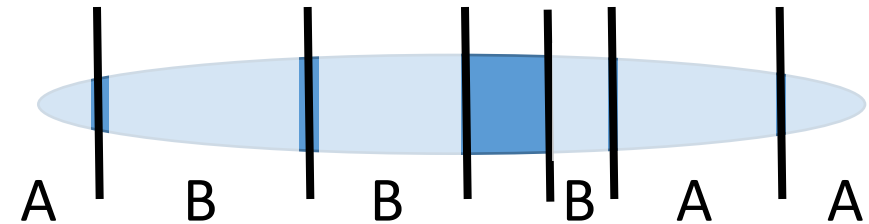
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end
- They alternate choosing end pieces



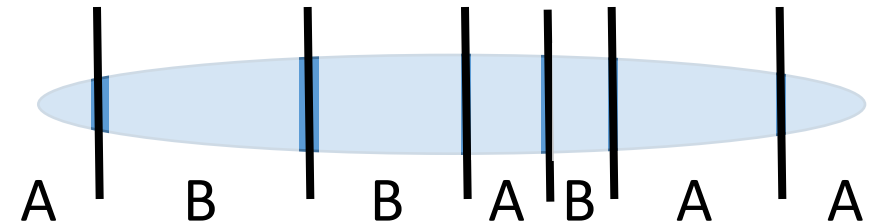
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end
- They alternate choosing end pieces



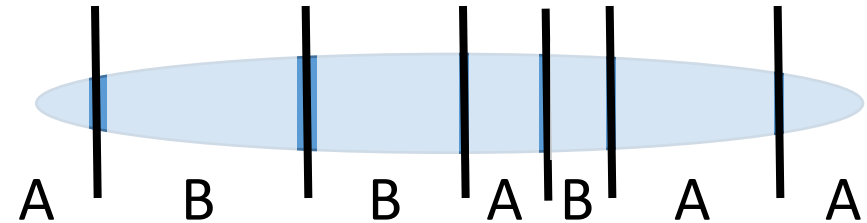
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end
- They alternate choosing end pieces



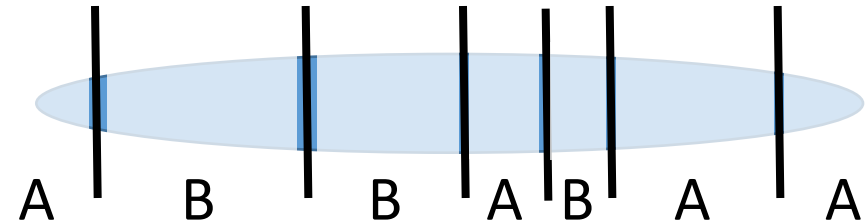
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end
- They alternate choosing end pieces
- Alice wins if she gets **at least** half



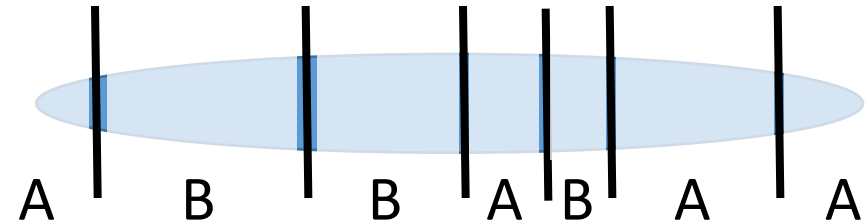
Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end
- They alternate choosing end pieces
- Alice wins if she gets **at least** half
- Bob wins if he gets **more** than half



Alice and Bob, I

- Alice and Bob are sharing a submarine sandwich
- Bob cuts it into pieces
- Alice chooses first piece from one end
- They alternate choosing end pieces
- Alice wins if she gets **at least** half
- Bob wins if he gets **more** than half

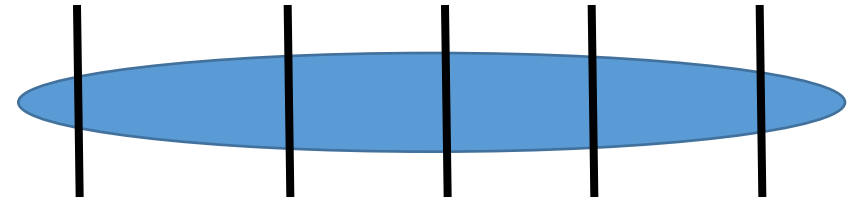


Who wins?

Alice and Bob, II

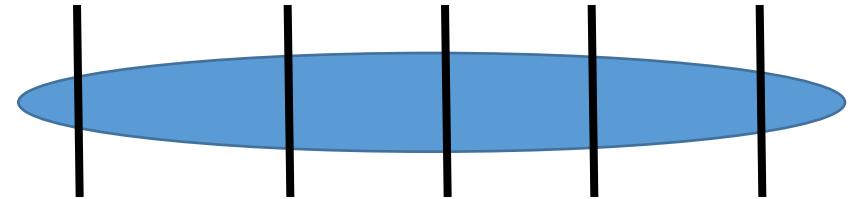
Alice and Bob, II

- Just like first Alice and Bob question, except Bob **must cut it into an even number of pieces**



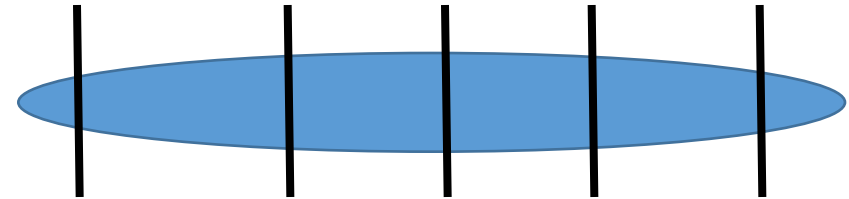
Alice and Bob, II

- Just like first Alice and Bob question, except Bob **must cut it into an even number of pieces**
- Alice wins if she gets **at least half**



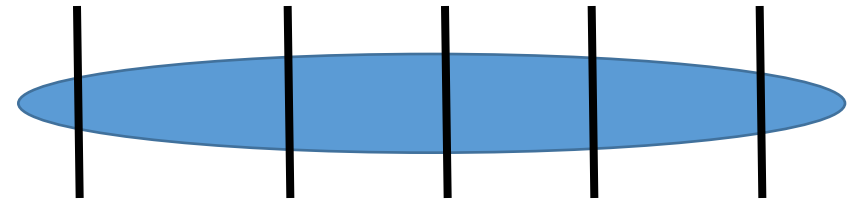
Alice and Bob, II

- Just like first Alice and Bob question, except Bob **must cut it into an even number of pieces**
- Alice wins if she gets **at least** half
- Bob wins if he gets **more** than half



Alice and Bob, II

- Just like first Alice and Bob question, except Bob **must cut it into an even number of pieces**
- Alice wins if she gets **at least** half
- Bob wins if he gets **more** than half

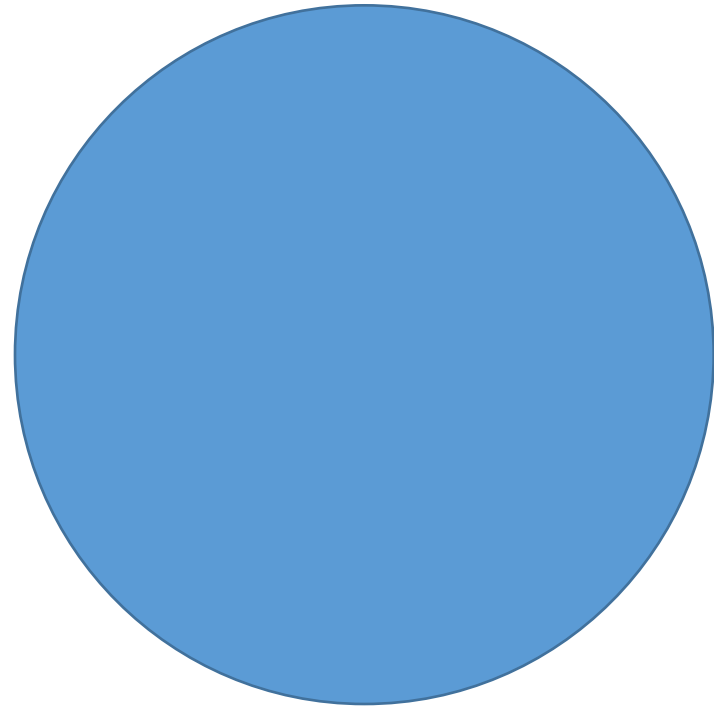


Who wins?

Alice and Bob, III

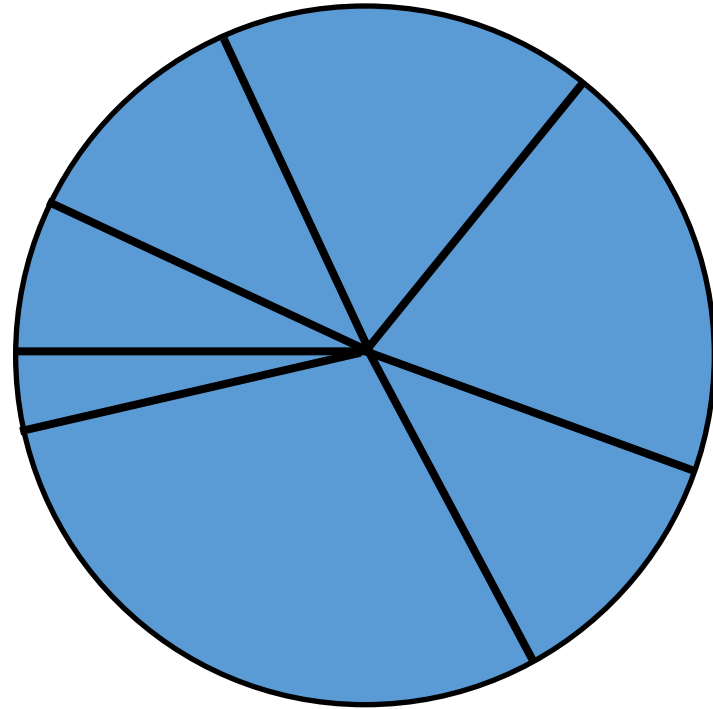
Alice and Bob, III

- Alice and Bob are sharing a circular pizza



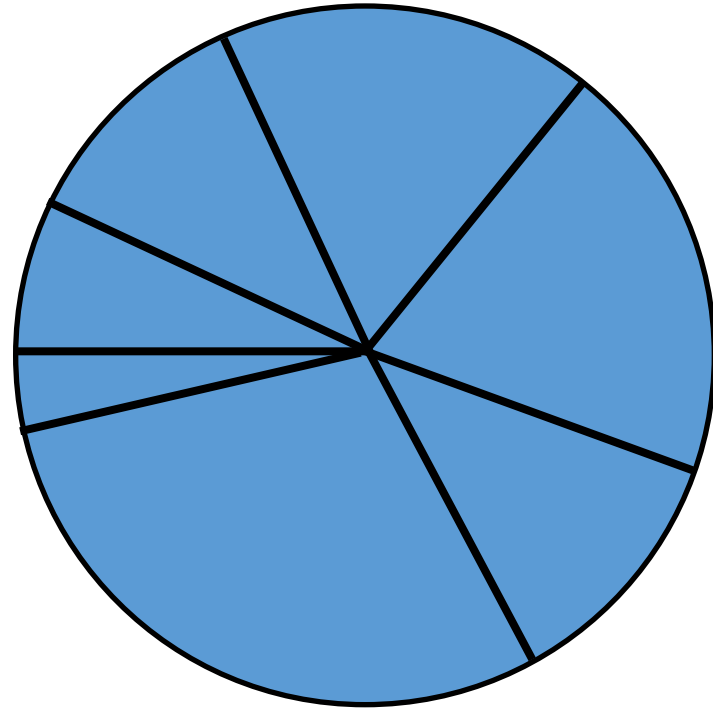
Alice and Bob, III

- Alice and Bob are sharing a circular pizza
- Bob cuts it into radial slices



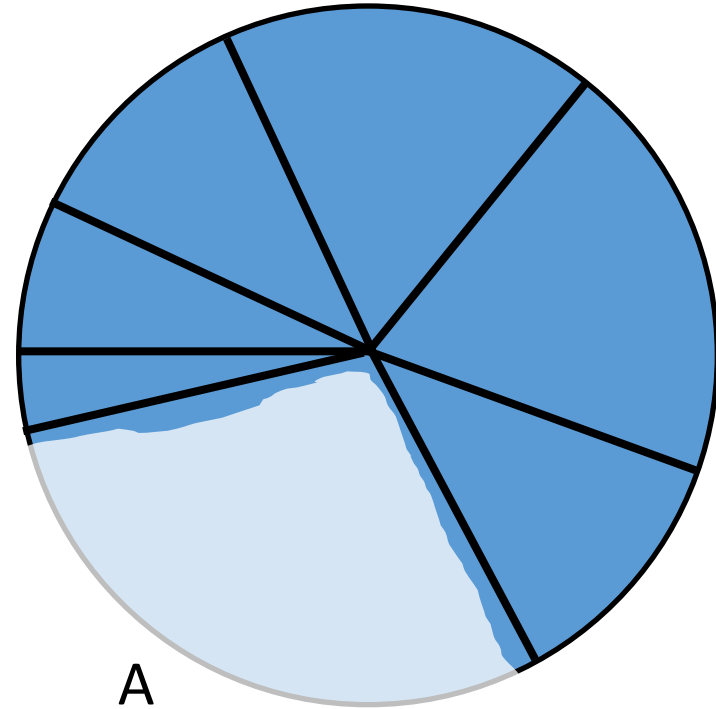
Alice and Bob, III

- Alice and Bob are sharing a circular pizza
- Bob cuts it into radial slices
- Alice chooses first slice



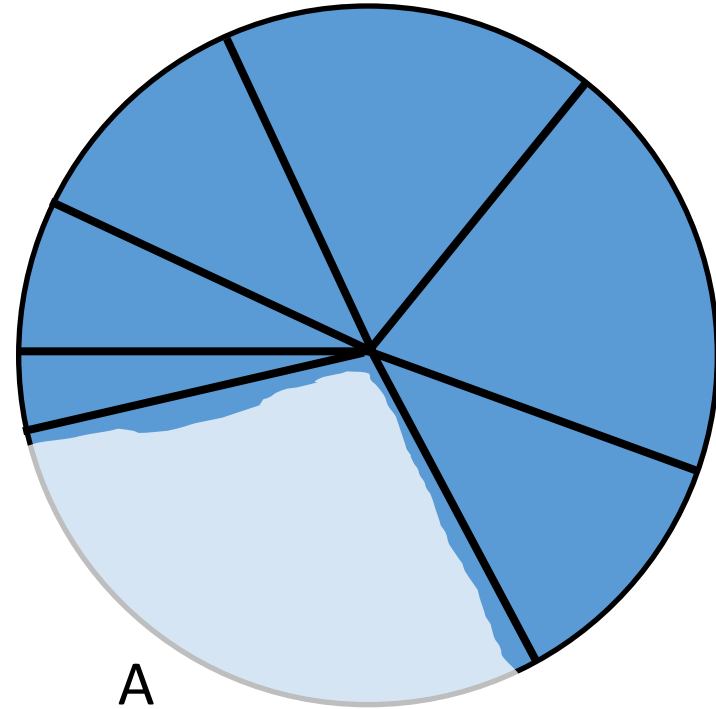
Alice and Bob, III

- Alice and Bob are sharing a circular pizza
- Bob cuts it into radial slices
- Alice chooses first slice



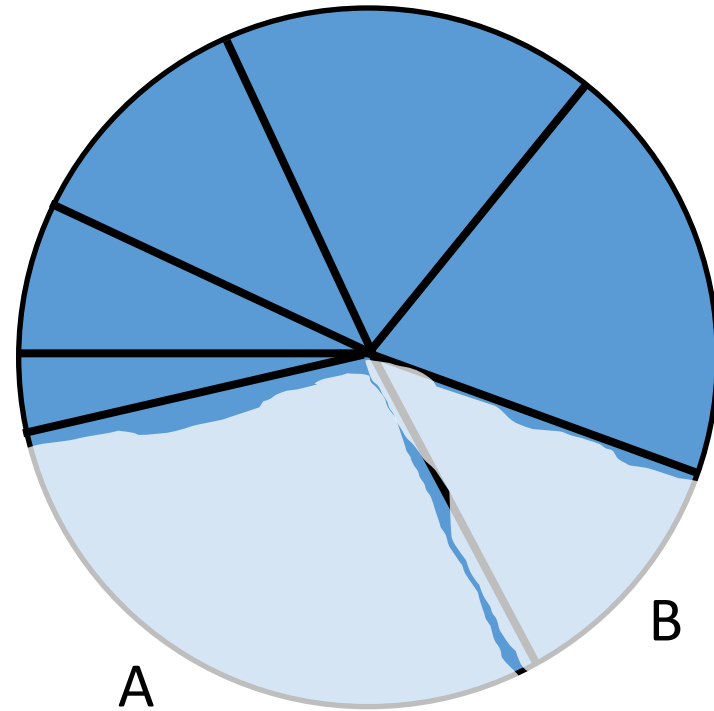
Alice and Bob, III

- Alice and Bob are sharing a circular pizza
- Bob cuts it into radial slices
- Alice chooses first slice
- They take turns from opening



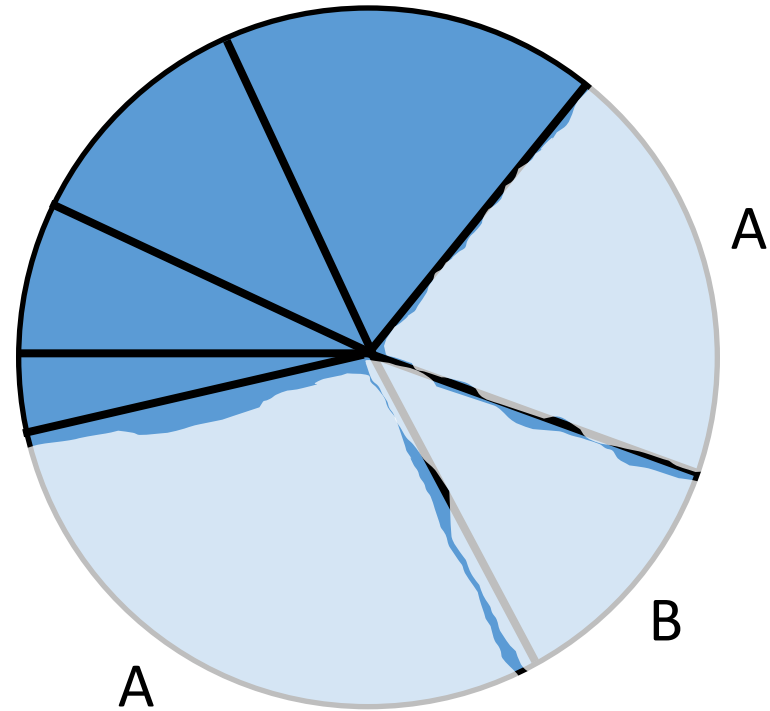
Alice and Bob, III

- Alice and Bob are sharing a circular pizza
- Bob cuts it into radial slices
- Alice chooses first slice
- They take turns from opening



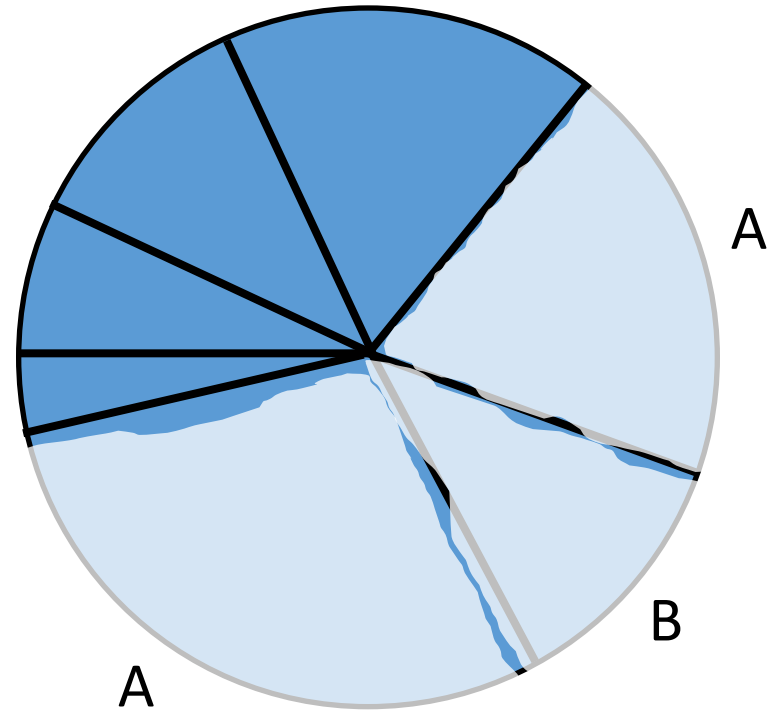
Alice and Bob, III

- Alice and Bob are sharing a circular pizza
- Bob cuts it into radial slices
- Alice chooses first slice
- They take turns from opening



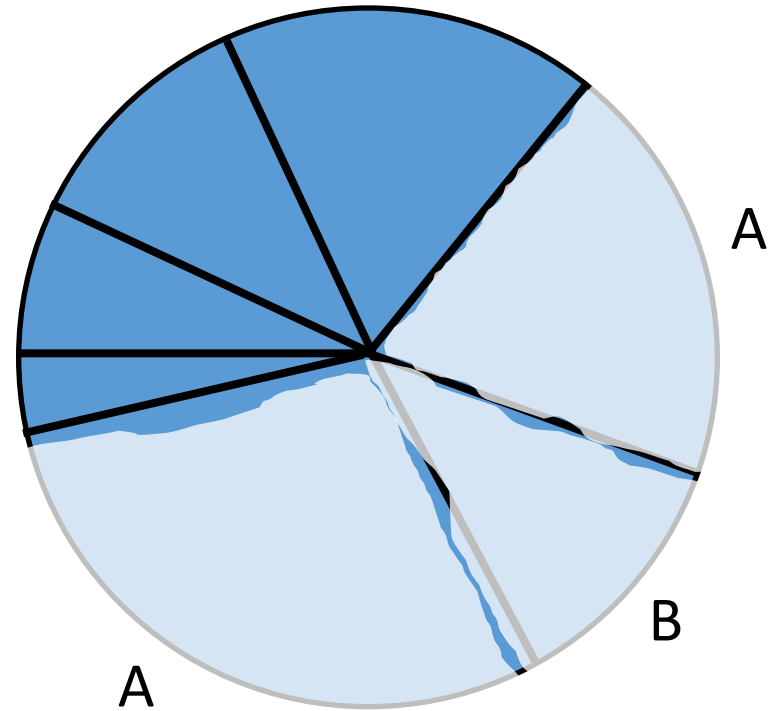
Alice and Bob, III

- Alice and Bob are sharing a circular pizza
- Bob cuts it into radial slices
- Alice chooses first slice
- They take turns from opening
- Alice wins if she gets **at least** half



Alice and Bob, III

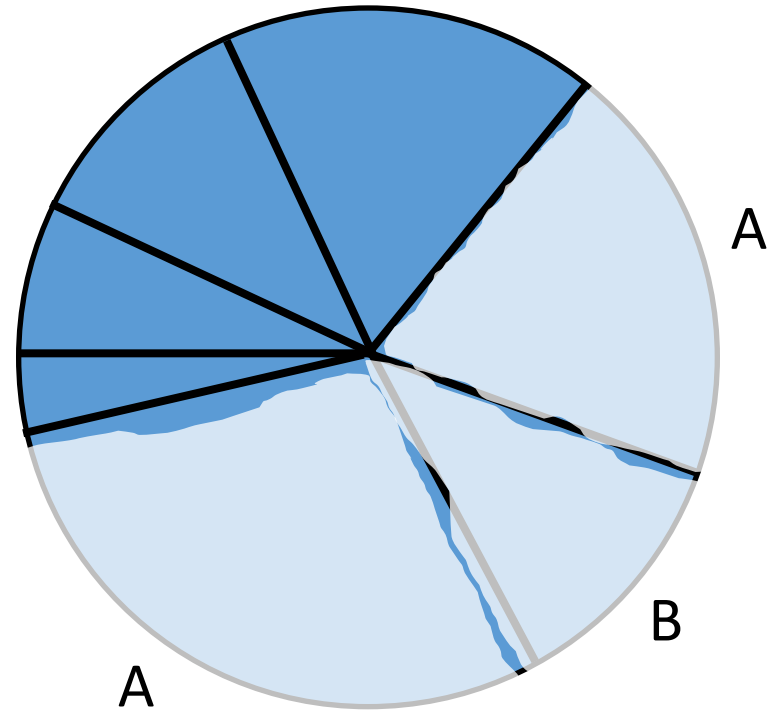
- Alice and Bob are sharing a circular pizza
- Bob cuts it into radial slices
- Alice chooses first slice
- They take turns from opening
- Alice wins if she gets **at least** half
- Bob wins if he gets **more** than half



Alice and Bob, III

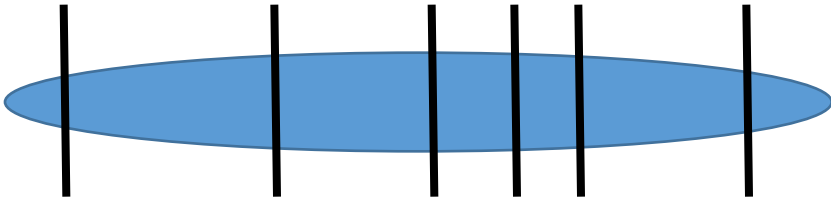
- Alice and Bob are sharing a circular pizza
- Bob cuts it into radial slices
- Alice chooses first slice
- They take turns from opening
- Alice wins if she gets **at least** half
- Bob wins if he gets **more** than half

Who wins?

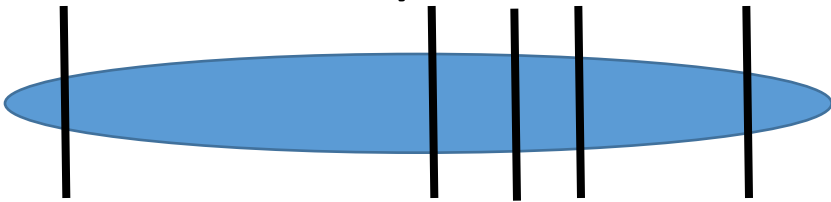


Alice and Bob have lunch

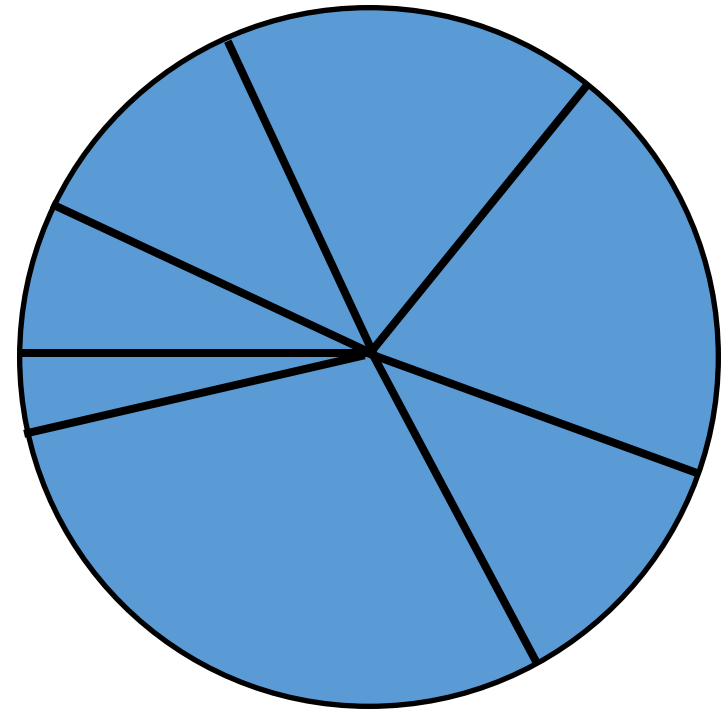
Version I: Submarine Sandwich



Version II: Submarine Sandwich
(even number)



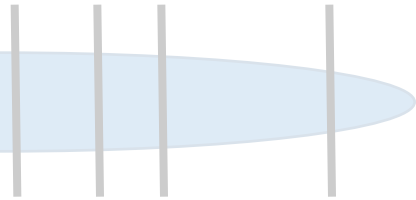
Version III: Pizza



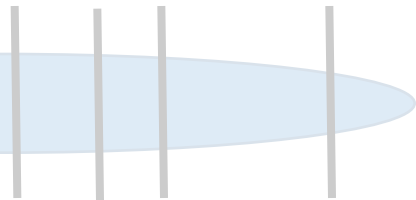
Alice and Bob have lunch



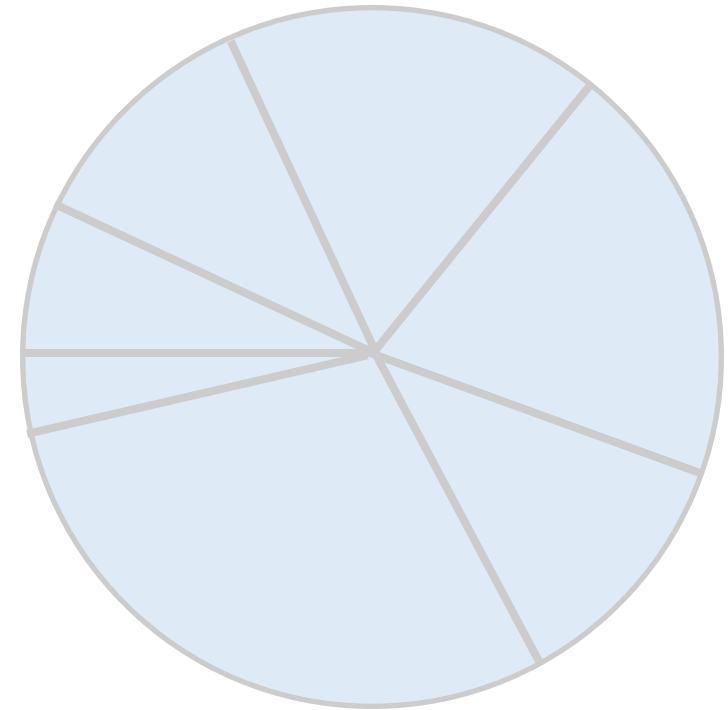
marine Sandwich



marine Sandwich



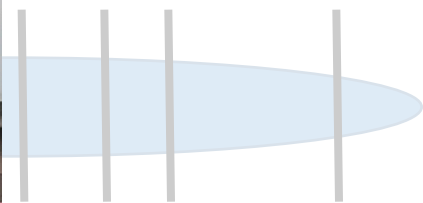
Version III: Pizza



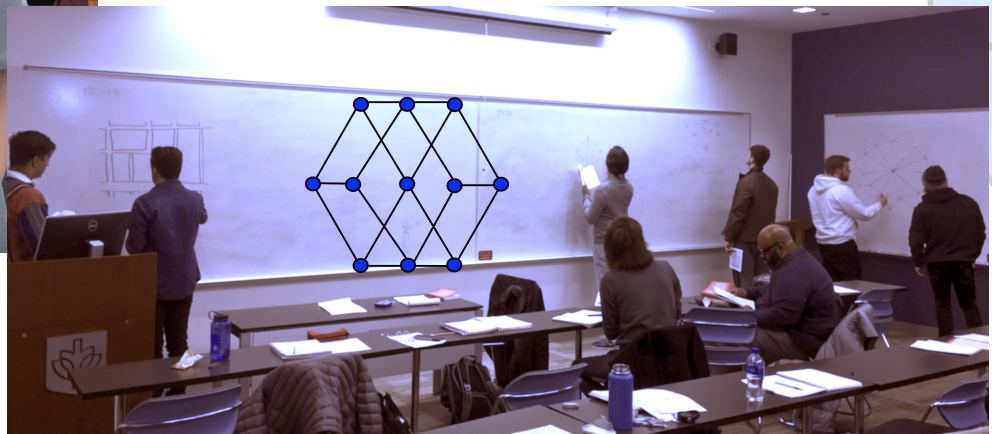
Alice and Bob have lunch



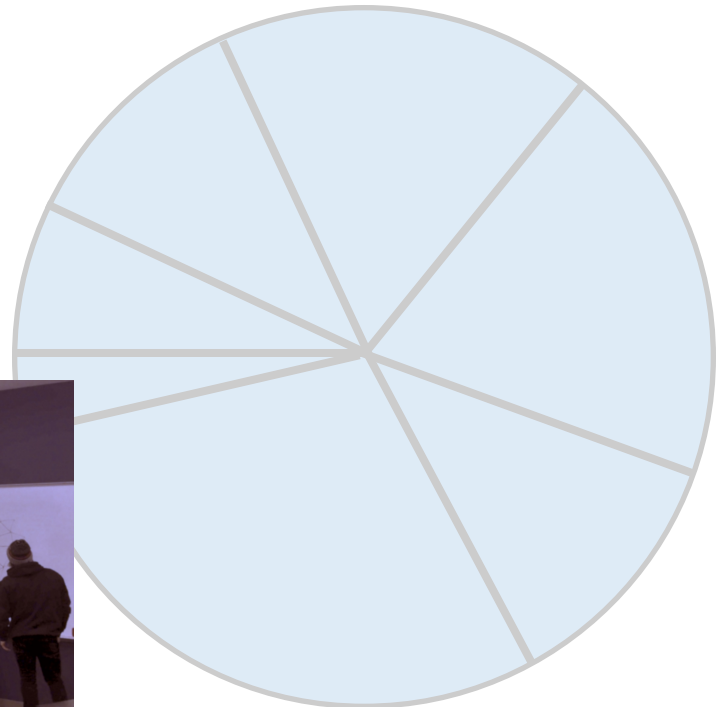
marine Sandwich



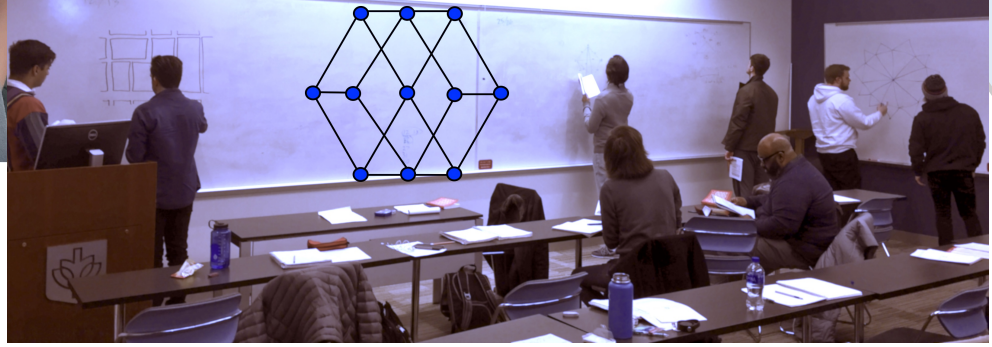
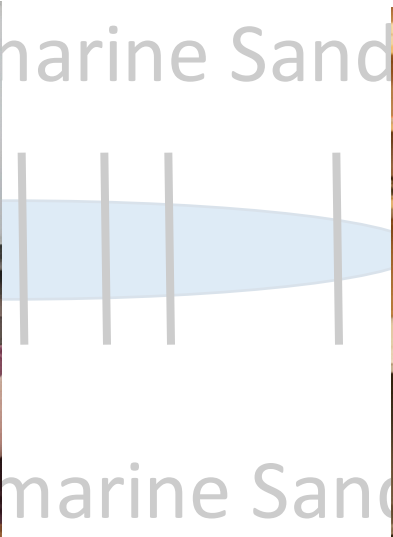
marine Sandwich



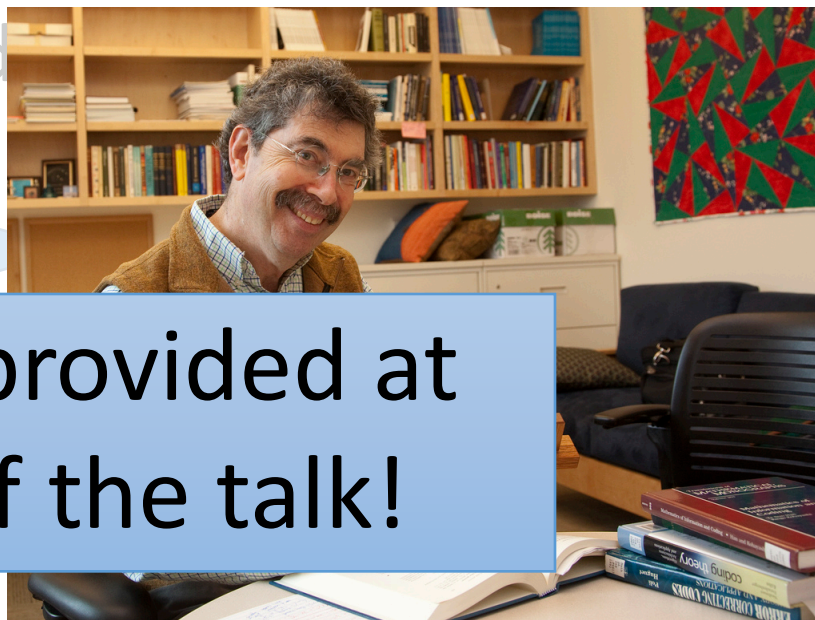
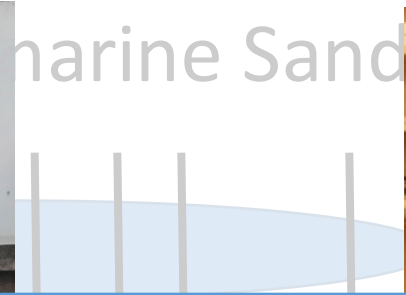
Version III: Pizza



Alice and Bob have lunch

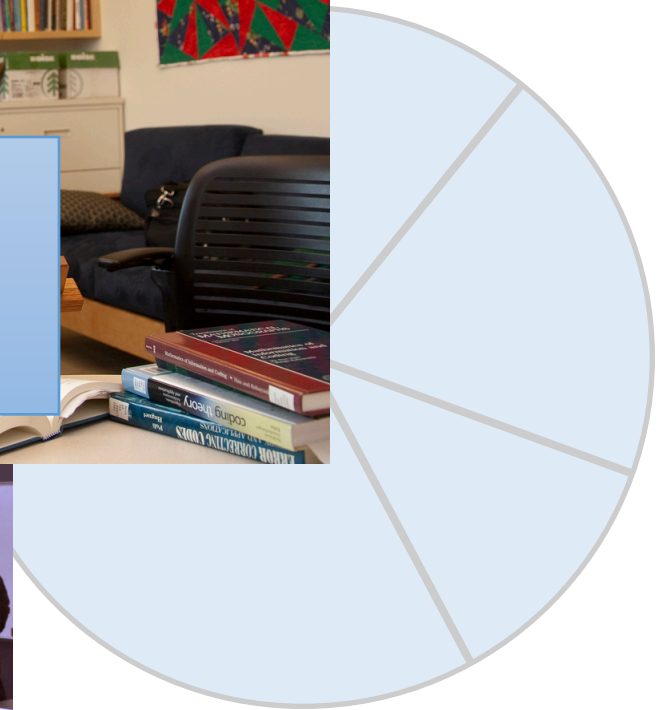
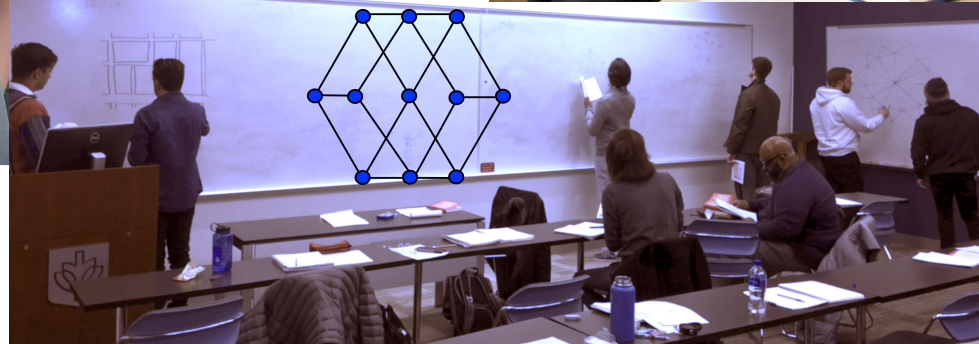


Alice and Bob have lunch



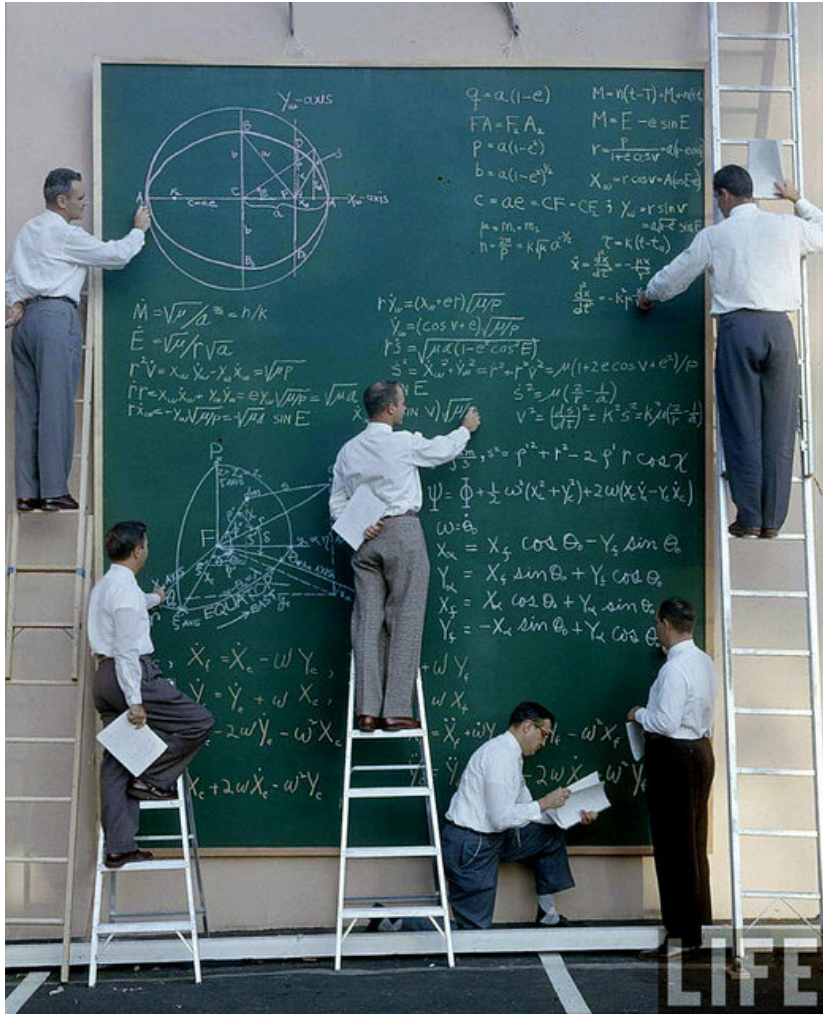
l: Pizza

Answers provided at
the end of the talk!

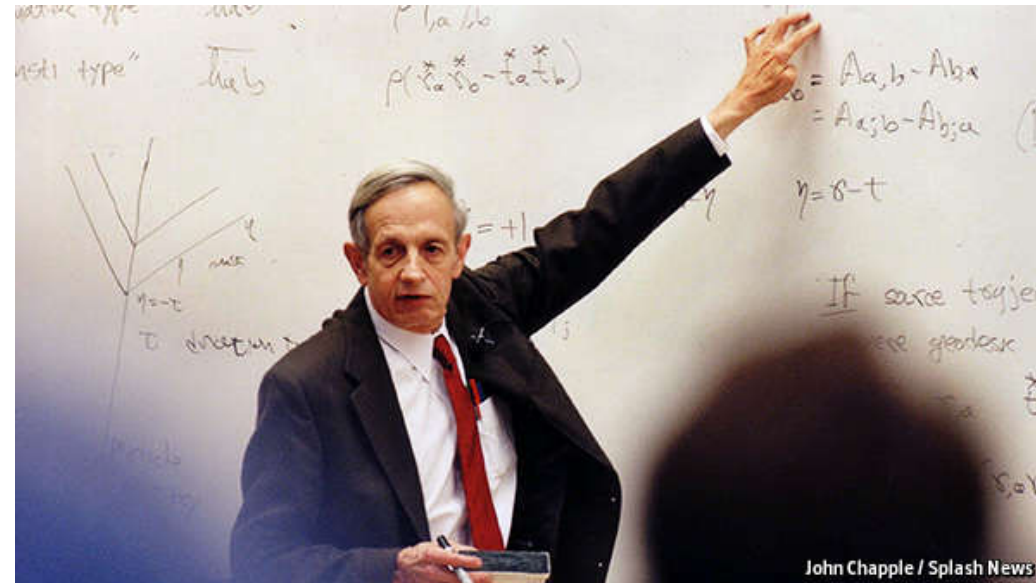
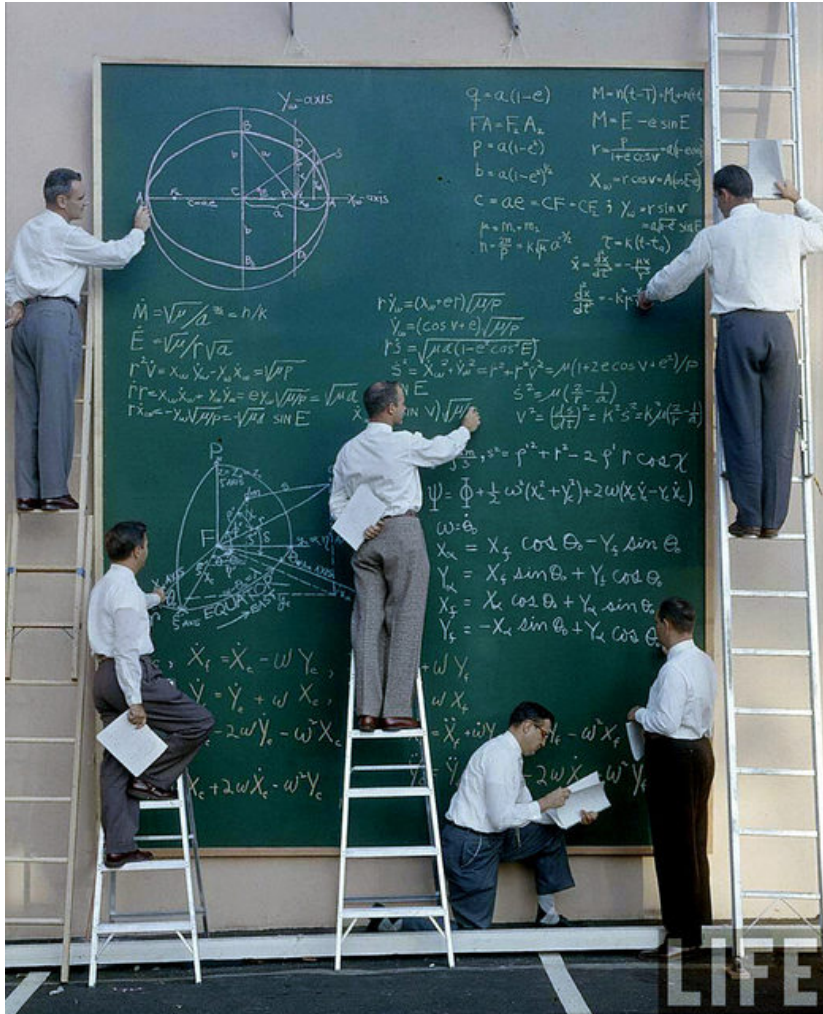


What is Mathematics Research?

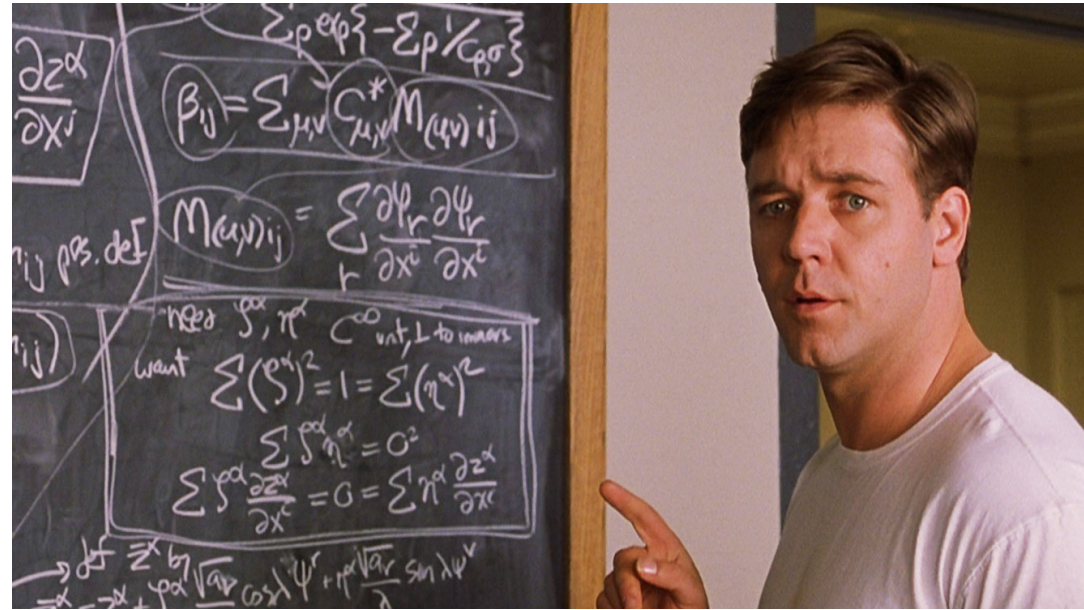
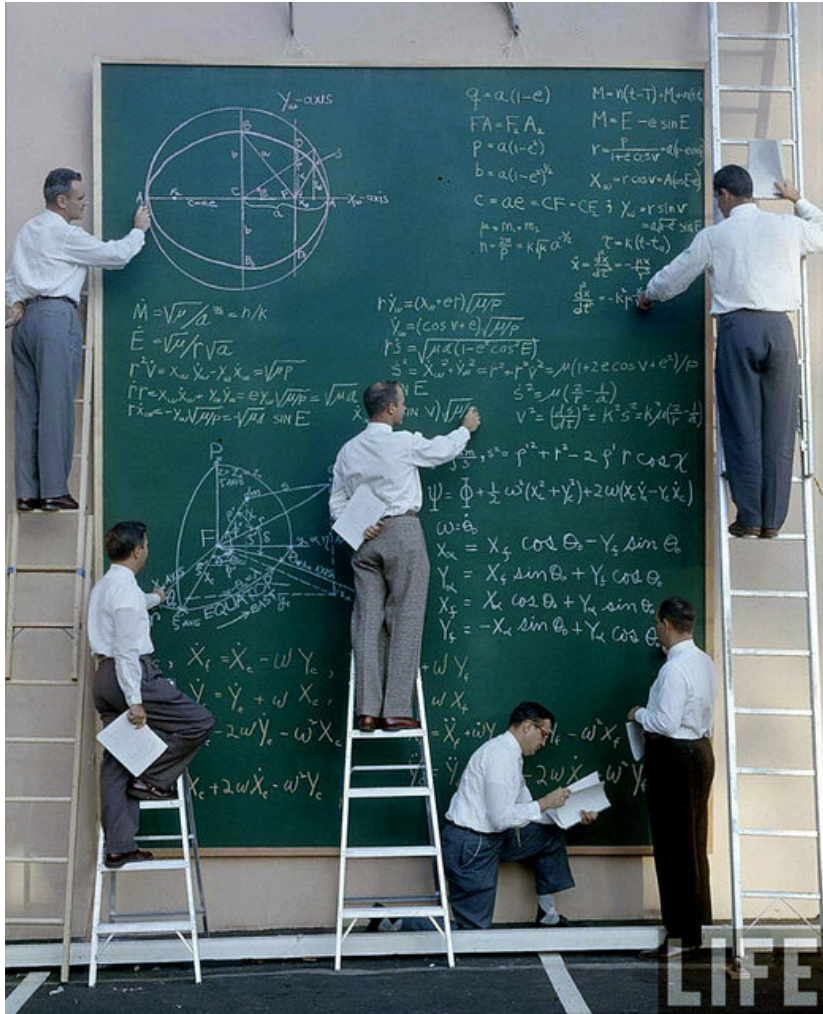
What is Mathematics Research?



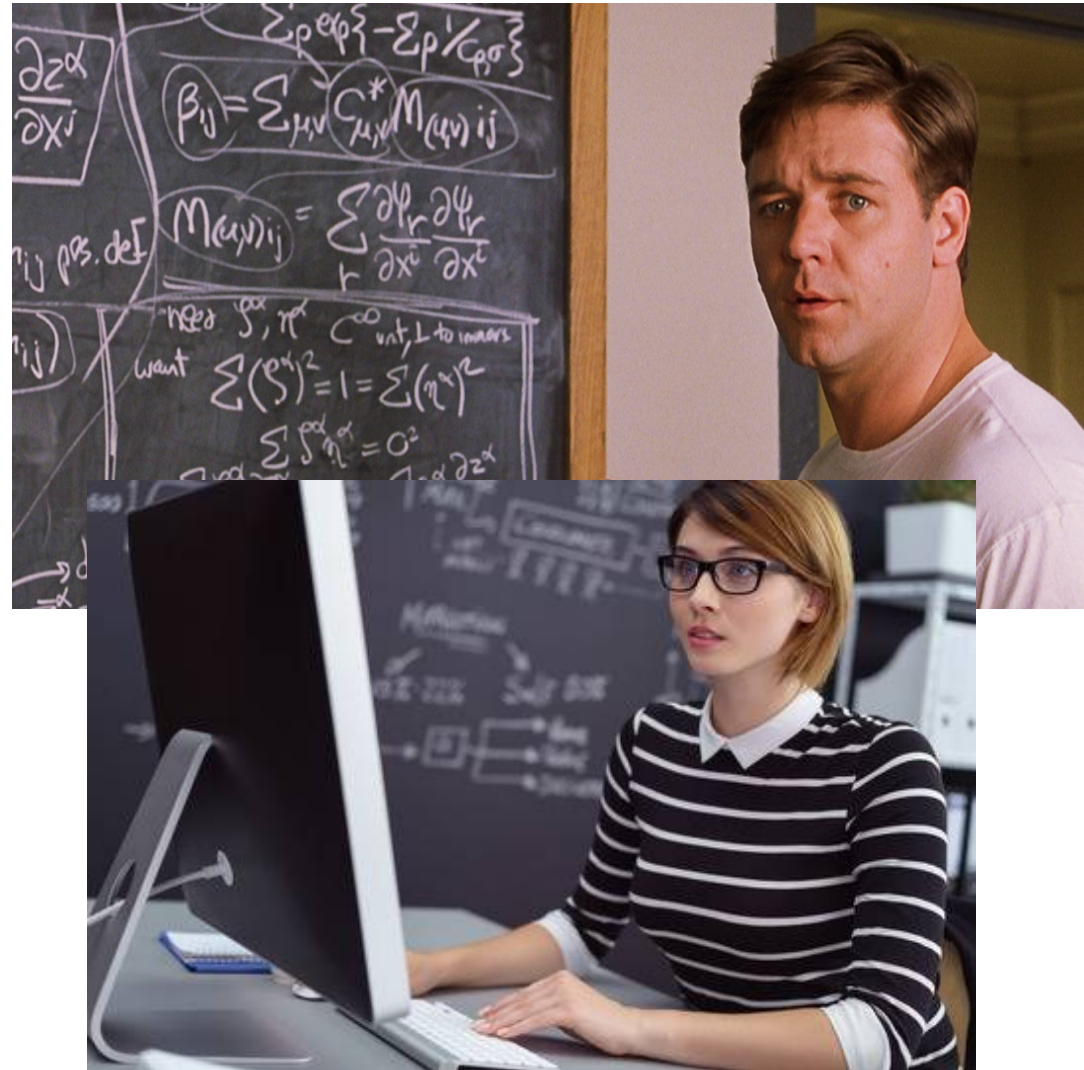
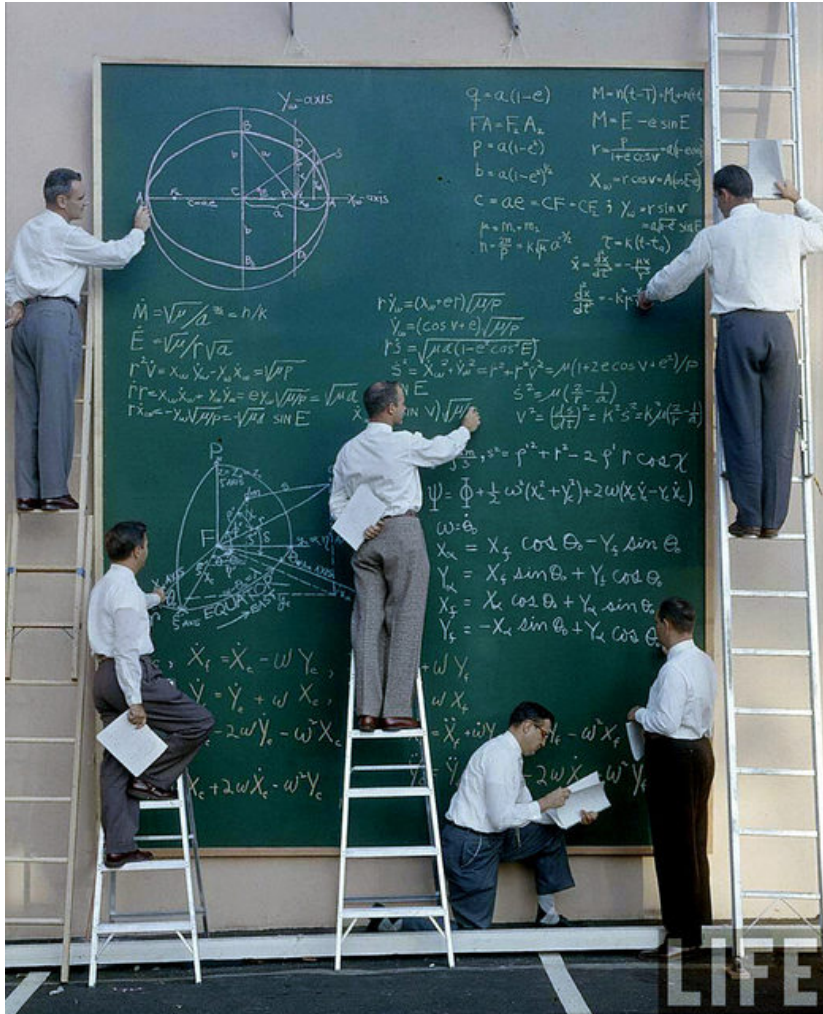
What is Mathematics Research?



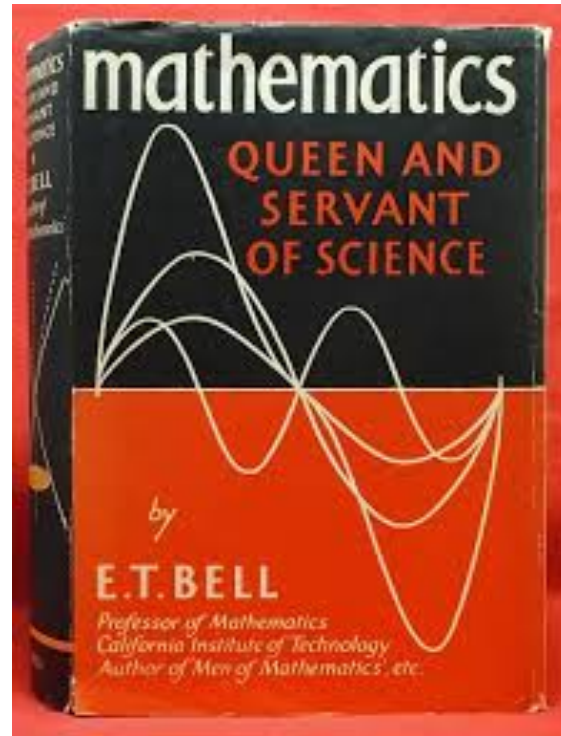
What is Mathematics Research?



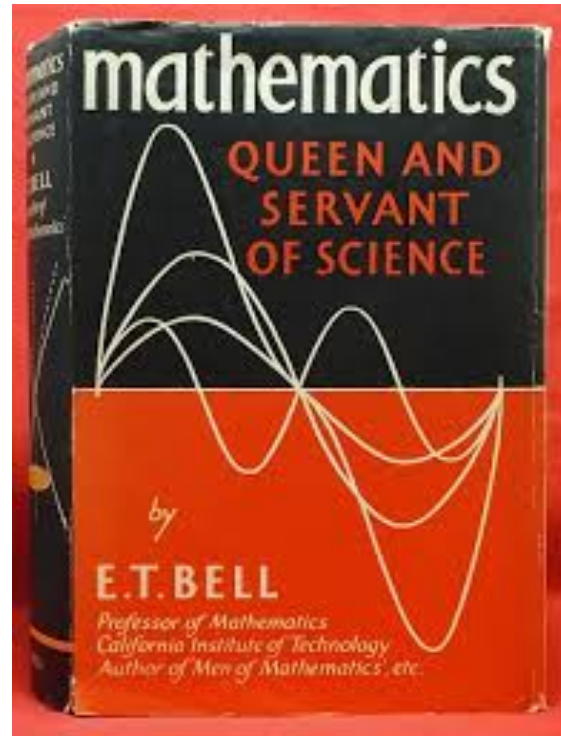
What is Mathematics Research?



Mathematics, Queen and Servant of Science



Mathematics, Queen and Servant of Science



“Pure” versus “Applied” Math

Applied Math

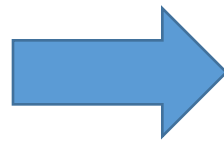
Applied Math

- starts from a real world problem, e.g., political gerrymandering

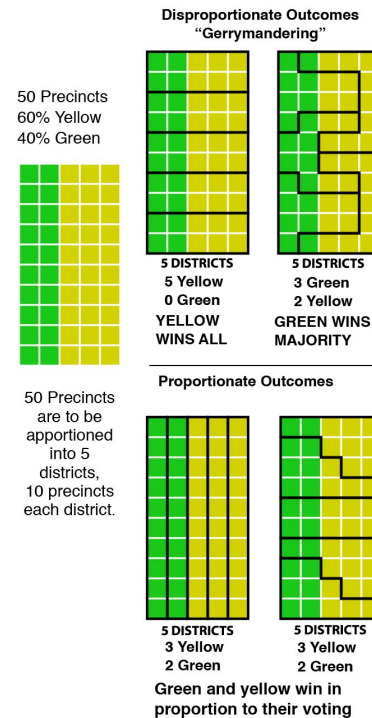


Applied Math

- starts from a real world problem, e.g., political gerrymandering
- constructs a mathematical model

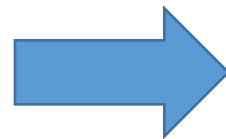


Gerrymandering: drawing different maps for electoral districts produces different outcomes

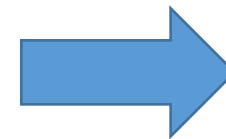
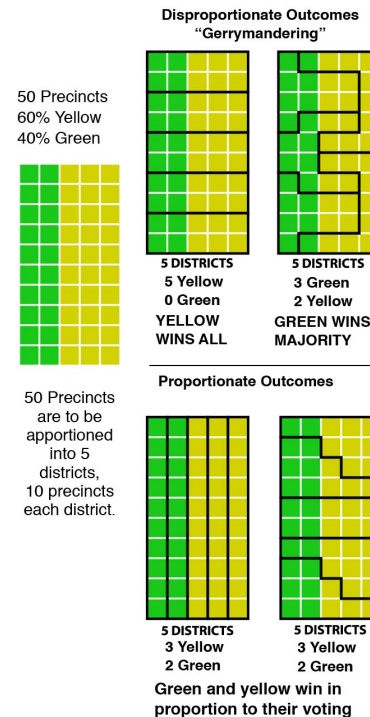


Applied Math

- starts from a real world problem, e.g., political gerrymandering
- constructs a mathematical model
- analyzes that model



Gerrymandering: drawing different maps for electoral districts produces different outcomes



$$PP(D) = \frac{4\pi A}{p^2}$$

Applied Math

- starts from a real world problem, e.g., political gerrymandering
- constructs a
- analyzes that



$$PP(D) = \frac{4\pi A}{p^2}$$

5 DISTRICTS
3 Yellow
2 Green
Green and yellow win in
proportion to their voting

5 DISTRICTS
3 Yellow
2 Green

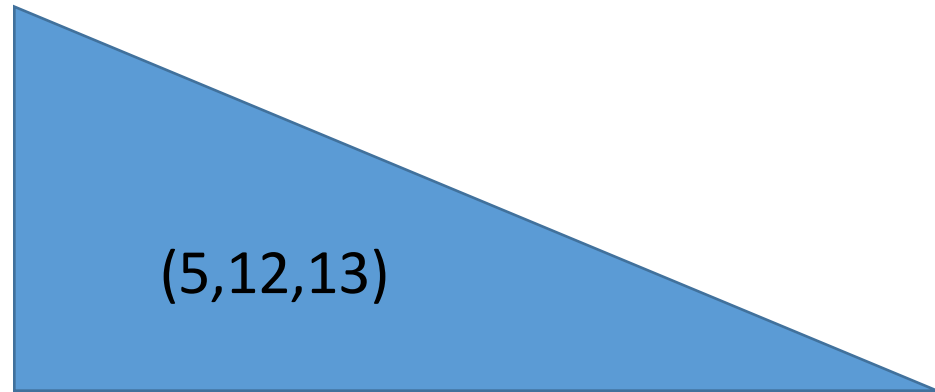
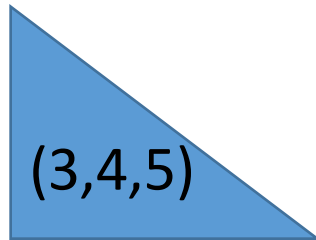
Pure Math

Pure Math

- starts from a question posed or observation made, e.g., which numbers can appear as the sides of a right triangle?

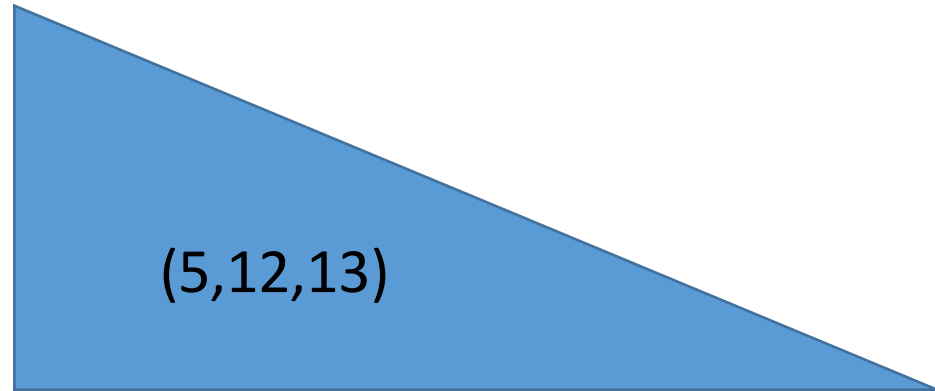
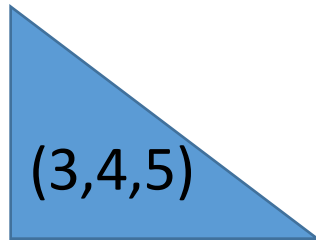
Pure Math

- starts from a question posed or observation made, e.g., which numbers can appear as the sides of a right triangle?



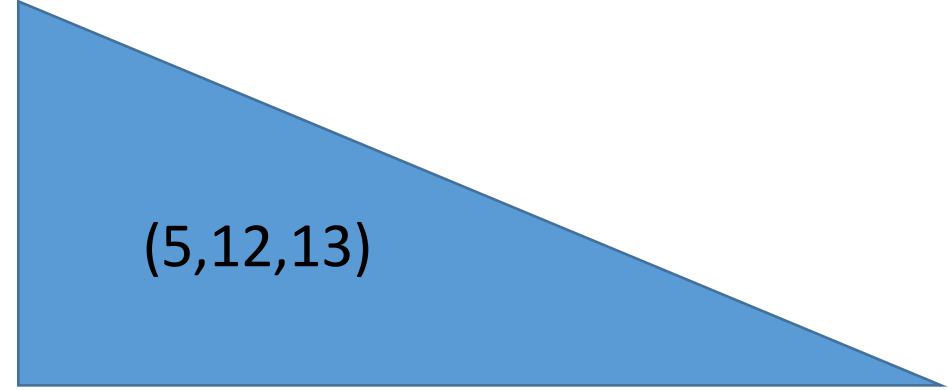
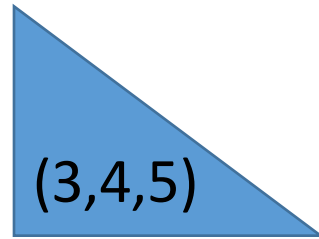
Pure Math

- starts from a question posed or observation made, e.g., which numbers can appear as the sides of a right triangle?

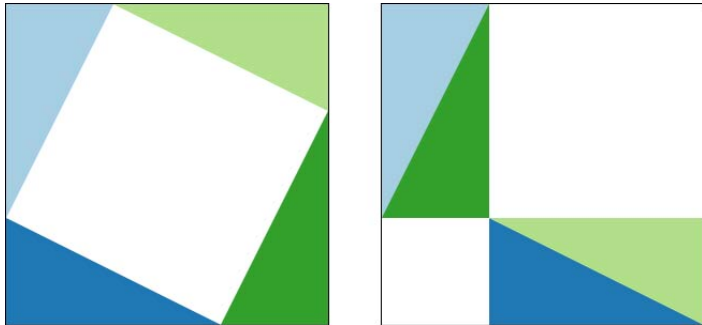


...but not (3,4,7)

Pure Math

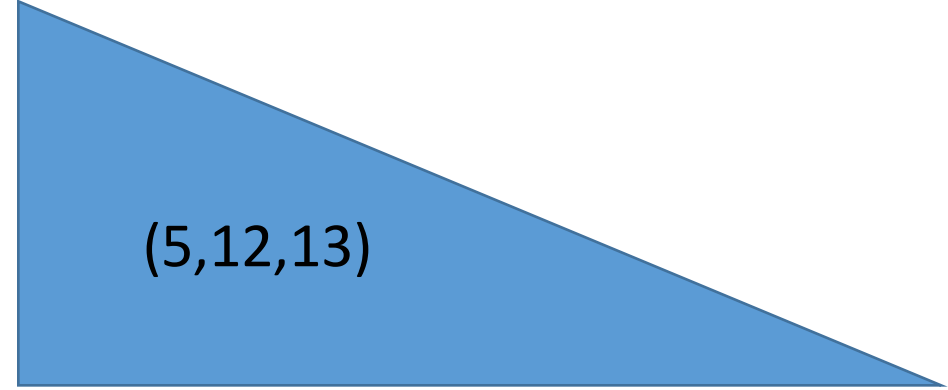
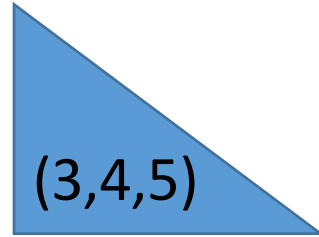


- starts from a question posed or observation made, e.g., which numbers can appear as the sides of a right triangle?

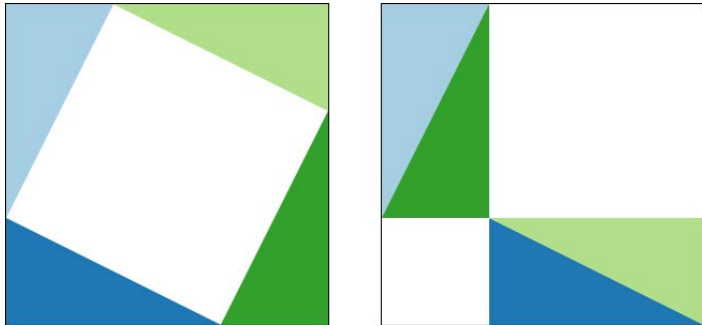


$$a^2 + b^2 = c^2$$

Pure Math

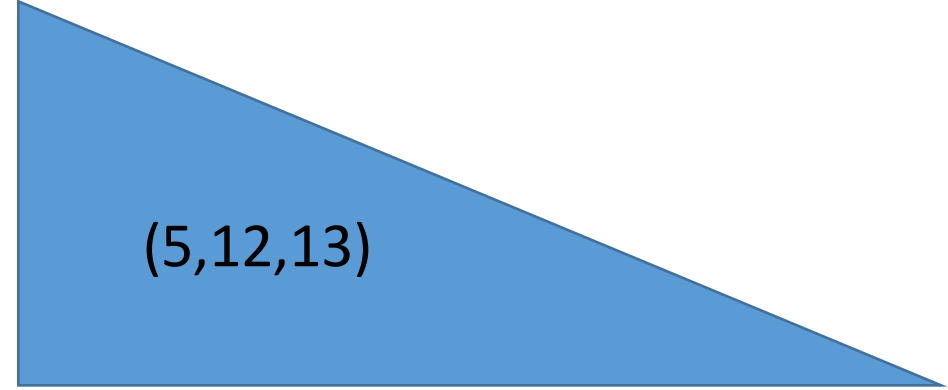
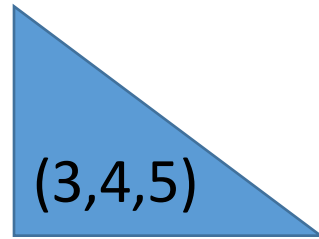


- starts from a question posed or observation made, e.g., which numbers can appear as the sides of a right triangle?
- a focus on justification (why is the answer correct?)

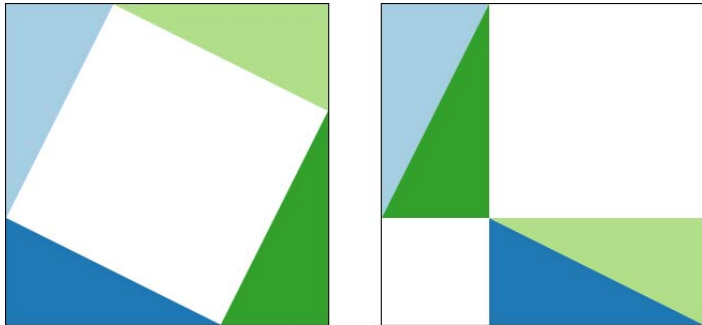


$$a^2 + b^2 = c^2$$

Pure Math

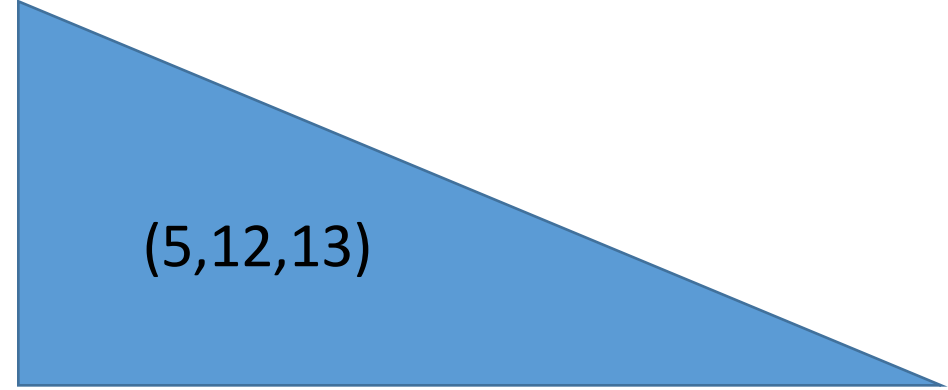
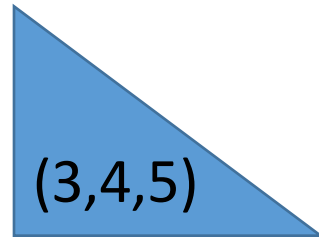


- starts from a question posed or observation made, e.g., which numbers can appear as the sides of a right triangle?
- a focus on justification (why is the answer correct?)
- style matters! Good questions and solutions breed more questions

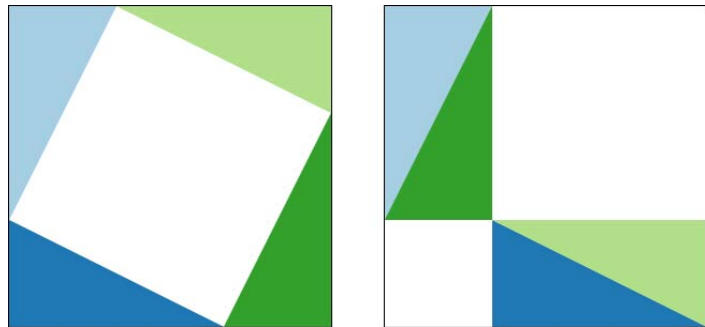


$$a^2 + b^2 = c^2$$

Pure Math



- starts from a question posed or observation made, e.g., which numbers can appear as the sides of a right triangle?
- a focus on justification (why is the answer correct?)
- style matters! Good questions and solutions breed more questions



$$a^2 + b^2 = c^2$$



Do these other equations have any whole number solutions?

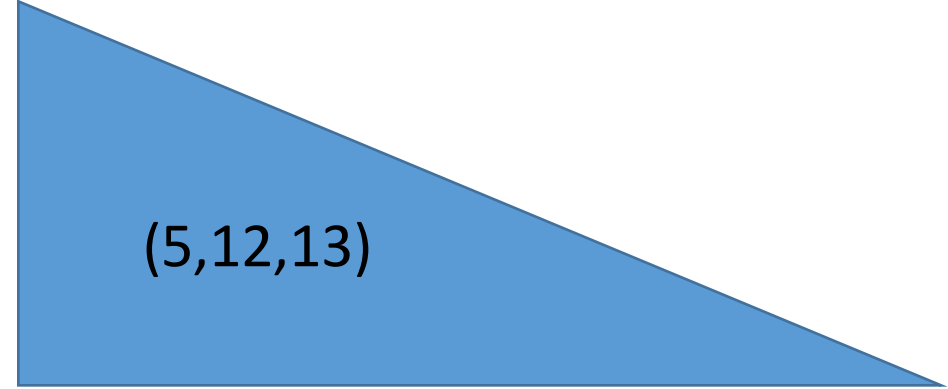
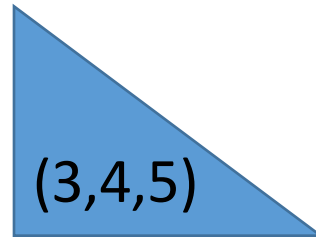
$$a^3 + b^3 = c^3$$

$$a^4 + b^4 = c^4$$

$$a^5 + b^5 = c^5$$

⋮

Pure Math



- starts from a question posed or observation made, e.g., which numbers can appear as the sides of a right triangle?
- a focus on justification (why is the answer correct?)
- style matters! Good questions and solutions breed more questions



Gauss: “Mathematics is queen of the sciences and ***Number Theory*** is the queen of mathematics”

Do these other equations have any whole number solutions?

$$a^3 + b^3 = c^3$$

$$a^4 + b^4 = c^4$$

$$a^5 + b^5 = c^5$$

⋮

Mathematics, Queen and Servant of Science



G. H. Hardy

Mathematics, Queen and Servant of Science



G. H. Hardy

*“The mathematician’s patterns, like the painter’s or the poet’s must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: **there is no permanent place in the world for ugly mathematics.**” –A Mathematician’s Apology*

Mathematics, Queen and Servant of Science



G. H. Hardy

“I have never done anything `useful’. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.” –*A Mathematician’s Apology*

Mathematics, Queen and Servant of Science



G. H. Hardy

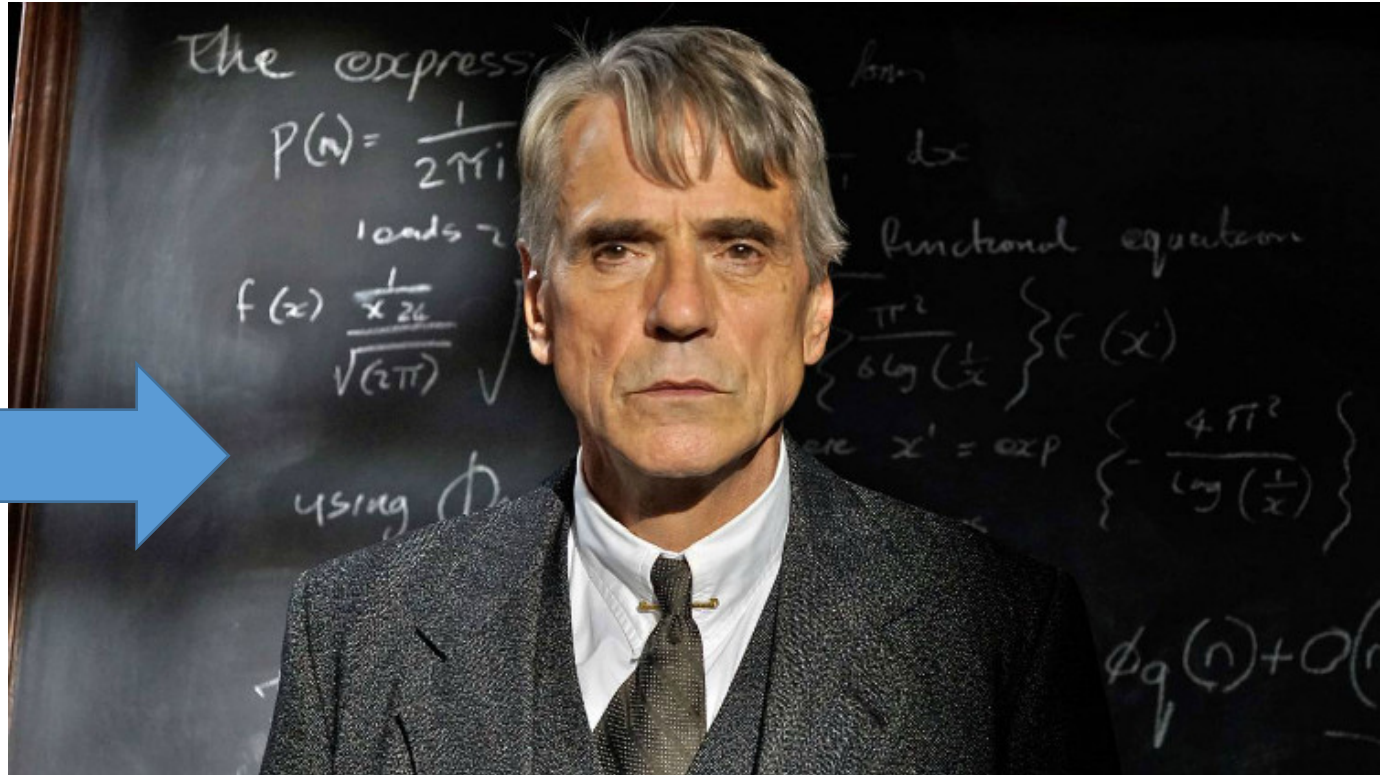
“I have never done anything `useful’. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.” –*A Mathematician’s Apology*



Mathematics, Queen and Servant of Science



G. H. Hardy



What do I do?

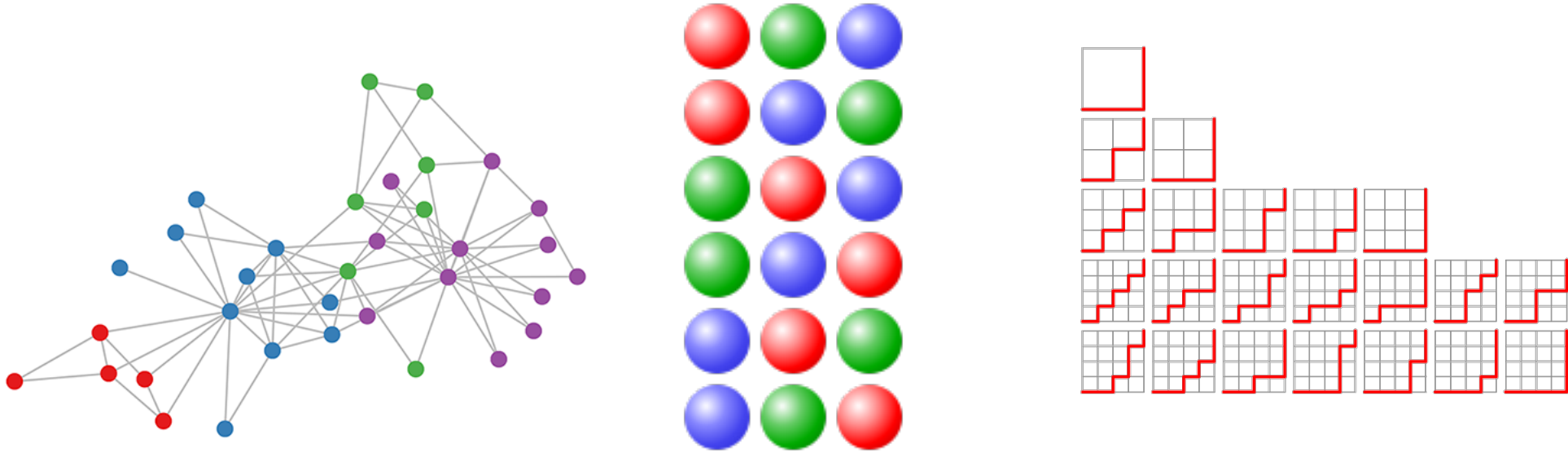
What do I do?

Combinatorics

Combinatorics: art and science of counting

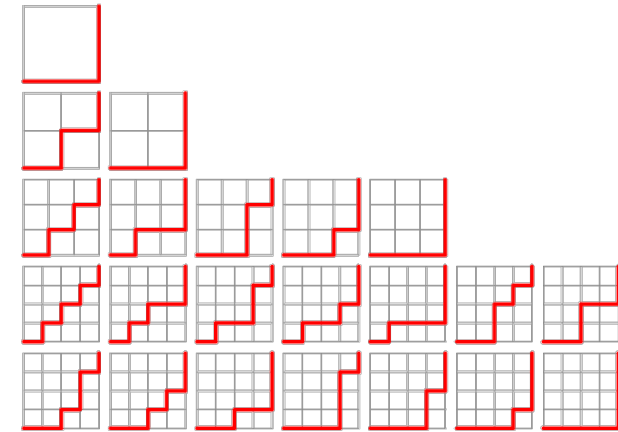
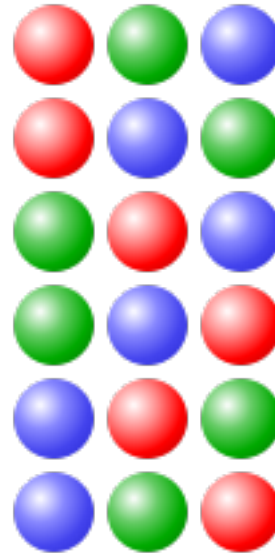
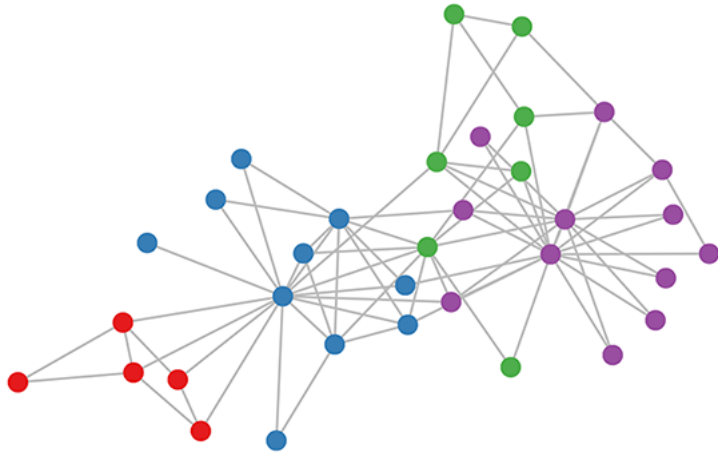
Combinatorics: art and science of counting

- Mostly deals in discrete mathematical models (originating in “real-world” applications, or in other parts of math)



Combinatorics: art and science of counting

- Mostly deals in discrete mathematical models (originating in “real-world” applications, or in other parts of math)
- Search for patterns, count things, make pictures!



Combinatorics: art and science of counting

14

S. K. HSIAO AND T. K. PETERSEN

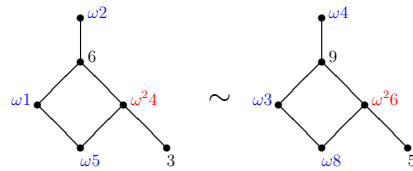


FIGURE 2. Two equivalently labeled colored posets.

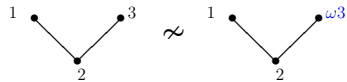


FIGURE 3. Inequivalently labeled colored posets.

Definition 3. An m -colored poset of n elements, or an (m, n) -poset, is a poset P whose elements form a subset of \mathbb{P}_m with distinct absolute values.

We say that two colored posets P and Q have equivalent labelings, written $P \sim Q$, if there is an isomorphism of posets $\phi : P \rightarrow Q$ such that:

- (1) the map ϕ preserves colors, i.e., $\varepsilon(a) = \varepsilon(\phi(a))$ for any $a \in P$, and
- (2) for all $a <_P b$, $\phi(a) <_{\mathbb{P}_m} \phi(b)$ if and only if $a <_{\mathbb{P}_m} b$.

See Figures 2 and 3. Let $\mathcal{P}_n^{(m)}$ denote the vector space over \mathbb{Q} with basis consisting of all (m, n) -posets, modulo equivalence of labelings, and define

$$\mathcal{P}^{(m)} = \bigoplus_{n \geq 0} \mathcal{P}_n^{(m)}.$$

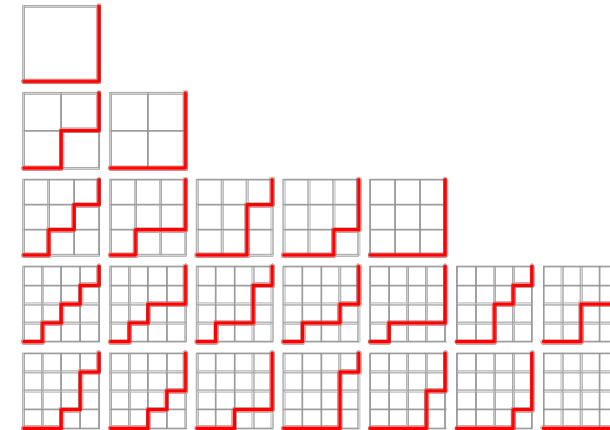
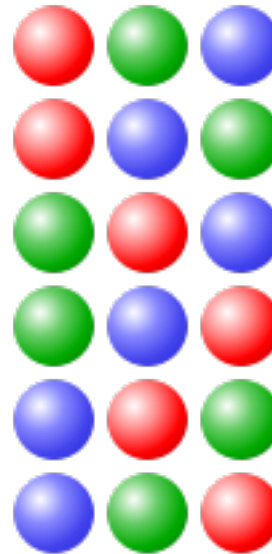
We will define a product \sqcup_m and coproduct δ_m that make $\mathcal{P}^{(m)}$ into a graded Hopf algebra.

If P is an (m, n) -poset and Q is an (m, r) -poset then let $P \sqcup_m Q$ be the $(m, n+r)$ -poset defined as follows. If as sets P and Q have any elements of the same absolute value, then replace Q by another (m, r) -poset that is label-equivalent to Q and whose elements have absolute values distinct from those of P . Again, this is easy to do since P and Q are finite sets. Now let $P \sqcup_m Q$ be the poset formed by taking the union of P and Q as posets. We have

$$|P| \cap |Q| = \emptyset, \text{ and}$$

$$x <_{P \sqcup_m Q} y \iff x <_P y \text{ or } x <_Q y.$$

mathematical models (originating in
physics, or in other parts of math)
count things, make pictures!



Combinatorics: art and science of counting

14

ω_1

FIGURE

1

FIGURE

Definition 3. An m -colored poset is a poset whose elements form a subset of

We say that two colored posets are isomorphic if there is an isomorphism of posets

- (1) the map ϕ preserve
- (2) for all $a <_P b$, $\phi(a) <_Q \phi(b)$

See Figures 2 and 3. Let (m, n) -posets, modulo equivalence

We will define a product \sqcup_m of m posets. If P is an (m, n) -poset defined as follows. If as Q is an (m, n) -poset, we replace Q by another (m, n) -poset Q' with the same absolute values distinct from Q . Now let $P \sqcup_m Q$ be the poset with

10

T. K. PETERSEN AND D. SPEYER

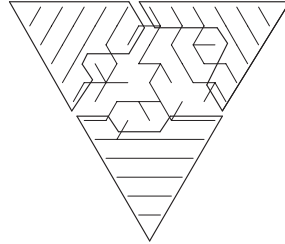


FIGURE 8. Frozen regions of a random grove of order 12.

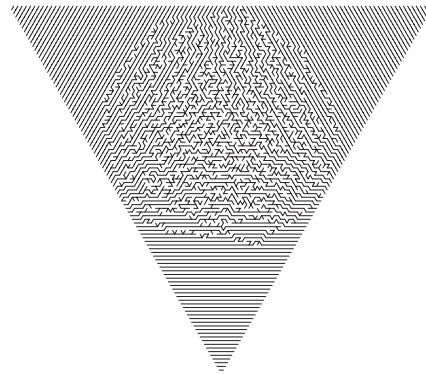
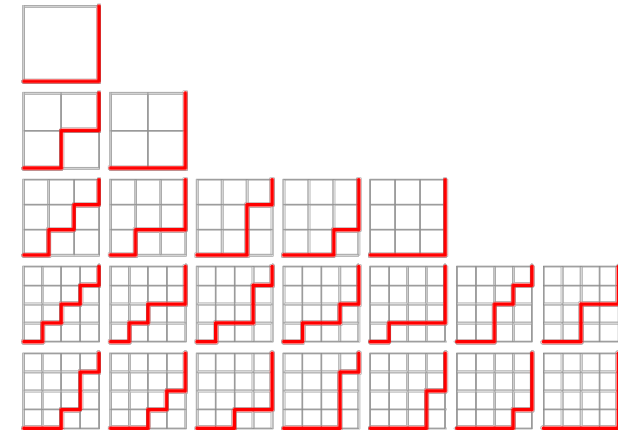
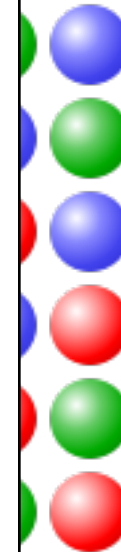


FIGURE 9. A grove on standard initial conditions of order 100.

there is homogeneity of the edges in an appropriately scaled random grove of order n , with probability approaching 1 as $n \rightarrow \infty$. Specifically, we will examine the limiting probability of finding a particular type of edge in a given location outside of the inscribed circle.

2.1. Edge probabilities. Let $p_n(i, j) = p(i, j, k)$, $k = -n - i - j$, be the probability that $a_{i, j}(n)$, the horizontal edge on triangle $x_{i, j, k+1}$, is present in a random grove

Mathematical models (originating in other parts of math) make pictures!



Combinatorics: art and science of counting

14

 ω_1

FIGURE

1

FIGU

Definition 3. *An m -cold elements form a subset of*

We say that two colored

is an isomorphism of pose

- (1) the map ϕ preserve
- (2) for all $a <_P b$, $\phi(a)$

See Figures 2 and 3. Let (m, n) -posets, modulo equ

If P is an (m, n) -poset defined as follows. If as s replace Q by another (m, n) -poset with absolute values distinct from P and Q and disjoint sets. Now let $P \sqcup_m Q$ be the disjoint union of P and Q . We have

10

T. K. PETERSEN AND

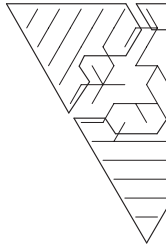


FIGURE 8. Frozen regions of a μ

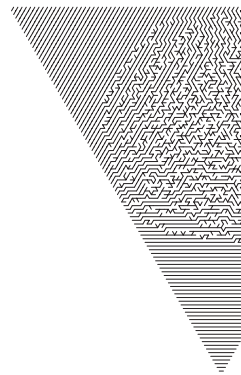


FIGURE 9. A grove on standard in

there is homogeneity of the edges in an approach n , with probability approaching 1 as $n \rightarrow \infty$. The limiting probability of finding a particular type of edge of the inscribed circle.

2.1. Edge probabilities. Let $p_n(i, j) = p(i, j | \mathcal{F}_n)$ denote the probability that $a_{i,j}(n)$, the horizontal edge on triangle

COMPUTING AFFINE REFLECTION LENGTH

25

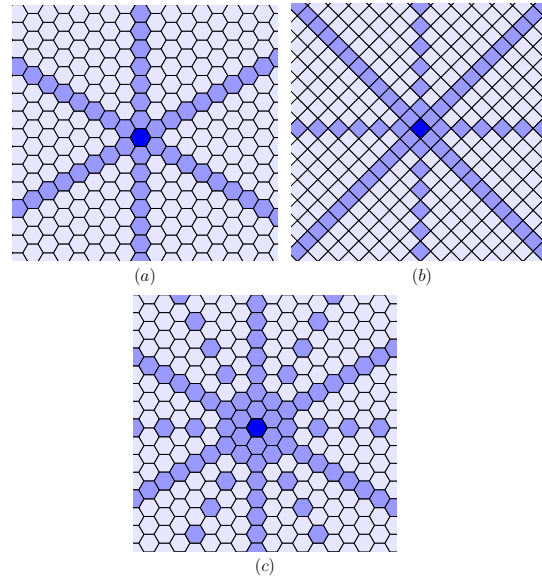
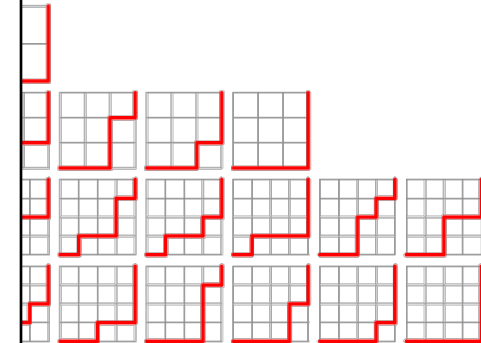


FIGURE 7. The translates $t_\lambda \cdot P$ in (a) type A_2 , (b) type B_2 , and (c) type C_2 , colored according to the local distribution of reflection length.

contains λ and $\text{MOV}(u)$. But by hypothesis, every such U contains μ and $\text{MOV}(u)$, and so contains $\text{MOV}(t_\mu u)$. Then the result follows immediately from the definition of f_λ . \square

Unfortunately, while Theorem 3.3 and Proposition 3.2 imply bounds on the number of local generating functions in terms of the number of W_0 -orbits of intersections of root subspaces, it is probably intractable to compute all $f_\lambda(s, t)$, or even all $f_\lambda(t)$, in general. We show in Appendix

originating in
(th)



Combinatorics: art and science of counting

in

14

$\omega 1$

FIGURE 1

FIGURE 1

Definition 3. An m -colored poset is a poset whose elements form a subset of $\{1, \dots, m\}$.

We say that two colored posets P and Q are isomorphic if there is an isomorphism of posets ϕ such that

- (1) the map ϕ preserve colors
- (2) for all $a <_P b$, $\phi(a) <_Q \phi(b)$.

See Figures 2 and 3. Let (m, n) -posets, modulo equivalence.

We will define a product $P \times Q$ of two colored posets. If P is an (m, n) -poset and Q is an (m', n') -poset, we replace Q by another (m', n') -poset Q' with the same absolute values distinct from P . Now let $P \sqcup_m Q$ be the disjoint union of P and Q' .

10

T. K. PETERSEN AND

FIGURE 8. Frozen regions of a tiling.

FIGURE 9. A grove on standard in B_2 , and tribut

there is homogeneity of the edges in an approx n , with probability approaching 1 as $n \rightarrow \infty$. The limiting probability of finding a particular type of edge is the same for all edges of the inscribed circle.

2.1. Edge probabilities. Let $p_n(i, j) = p(i, j | a_{i,j}(n))$, the horizontal edge on triangle T_n .

FIGURE 8. The translates $t_\lambda \cdot P$ in type A_3 colored according to the local distribution of reflection length.

FIGURE 9. A grove on standard in B_2 , and tribut

contains λ as a μ and $\text{MOV}(\lambda, \mu)$ immediately.

Unfortunately, the number of W_0 -orbits of f is infinite, so we compute all f .

26

J. B. LEWIS, J. MCCAMMOND, T. K. PETERSEN, AND P. SCHWER

FIGURE 8. The translates $t_\lambda \cdot P$ in type A_3 colored according to the local distribution of reflection length.

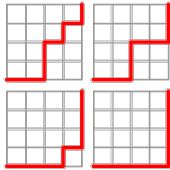
λ	A_3	$f_\lambda(s, t)$
\emptyset		$(1+t)(1+2t)(1+3t)$
α_1^\vee		$(s+t)(1+2t)(1+3t)$
$\alpha_2^\vee + \alpha_3^\vee$		$2t^2 + 6t^3 + 4st + 9st^2 + s^2 + 2s^2t$
generic span of $\alpha_1^\vee, \alpha_2^\vee$		$(s+t)(s+2t)(1+3t)$
generic span of $\alpha_1^\vee, \alpha_3^\vee$		$(s+t)(t+6t^2+s+4st)$
generic		$(s+t)(s+2t)(s+3t)$

TABLE 3. Local generating functions for affine A_3 . Here α_1 and α_3 are any two orthogonal roots, while α_2 and α_1 are not orthogonal.

A that computing $d(t_\lambda)$ for an element λ of the type A_n coroot lattice is essentially equivalent to the NP-complete problem **SubsetSum**.

A different approach to understanding the distribution of reflection length would be to introduce a new statistic that grows with λ and take a bivariate generating function, either over the whole group W or over the elements with fixed elliptic part (that is, over a coset of the translations). Thus, for a given element $u \in W_0$ one could consider the generating function

$$g_u(q, t) = \sum_{\lambda \in L} t^{d(t_\lambda u)} q^{\text{stat}(\lambda, u)}$$



Combinatorics: art and science of counting

14



FIGURE 1

1

FIGURE 2

Definition 3. An m -colored elements form a subset of

We say that two colored is an isomorphism of pose

- (1) the map ϕ preserve
- (2) for all $a <_P b$, $\phi(a)$

See Figures 2 and 3. Let (m, n) -posets, modulo equ

We will define a product L

If P is an (m, n) -poset defined as follows. If as s replace Q by another $(m$ absolute values distinct fr sets. Now let $P \sqcup_m Q$ be have

10

T. K. PETERSEN AND

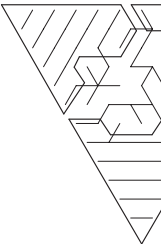


FIGURE 8. Frozen regions of a



FIGURE 9. A grove on standard in

there is homogeneity of the edges in an appr n , with probability approaching 1 as $n \rightarrow$ limiting probability of finding a particular ty of the inscribed circle.

2.1. Edge probabilities. Let $p_n(i, j) = p(i$ that $a_{i,j}(n)$, the horizontal edge on triangle

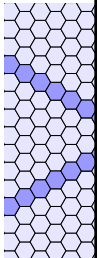


FIGURE 10. B_2 , an tribut

contains λ a μ and $\text{MOV}(\lambda, \mu)$ immediately

Unfortunately on the numbe W_Q -orbits of f compute all f

26

J. B. LEWIS, J. MCCAMMOND, T.

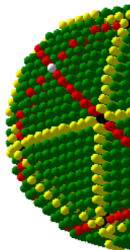


FIGURE 8. The translates t cording to the local distribu

λ	A
\bullet	0
\circ	α_1^\vee
\circ	$\alpha_2^\vee + \alpha_3^\vee$
\circ	$\text{generic span of } \alpha_1^\vee, \alpha_2^\vee$
\circ	$\text{generic span of } \alpha_1^\vee, \alpha_3^\vee$
\bullet	generic

TABLE 3. Local generating α_1 and α_3 are any two orthog are not orthogonal.

A that computing $d(t_\lambda)$ for an elem is essentially equivalent to the NP-d

A different approach to understa length would be to introduce a ne take a bivariate generating function over the elements with fixed ellipti translations). Thus, for a given elem generating function

$$g_u(q, t) = \sum_{\lambda \in L}$$

AREA AND RANK OF A DYCK PATH

7

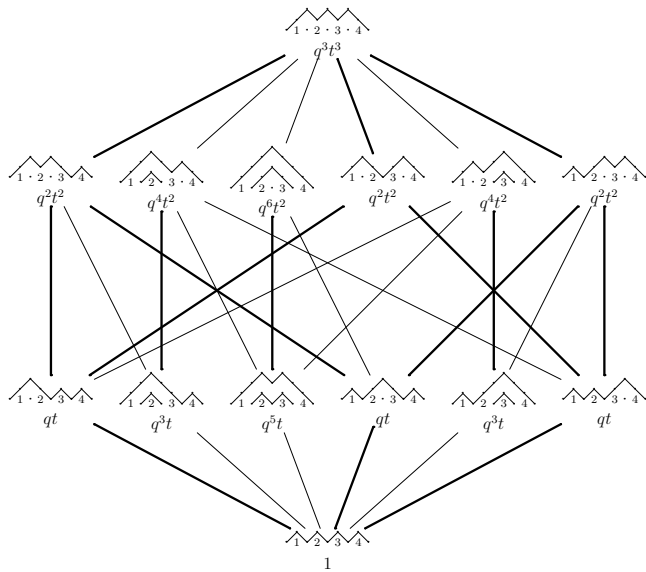
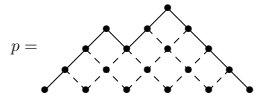


FIGURE 3. The noncrossing partitions of $\{1, 2, 3, 4\}$, drawn as nested Dyck paths. Weights are given with respect to area and rank. Boldface lines indicate Simion and Ullmann's symmetric boolean decomposition of the poset.

have $\text{area}(p) = 8$ for the following path:



We use the term “area” because this is the number of $1/\sqrt{2} \times 1/\sqrt{2}$ diamonds that can fit below the path and above the x -axis. (Just put the bottom of the diamonds on the unused lattice points.) This is indicated with dashed lines in the picture above. The distribution of area for Dyck paths has been studied by many, starting with Carlitz and Riordan [11].

Combinatorics and science of counting

and

$$x_2 < x_4 = x_6 < x_1 = x_3 = x_5 < x_2 + 1,$$

respectively.

The faces in $\bar{\Sigma}(A_2)$, labeled with spin necklaces, are shown in Figure 19.

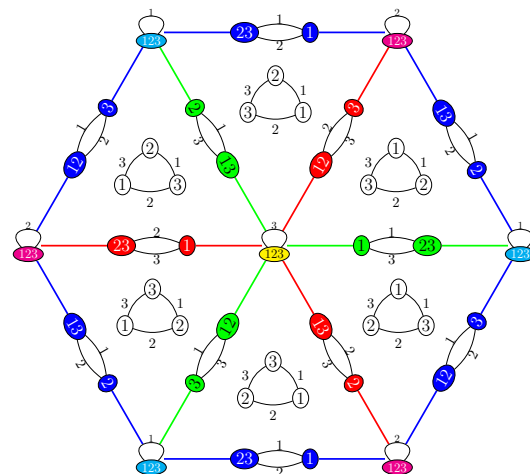


FIGURE 19. The faces of the Steinberg torus $\bar{\Sigma}(A_2)$, with colors corresponding to W -orbits. Note the identifications along the boundary.

The partial order on faces (given by inclusion of closures) corresponds to the partial order on spin necklaces given by edge contraction. On the other hand, sliding consecutive beads (blocks) past each other in the necklace corresponds to walking between adjacent chambers.

A permutation w acts on a spin necklace by changing each block B into $w(B)$, and keeping the edge labels. This corresponds to the action of the Weyl group on faces of the torus, and the set of edge labels of the necklace corresponds to the color set of the face (under the identification between $\bar{\Delta}$ and $\{1, \dots, n\}$.) The orbits are thus parametrized by nonempty subsets of $\{1, \dots, n\}$, with the orbit $\bar{\Sigma}_J$ consisting of the spin necklaces with edge label set J . Figure 19 shows the orbits in $\bar{\Sigma}(A_2)$. For example, the edges in red constitute the orbit of color set $\{2, 3\}$.

The permutation associated to the torus face as in (16) (or as in Corollary 6) is obtained by listing the blocks in the split necklace from left to right, and writing the elements in each block in increasing order. For example, the permutations associated to the faces (spin necklaces) (30) and (31) are 524613 and 246135, respectively.

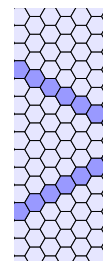


FIGURE 20. The translates t corresponding to the local distribution.

contains λ as μ and $\text{MOV}(\mu)$ immediately.

Unfortunately, the number of W_0 -orbits of f is too large to compute all f .

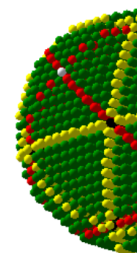


FIGURE 8. The translates t corresponding to the local distribution.

λ	A
\bullet	0
\circ	α_1^\vee
\circ	$\alpha_3^\vee + \alpha_3^\vee$
\circ	$\text{generic span of } \alpha_1^\vee, \alpha_3^\vee$
\circ	$\text{generic span of } \alpha_1^\vee, \alpha_3^\vee$
\bullet	generic

TABLE 3. Local generating α_1 and α_3 are any two orthogonal elements.

A that computing $d(t_\lambda)$ for an element λ is essentially equivalent to the NP-complete problem.

A different approach to understanding the length would be to introduce a new variable t and take a bivariate generating function over the elements with fixed elliptic translations). Thus, for a given element λ , the generating function

$$g_\lambda(q, t) = \sum_{\lambda \in L} q^{d(\lambda)} t^{\text{length}(\lambda)}$$

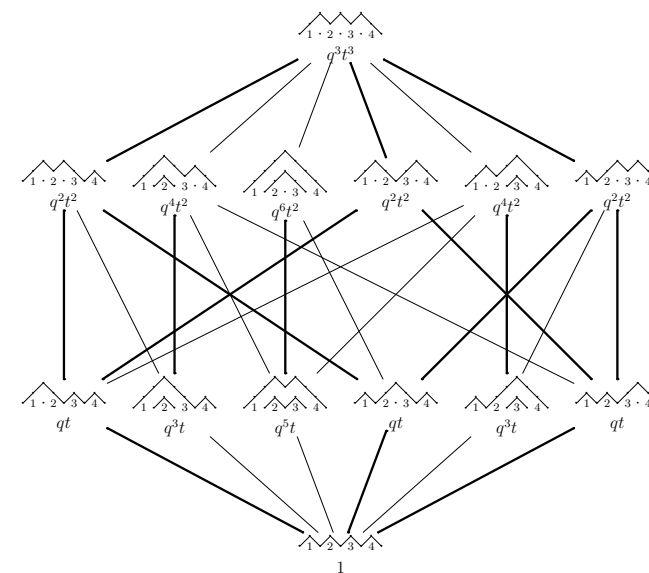
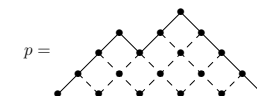


FIGURE 3. The noncrossing partitions of $\{1, 2, 3, 4\}$, drawn as nested Dyck paths. Weights are given with respect to area and rank. Boldface lines indicate Simion and Ullmann's symmetric boolean decomposition of the poset.

have $\text{area}(p) = 8$ for the following path:



We use the term "area" because this is the number of $1/\sqrt{2} \times 1/\sqrt{2}$ diamonds that can fit below the path and above the x -axis. (Just put the bottom of the diamonds on the unused lattice points.) This is indicated with dashed lines in the picture above. The distribution of area for Dyck paths has been studied by many, starting with Carlitz and Riordan [11].

science of counting

and
respectively.
The faces in $\overline{\Sigma}(A_2)$, lab

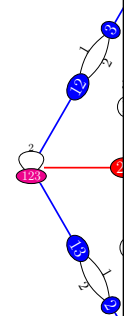
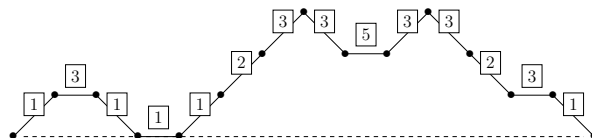


FIGURE 19. The
to W -orbits. Note

The partial order on fa
on spin necklaces given b
(blocks) past each other
A permutation w acts o
the edge labels. This co
and the set of edge labels
identification between Δ
subsets of $\{1, \dots, n\}$, wit
 J . Figure 19 shows the o
color set $\{2, 3\}$.

The permutation assoc
by listing the blocks in
each block in increasing o
necklaces (30) and (31).

As a larger example, the path $q = UHDDHUUDHUDDHD$ has the step weights:



so $\omega(q) = 2^2 \cdot 3^6 \cdot 5 = 14580$.

Remark 4.3. In [4, Proposition 3.3], the first author and Tenner prove that the number of permutations $w \in S_n$ which achieve the maximal depth of $\lfloor n^2/4 \rfloor$ is

$$|\{w \in S_n : \text{dep}(w) = \lfloor n^2/4 \rfloor\}| = \begin{cases} (k!)^2 & \text{if } n = 2k, \\ n(k!)^2 & \text{if } n = 2k + 1. \end{cases}$$

We can recover this result as a corollary of 4.2 by noting that this is the weight of the Motzkin path with maximal area, namely $p = U^k D^k$ if $n = 2k$ is even, and $p = U^k H D^k$ if $n = 2k + 1$ is odd.

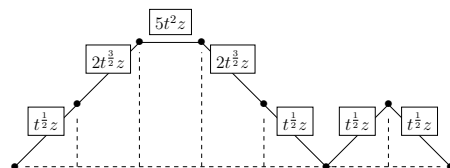
Statements equivalent to [4, Proposition 3.2] and [4, Proposition 3.3] can be found in the paper of Diaconis and Graham, although without proof (see Table 1 and Remark 2 of [1]). They are also mentioned in the remarks (and links therein) for entry A062870 of [5].

5 Counting weighted Motzkin paths by area

Taking Propositions 3.2 and 4.2 into account, we can express the generating function for permutations with respect to depth as

$$F(t, z) = \sum_{n \geq 0} \sum_{w \in S_n} t^{\text{dep}(w)} z^n = \sum_{p \in \text{Motz}} \omega(p) t^{\text{area}(p)} z^{|p|},$$

where $|p|$ is the number of steps in the path p . Furthermore, if we decompose each Motzkin path into vertical strips (instead of horizontal strips as in section 3) to compute its area, we can rewrite the whole term $\omega(p) t^{\text{area}(p)} z^{|p|}$ as a product over the steps of p . For example, if $p = UHDDUD$, we would have the modified weights



J. B. LEWIS, J. MCCAMMOND, T.

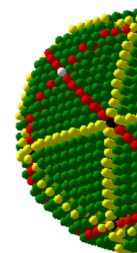


FIGURE 8. The translates t
according to the local distribu

λ	A
\bullet	0
\circ	α^\vee
\circ	$\alpha_1^\vee + \alpha_3^\vee$
\circ	$\text{generic span of } \alpha_1^\vee, \alpha_3^\vee$
\circ	$\text{generic span of } \alpha_1^\vee, \alpha_3^\vee$
\circ	generic

TABLE 3. Local generating
 α_1 and α_3 are any two orthog
are not orthogonal.

that computing $d(t_\lambda)$ for an elem
essentially equivalent to the NP-c
A different approach to understa
length would be to introduce a ne
ake a bivariate generating function
ver the elements with fixed ellipti
anslations). Thus, for a given elem
generating function

$$g_u(q, t) = \sum_{\lambda \in L}$$

AREA AND RANK OF A DYCK PATH

7

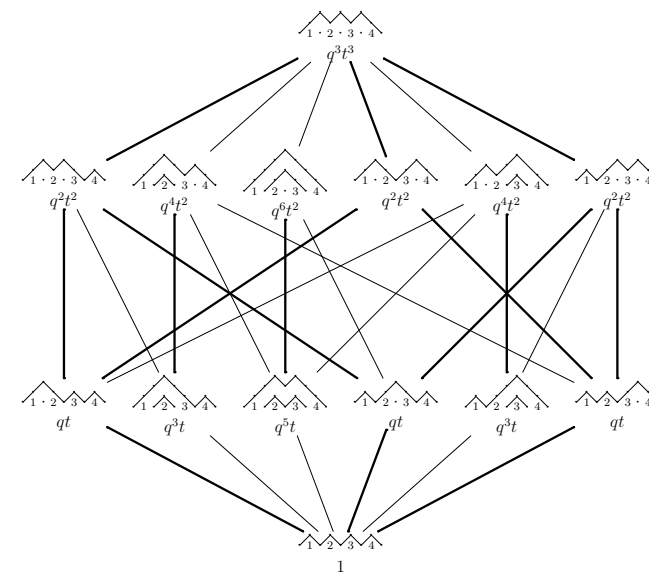
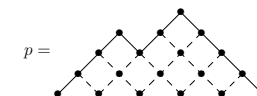


FIGURE 3. The noncrossing partitions of $\{1, 2, 3, 4\}$, drawn as nested Dyck paths. Weights are given with respect to area and rank. Boldface lines indicate Simion and Ullmann's symmetric boolean decomposition of the poset.

have $\text{area}(p) = 8$ for the following path:



We use the term “area” because this is the number of $1/\sqrt{2} \times 1/\sqrt{2}$ diamonds that can fit below the path and above the x -axis. (Just put the bottom of the diamonds on the unused lattice points.) This is indicated with dashed lines in the picture above. The distribution of area for Dyck paths has been studied by many, starting with Carlitz and Riordan [11].

and
respectively.
The faces in $\bar{\Sigma}(A_2)$, lab

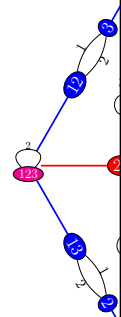
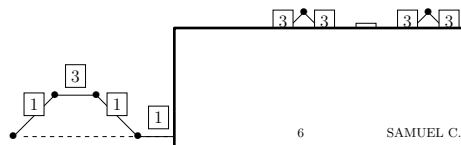


FIGURE 19. The
to W -orbits. Note

The partial order on fa
on spin necklaces given b
(blocks) past each other
A permutation w acts o
the edge labels. This co
and the set of edge labels
identification between $\bar{\Delta}$
subsets of $\{1, \dots, n\}$, wit
 J . Figure 19 shows the o
color set $\{2, 3\}$.
The permutation assoc
by listing the blocks in
each block in increasing o
necklaces) (30) and (31)

As a larger example, the path $q = UH\bar{D}HUU\bar{D}HUU\bar{D}\bar{D}H\bar{D}$ has the step weights:



so $\omega(q) = 2^2 \cdot 3^6 \cdot 5 = 14580$

Remark 4.3. In [4, Propo
of permutations $w \in S_n$ wh

$$|\{w \in S_n :$$

We can recover this result
Motzkin path with maxim
if $n = 2k + 1$ is odd.

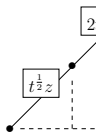
Statements equivalent t
the paper of Diaconis and
of [1]). They are also men
of [5].

5 Counting we

Taking Propositions 3.2 an
permutations with respect

$$F(t, z)$$

where $|p|$ is the number o
Motzkin path into vertica
its area, we can rewrite th
For example, if $p = UUUH$



THE ELECTRONIC JOURNAL OF COMBIN

6

SAMUEL C. GUTEKUNST, KAROLA MÉSZÁROS, AND

(a)

(b)

FIGURE 1. Two views of the threshold arrangement
we can see seven chambers above V_0 (the other se
these), thus there are $T_2 = 14$ threshold functions
(b) the six regions of the resonance arrangement \mathcal{R}
restrictions of V_1, V_2 , and V_{12} to the subspace V_0 .

The sign vector of a point in \mathbb{R}^n with respect to \mathcal{T}_n is de

$$\tau(x) = (\tau_S(x))_{S \subseteq [n-1]},$$

where $\tau_S(x) = \text{sgn}(\langle x, v_S \rangle)$.

For example, the point $x = (1, 2, 1)$ has $\tau(x)$ given by

$$(\tau_0, \tau_1, \tau_2, \tau_{12}) = (-, -, 0, +).$$

2.2. The resonance arrangement. For any subset $S \subseteq$
vector of length n in which the elements of S denote the ent
if $n = 8$,

$$u_{\{1,3,4,6\}} = (1, 0, 1, 1, 0, 1, 0, 0).$$

PROMOTION AND CYCLIC SIEVING VIA WEBS

13

examination of the growth rules in Figure 3, we see that any right turn from a downward-pointing 0-edge takes us on an upward-pointing 1-labeled edge. Any left turn from an upward-pointing 1-edge leads to another downward-pointing 0-edge and so on, as shown in Figure 7. Because the path must have even length in order to end up on the boundary, we know that the final edge traversed is labeled with a 0. Similarly, by examination of the local moves we have that C^r alternates $\bar{0}\bar{1}\bar{0}\bar{1}\dots$ upon leaving v^* , terminating at v^r , which, by parity considerations, must be labeled with $\bar{1}$.

We define $\mathcal{L}(D)$ to be the collection of faces to the left of C^l (when moving from v^* to v^l). Similarly, $\mathcal{R}(D)$ denotes the faces to the right of C^r (notice that this includes the outer face f_0). Let $\mathcal{M}(D)$ denote the faces to the right of C^l and to the left of C^r . See Figure 7.

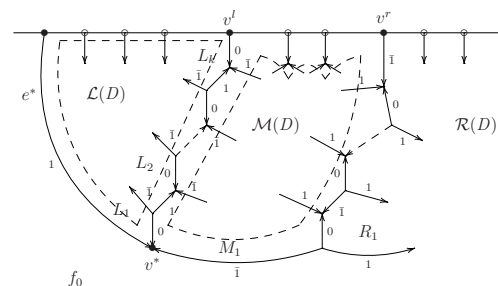


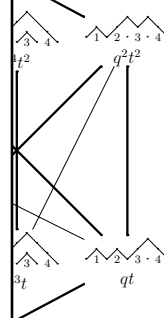
FIGURE 7

Lemma 3.4. Let D be a web. After moving the leftmost boundary vertex to the right,

- (1) the depth of every face in $\mathcal{L}(D)$ decreases by 1,
- (2) the depth of every face in $\mathcal{R}(D)$ increases by 1, and
- (3) the depth of every face in $\mathcal{M}(D)$ remains unchanged.

Proof. Let L_1 denote the face separated from the outer face by e^* . This face will be the outer face once the leftmost boundary vertex moves to the right. Let L_2, \dots, L_k denote the other faces of $\mathcal{L}(D)$ that border the left cut. By examining the edge labels (which by Lemma 3.3 are consistent with depth) every face L_i has a minimal path to f_0 that passes through L_1 . Thus, any face in $\mathcal{L}(D)$ has a minimal path to f_0 that goes through L_1 . Claim (1) then follows.

By examining the faces on the boundary of $\mathcal{M}(D)$, we see that no face in $\mathcal{M}(D)$ has a minimal length path through L_1 , but they all have such a path through M_1 . Since M_1

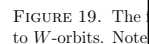


as nested Dyck
face lines indicate
the poset.

$\bar{2}$ diamonds that can fit
diamonds on the unused
ove. The distribution of

area for Dyck paths has been studied by many, starting with Carlitz and Riordan [11].

The faces in $\overline{\Sigma}(A_2)$, lab



The permutation associated with \mathbf{b} is obtained by listing the blocks in \mathbf{b} in increasing order, and then listing each block in increasing order (see the necklaces) (30) and (31).

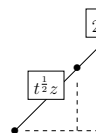
Remark 4.3. In [4, Propo of permutations $w \in S_n$ wh

We can recover this result
Motzkin path with maxim.
if $n = 2k + 1$ is odd.

Statements equivalent to (1) are given in the paper of Diaconis and Freedman (1980, see also Freedman et al. of [1]). They are also mentioned in the paper of Diaconis and Freedman of [5].

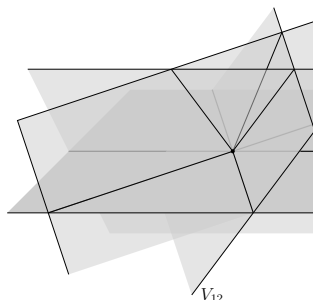
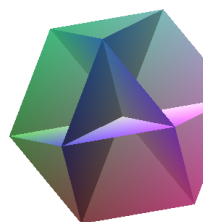
Taking Propositions 3.2 and 3.3 into account, we obtain the following result.

where $|p|$ is the number of vertical steps in the Motzkin path into vertical steps. Since the area of the path is $|p|$, we can rewrite the formula as $\frac{1}{n} \sum_{p \in \mathcal{M}_n} |p|$. For example, if $p = UUHL$, then $|p| = 2$.



6

SAMUEL C. GUTEKUNST, KAROLA MÉSZÁROS, AND



(b)

FIGURE 1. Two views of the threshold arrangement we can see seven chambers above V_0 (the other see these), thus there are $T_2 = 14$ threshold functions (b) the six regions of the resonance arrangement \mathcal{R} restrictions of V_1, V_2 , and V_3 to the subspace V_0 .

The sign vector of a point in \mathbb{R}^n with respect to \mathcal{T}_n is de

where $\tau_S(x) = \text{sgn}(\langle x, v_S \rangle)$.

For example, the point $x = (1, 2, 1)$ has $\tau(x)$ given by

$$(\tau_\emptyset, \tau_1, \tau_2, \tau_{12}) = (-, -, 0, +).$$

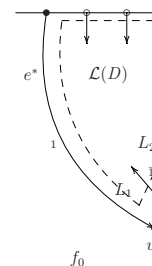
2.2. The resonance arrangement. For any subset $S \subseteq \mathbb{Z}^n$ a vector of length n in which the elements of S denote the entries, if $n = 8$.

$$u_{\{1,3,4,6\}} = (1, 0, 1, 1, 0, 1, 0, 0).$$

PROMC

examination of the growth rule, an upward-pointing 0-edge takes us on a path of upward-pointing 1-edges leading to v_j in Figure 7. Because the path is unique, we know that the final edge is the edge (v_i, v_j) . By the same argument, the local moves we have that $(v_i, v_j) \in \mathcal{R}(D)$ if and only if (v_i, v_j) is a 1-edge in D , which, by parity consideration, is the case if and only if $i \equiv j \pmod{2}$.

We define $\mathcal{L}(D)$ to be the collection of all v_j such that $(v_0, v_j) \in \mathcal{L}(D)$. Similarly, $\mathcal{R}(D)$ denotes the collection of all v_j such that $(v_0, v_j) \in \mathcal{R}(D)$. Let f_0 denote the outer face f_0 . Let $\mathcal{M}(D)$ denote the collection of all v_j such that $(v_0, v_j) \in \mathcal{M}(D)$. Figure 7.



Lemma 3.4. *Let D be a well*

- (1) the depth of every face
- (2) the depth of every face
- (3) the depth of every face

Proof. Let L_1 denote the face on the outer face once the leftmost edge of L is removed. The other faces of $\mathcal{L}(D)$ that are adjacent to L_1 by Lemma 3.3 are consistent with L and hence passes through L_1 . Thus, any face of $\mathcal{L}(D)$ that is adjacent to L_1 is consistent with L . Claim (1) then follows. \square

By examining the faces on a minimal length path through

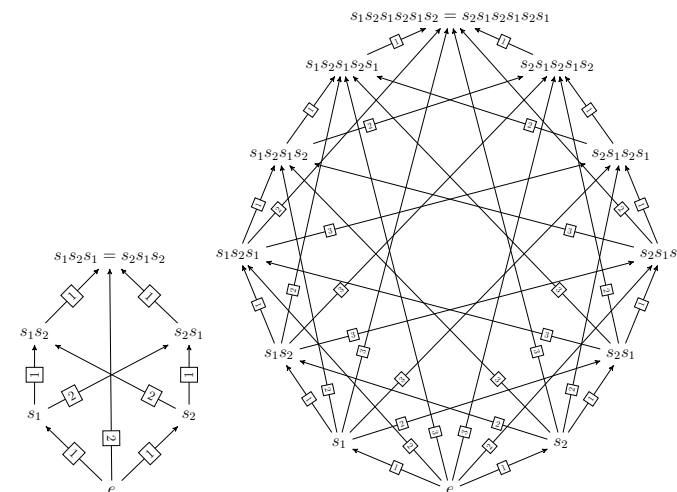


FIGURE 1. The edge-weighted Bruhat graphs of type A_2 and G_2

2.3. Depth for dihedral groups. For dihedral groups, depth is straightforward. Let $I_2(m)$ denote the dihedral group of order $2m$, for $m \leq \infty$. Let $S = \{s_1, s_2\}$ denote the simple reflections.

Proposition 2.7. *For an element $w \in I_2(m)$, we have*

$$\text{dp}(w) = \left\lceil \frac{\ell_S(w) + 1}{2} \right\rceil.$$

Hence,

$$\sum_{w \in I_2(m)} q^{\ell_{\mathbb{S}}(w)} t^{\text{dp}(w)} = \begin{cases} 1 + 2qt + q^{m-1} t^{\frac{m}{2}+1} + 2(1+q)t \sum_{i=1}^{\frac{m}{2}-1} q^{2it^i} & \text{if } m \text{ is even,} \\ 1 + 2qt + q^{m-1} t^{\frac{m+1}{2}} (2+q) + 2(1+q)t \sum_{i=1}^{\frac{m-3}{2}} q^{2it^i} & \text{if } m \text{ is odd, and} \\ 1 + 2qt \cdot \frac{1+qt}{1-q^2t} & \text{if } m = \infty. \end{cases}$$

$\bar{2}$ diamonds that can fit
diamonds on the unused
ove. The distribution of

area for Dyck paths has been studied by many, starting with Carlitz and Riordan [11].

Example 8.11. Let us take $m = 5$ and take the shape L with sinks $(1, 4)$ and $(3, 2)$. Then the associated complex $\mathbf{m}_L^{(nc)*}$ and a realization of the dual polytope are shown

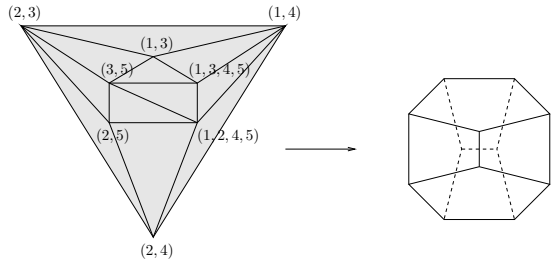


FIGURE 7

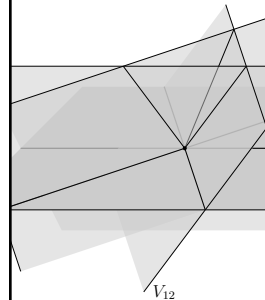
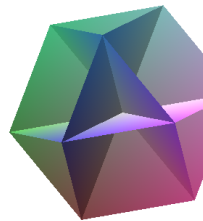
on Figure 7. Here the outer triangle $\{(2, 3), (1, 4), (2, 4)\}$ should also be understood as a face.

Example 8.12. If L is a $2 \times n$ (or an $n \times 2$) rectangle, $\mathbf{m}_L^{(nc)*}$ is the type A cluster complex of $[\mathbb{F}Z]$. It is known to be polytopal, and the dual polytope is known as the *associahedron*. However, Γ_L is usually not simplicial. The first counter-example is when $n = 4$ (so $m = 6$). Here Γ_L has a square face whose vertices correspond to 14, 15, 25 and 24. In $\mathbf{m}_L^{(nc)*}$, this square is subdivided into two triangles, along the diagonal joining $(1, 5)$ and $(2, 4)$.

Example 8.13. Let L be a 3×3 rectangle (so $m = 6$). In this example, we will explore the difference between $\mathbf{m}_L^{(nc)}$ and $\mathbf{m}_L^{(ws)}$. There are 6 solid paths and $N(L) = 9$, so $\mathbf{m}_L^{(nc)*}$ is a 3-sphere. We write $\mathbf{m}_L^{(ws)*}$ for the subcomplex of $\mathbf{m}_L^{(nc)*}$ corresponding to weakly separated paths. There are two pairs of 3-element subsets of $[6]$ which are non-crossing but not weakly separated, namely the pairs $(145, 236)$ and $(124, 356)$. (The first pair of paths crosses twice; the second pair has an hourglass.) Each of these pairs corresponds to an edge in $\mathbf{m}_L^{(nc)*}$. Each of these edges is surrounded by four tetrahedra and these tetrahedra fit together to form an octahedron subdivided around a central axis. These two octahedra are disjoint from one another. In $\mathbf{m}_L^{(ws)*}$, these two octahedra are removed, leaving behind a complex homeomorphic to $S^2 \times [0, 1]$. The endpoints of this product are a pair of 2-spheres, each triangulated as the boundary of the octahedron. The simplicial complex $\mathbf{m}_L^{(ws)*}$ is a subcomplex of the D_4 -cluster complex, which is again a 3-sphere. In the D_4 cluster complex, two new vertices are added. One of these vertices is compatible

the step weights:

TEKUNST, KAROLA MÉSZÁROS, AND



views of the threshold arrangement chambers above V_0 (the other set are $T_2 = 14$ threshold functions of the resonance arrangement \mathcal{R} V_2 , and V_{12} to the subspace V_0 .

point in \mathbb{R}^n with respect to \mathcal{T}_n is de

$$\tau(x) = (\tau_S(x))_{S \subseteq [n-1]},$$

)).

$x = (1, 2, 1)$ has $\tau(x)$ given by

$$(\tau_0, \tau_1, \tau_2, \tau_{12}) = (-, -, 0, +).$$

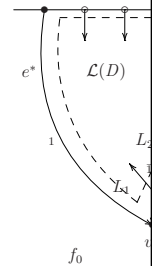
2.2. The resonance arrangement. For any subset $S \subseteq$ vector of length n in which the elements of S denote the ent if $n = 8$,

$$u_{(1,3,4,6)} = (1, 0, 1, 1, 0, 1, 0, 0).$$

PROMO

examination of the growth rule, an upward-pointing 0-edge takes us on to an upward-pointing 1-edge leads in Figure 7. Because the path we know that the final edge of the local moves we have that which, by parity consideration

We define $\mathcal{L}(D)$ to be the dual of $\mathcal{R}(D)$. Similarly, $\mathcal{R}(D)$ denotes the outer face f_0 . Let $\mathcal{M}(D)$ denote the graph in Figure 7.

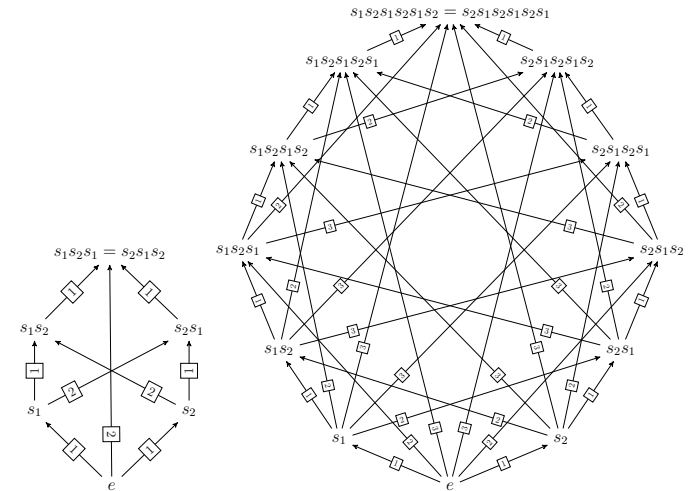


Lemma 3.4. Let D be a weakly separated diagram.

- (1) the depth of every face
- (2) the depth of every face
- (3) the depth of every face

Proof. Let L_1 denote the face of $\mathcal{L}(D)$ that is the outer face once the leftmost edge of D is removed. The other faces of $\mathcal{L}(D)$ that are adjacent to L_1 are consistent passes through L_1 . Thus, any path passing through L_1 is consistent. Claim (1) then follows.

By examining the faces on the boundary of $\mathcal{L}(D)$, one can find a minimal length path through

FIGURE 1. The edge-weighted Bruhat graphs of type A_2 and G_2 .

2.3. Depth for dihedral groups. For dihedral groups, depth is straightforward. Let $I_2(m)$ denote the dihedral group of order $2m$, for $m \leq \infty$. Let $S = \{s_1, s_2\}$ denote the simple reflections.

Proposition 2.7. For an element $w \in I_2(m)$, we have

$$\text{dp}(w) = \left\lceil \frac{\ell_S(w) + 1}{2} \right\rceil.$$

Hence,

$$\sum_{w \in I_2(m)} q^{\ell_S(w)} t^{\text{dp}(w)} = \begin{cases} 1 + 2qt + q^m t^{\frac{m}{2}+1} + 2(1+q)t \sum_{i=1}^{\frac{m}{2}-1} q^{2i} t^i & \text{if } m \text{ is even,} \\ 1 + 2qt + q^{m-1} t^{\frac{m+1}{2}} (2+q) + 2(1+q)t \sum_{i=1}^{\frac{m-3}{2}} q^{2i} t^i & \text{if } m \text{ is odd, and} \\ 1 + 2qt \cdot \frac{1+qt}{1-q^2 t} & \text{if } m = \infty. \end{cases}$$

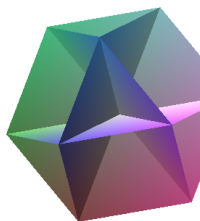
$\frac{1}{2}$ diamonds that can fit into the arrangement of diamonds on the unused side of the arrangement. The distribution of

area for Dyck paths has been studied by many, starting with Carlitz and Riordan [11].

on Figure 7. Here the outer triangle $\{(2, 3), (1, 4), (2, 4)\}$ should also be understood as a face.

Example 8.13. Let L be a 3×3 rectangle (so $m = 6$). In this example, we will explore the difference between $\mathbf{m}_L^{(nc)}$ and $\mathbf{m}_L^{(ws)}$. There are 6 solid paths and $N(L) = 9$, so $\mathbf{m}_L^{(nc)*}$ is a 3-sphere. We write $\mathbf{m}_L^{(ws)*}$ for the subcomplex of $\mathbf{m}_L^{(nc)*}$ corresponding to weakly separated paths. There are two pairs of 3-element subsets of $[6]$ which are non-crossing but not weakly separated, namely the pairs $(145, 236)$ and $(124, 356)$. (The first pair of paths crosses twice; the second pair has to an edge in $\mathbf{m}_L^{(nc)*}$. Each of these tetrahedra fit together to form an octahedron; two octahedra are disjoint from one another leaving behind a complex homeomorphic to a pair of 2-spheres, each triangulated as the complex $\mathbf{m}_L^{(ws)*}$ is a subcomplex of the D the D_4 cluster complex, two new vertices

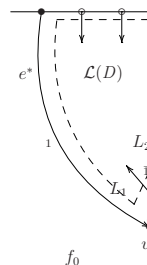
LITEKUNST. KARBOLA MÉSZÁROS. ANTON



PROMC

examination of the growth rule, a non-pointing 0-edge takes us on a downward-pointing 1-edge leads to a new vertex in Figure 7. Because the path is finite, we know that the final edge is non-pointing. The local moves we have that preserve the parity of the number of non-pointing edges, which, by parity consideration, implies that the number of non-pointing edges is even.

We define $\mathcal{L}(D)$ to be the collection of all D ’s such that $D \in \mathcal{L}(D)$. Similarly, $\mathcal{R}(D)$ denotes the collection of all D ’s such that $D \in \mathcal{R}(D)$. Let $\mathcal{M}(D)$ denote the collection of all D ’s such that $D \in \mathcal{M}(D)$. Figure 7.



Lemma 3.4. *Let D be a well ordered set. Then*

- (1) *the depth of every face of D is at most 1.*

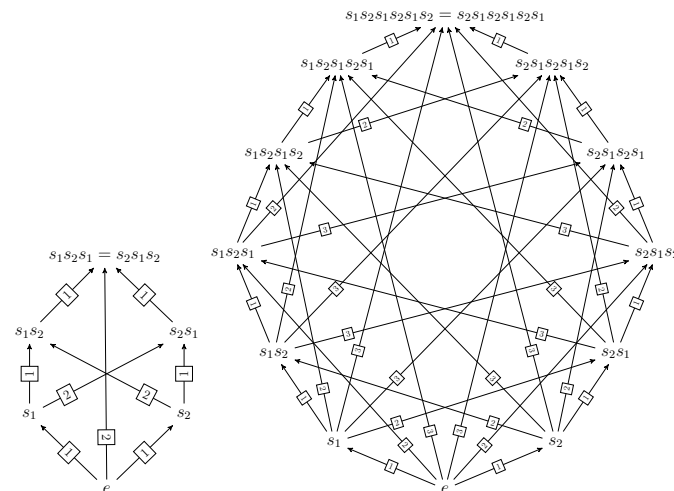


FIGURE 1. The edge-weighted Bruhat graphs of type A_2 and G_2 .

2.3. Depth for dihedral groups. For dihedral groups, depth is straightforward. Let $I_2(m)$ denote the dihedral group of order $2m$, for $m \leq \infty$. Let $S = \{s_1, s_2\}$ denote the simple reflections.

Proposition 2.7. *For an element $w \in I_2(m)$, we have*

$$\text{dp}(w) = \left\lceil \frac{\ell_S(w) + 1}{2} \right\rceil.$$

Hence,

$$\mathrm{dp}(w) = \begin{cases} 1 + 2qt + q^{m t \frac{m}{2} + 1} + 2(1+q)t \sum_{i=1}^{\frac{m}{2}-1} q^{2i t} & \text{if } m \text{ is even,} \\ 1 + 2qt + q^{m-1 t \frac{m+1}{2}} (2+q) + 2(1+q)t \sum_{i=1}^{\frac{m-3}{2}} q^{2i t} & \text{if } m \text{ is odd, and} \\ 1 + 2qt \cdot \frac{1+qt}{1-q^2 t} & \text{if } m = \infty. \end{cases}$$

Combinatorics: art and science of making pretty diagrams!

$$(\tau_\emptyset, \tau_1, \tau_2, \tau_{12}) = (-, -, 0, +).$$

2.2. The resonance arrangement. For any subset $S \subseteq$ vector of length n in which the elements of S denote the entries of \mathbf{v} if $n = 8$.

$$u_{\{1,3,4,6\}} = (1, 0, 1, 1, 0, 1, 0, 0).$$

2 diamonds that can fit
diamonds on the unused
ove. The distribution of

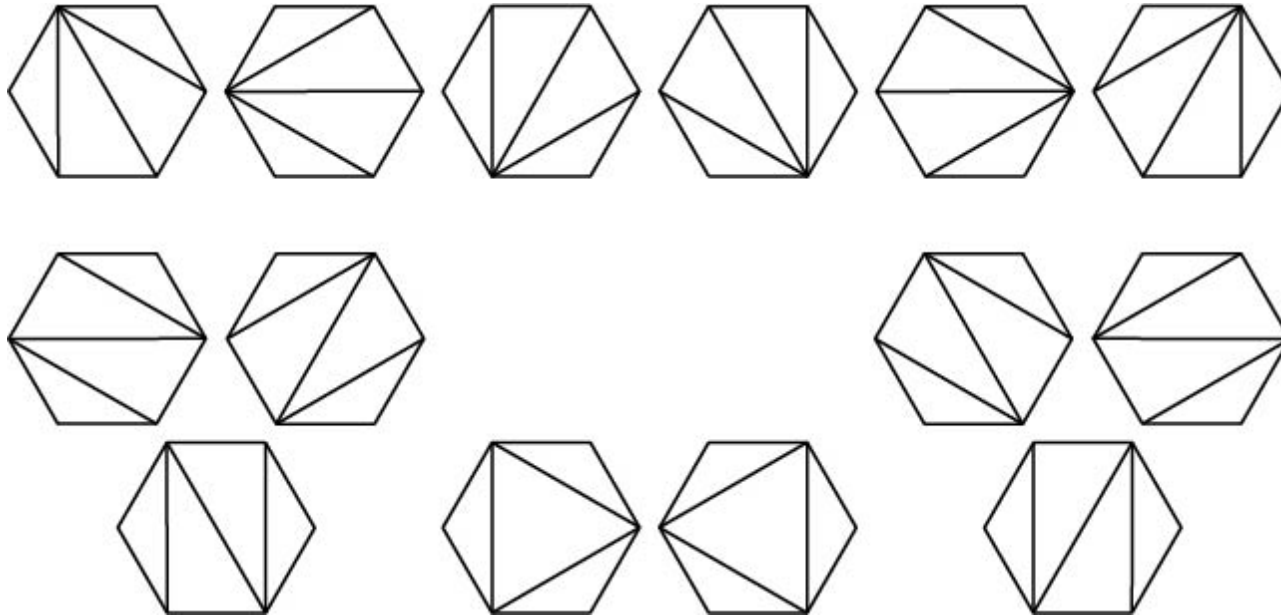
area for Dyck paths has been studied by many, starting with Carlitz and Riordan [11].

Combinatorics: art and science of counting

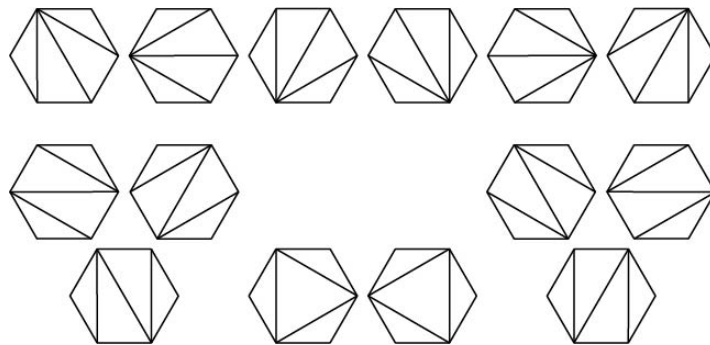
Leonhard Euler ca. 1750
asked: How many ways to
triangulate a polygon?

Combinatorics: art and science of counting

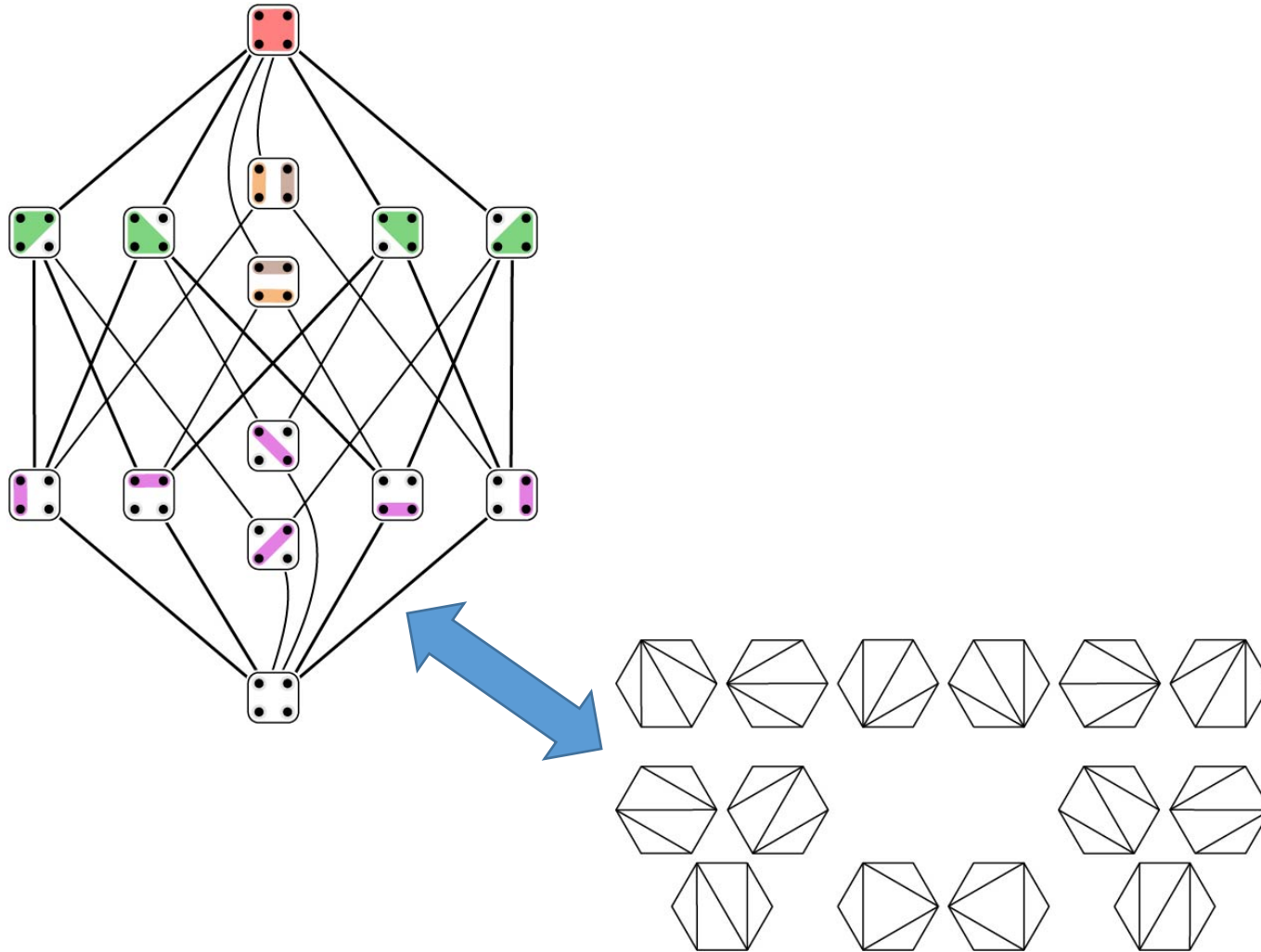
Leonhard Euler ca. 1750
asked: How many ways to
triangulate a polygon?



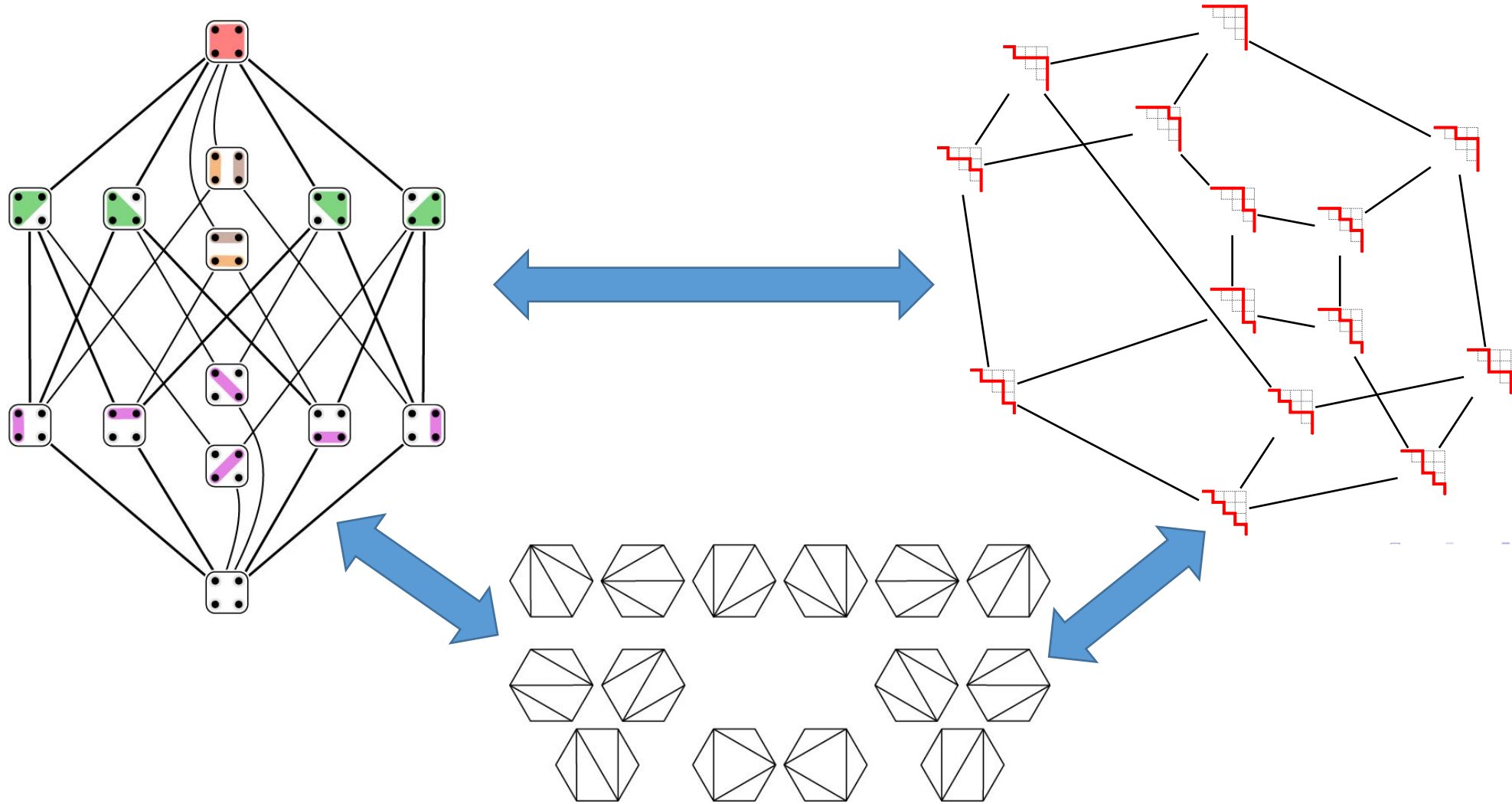
Combinatorics: art and science of counting



Combinatorics: art and science of counting

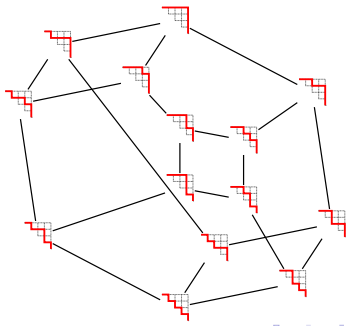


Combinatorics: art and science of counting



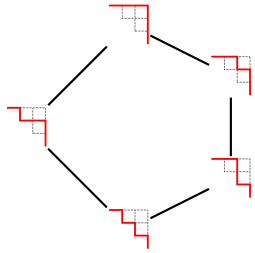
Combinatorics: art and science of counting

$n=4$

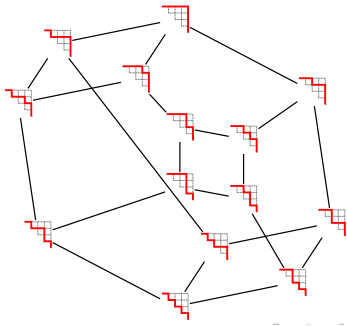


Combinatorics: art and science of counting

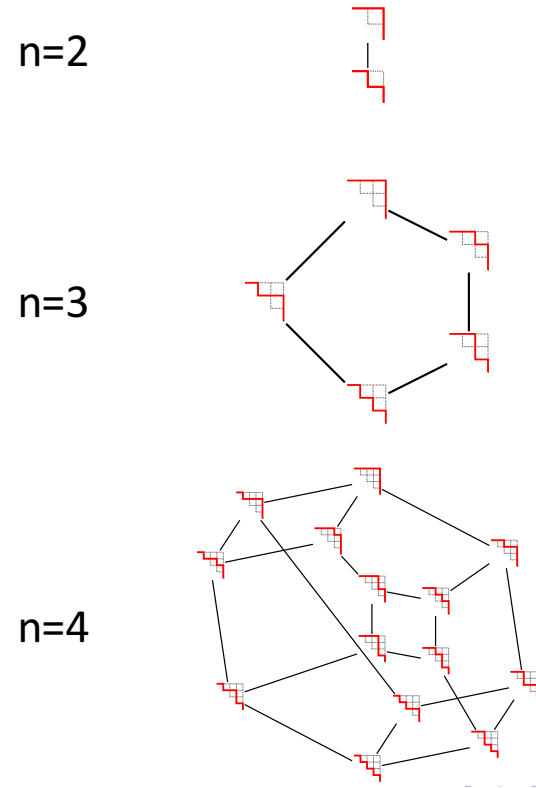
$n=3$



$n=4$



Combinatorics: art and science of counting



Combinatorics: art and science of counting

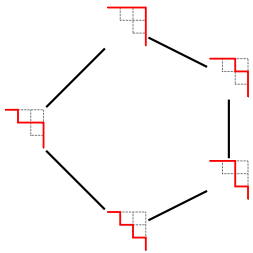
n=1



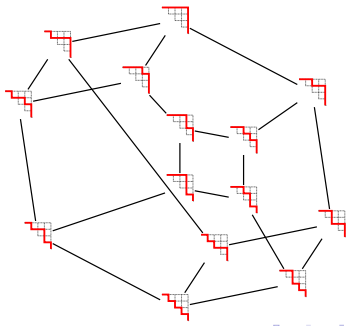
n=2



n=3



n=4



Combinatorics: art and science of counting

$n=0$



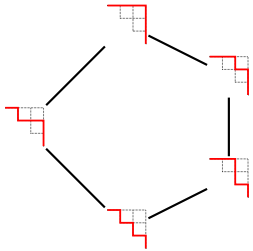
$n=1$



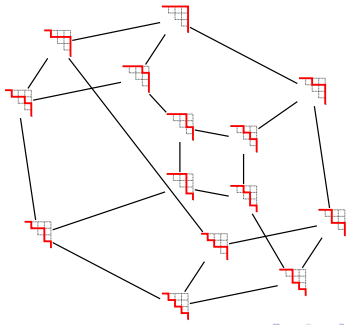
$n=2$



$n=3$



$n=4$

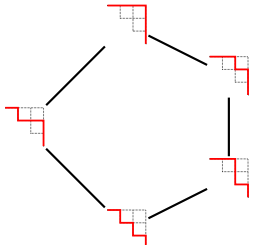


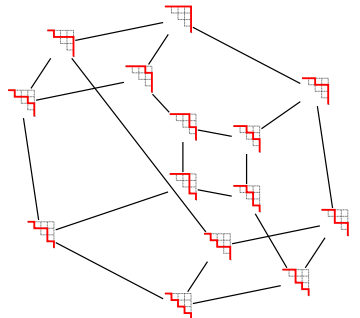
Combinatorics: art and science of counting

$n=0$  $C_0=1$

$n=1$ 

$n=2$ 

$n=3$ 

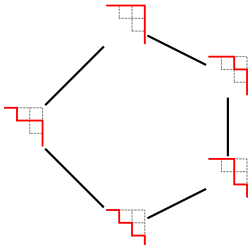
$n=4$ 

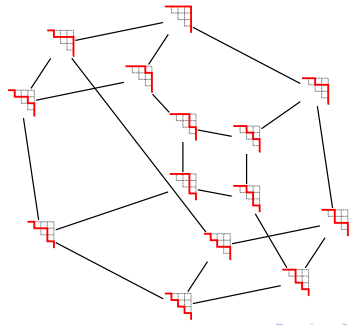
Combinatorics: art and science of counting

$n=0$  $C_0=1$

$n=1$  $C_1=1$

$n=2$ 

$n=3$ 

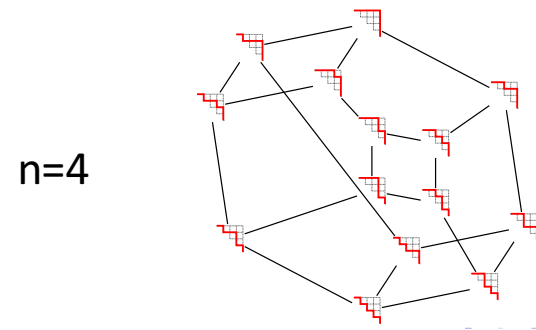
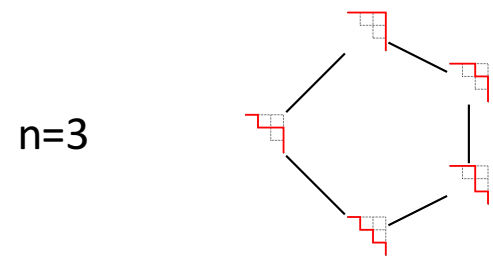
$n=4$ 

Combinatorics: art and science of counting

$n=0$  $C_0=1$

$n=1$  $C_1=1$

$n=2$  $C_2=2$

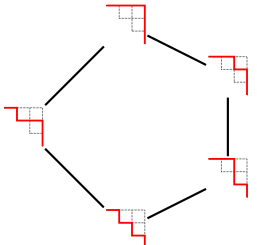


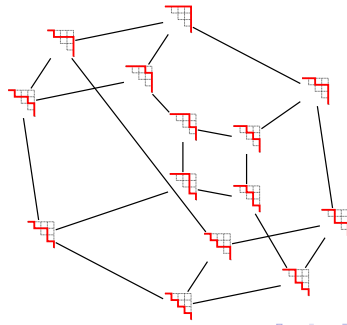
Combinatorics: art and science of counting

$n=0$  $C_0=1$

$n=1$  $C_1=1$

$n=2$  $C_2=2$

$n=3$  $C_3=5$

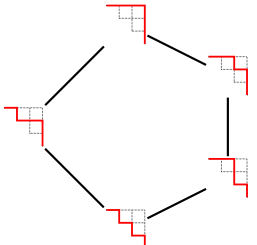
$n=4$ 

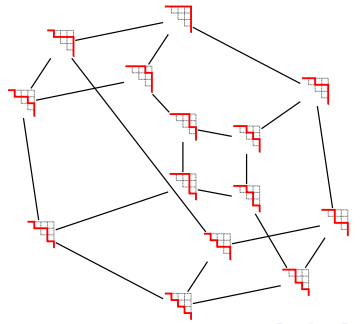
Combinatorics: art and science of counting

$n=0$  $C_0=1$

$n=1$  $C_1=1$

$n=2$  $C_2=2$

$n=3$  $C_3=5$

$n=4$  $C_4=14$

Combinatorics: art and science of counting

n=0



$$C_0=1$$

Formula:

n=1



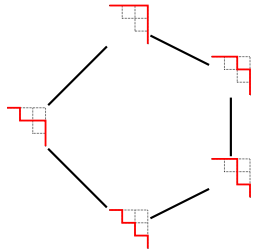
$$C_1=1$$

n=2



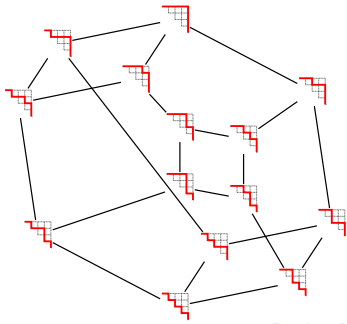
$$C_2=2$$

n=3



$$C_3=5$$

n=4



$$C_4=14$$

$$C_n = \frac{(2n)!}{n! (n+1)!}$$

Combinatorics: art and science of counting

n=0



$$C_0=1$$

Formula:

n=1



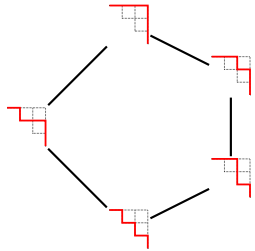
$$C_1=1$$

n=2



$$C_2=2$$

n=3

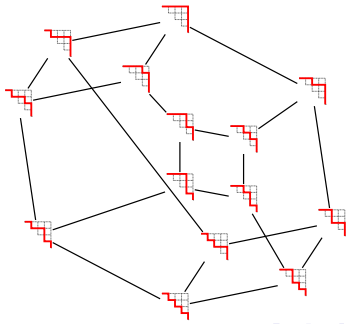


$$C_3=5$$

Generating Function:

$$\frac{1 - \sqrt{1 - 4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots$$

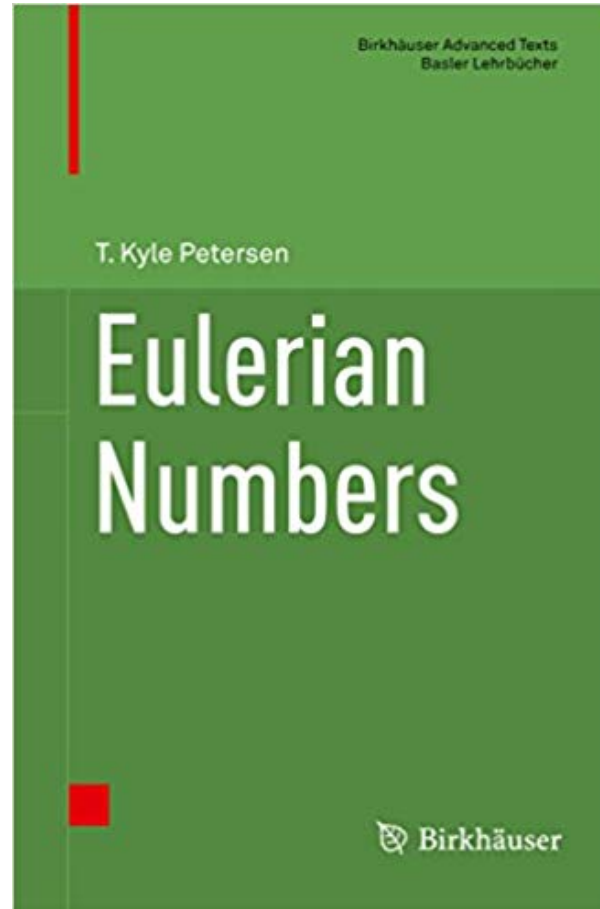
n=4



$$C_4=14$$

Number Triangles and Geometry

Number Triangles and Geometry



Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Pascal's Triangle

Sums

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Pascal's Triangle

Sums

1

1

1

1

1

2

1

1

3

3

1

1

4

6

4

1

Pascal's Triangle

Sums

1

1

1

1

2

1

2

1

1

3

3

1

1

4

6

4

1

Pascal's Triangle

Sums

1

1

1

1

2

1

2

1

4

1

3

3

1

1

4

6

4

1

Pascal's Triangle

Sums

1					1
1	1				2
1	2	1			4
1	3	3	1		8
1	4	6	4	1	

Pascal's Triangle

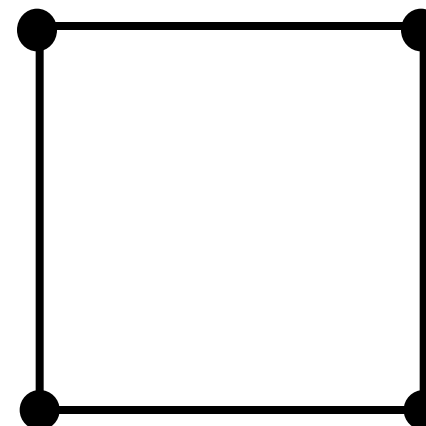
Sums

1					1
1	1				2
1	2	1			4
1	3	3	1		8
1	4	6	4	1	16

Pascal's Triangle

Sums

1					1
1	1				2
1	2	1			4
1	3	3	1		8
1	4	6	4	1	16



Pascal's Triangle

Sums

1

1

1

1

2

1

2

1

4

1

3

3

1

8

1

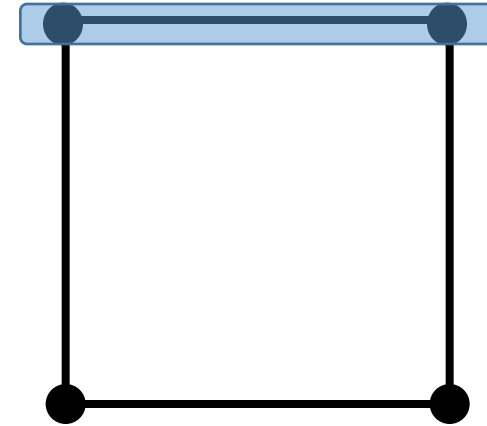
4

6

4

1

16



Pascal's Triangle

Sums

1

1

1

1

2

1

2

1

4

1

3

3

1

8

1

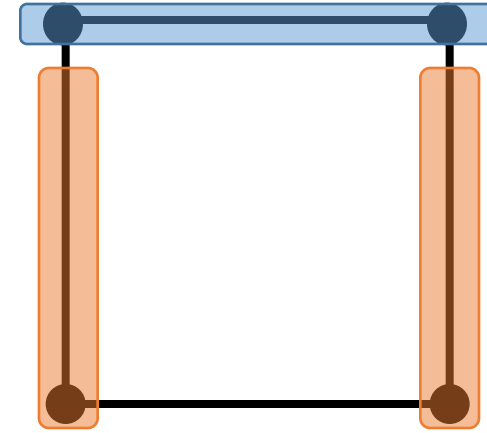
4

6

4

1

16



Pascal's Triangle

Sums

1

1

1

1

2

1

2

1

4

1

3

3

1

8

1

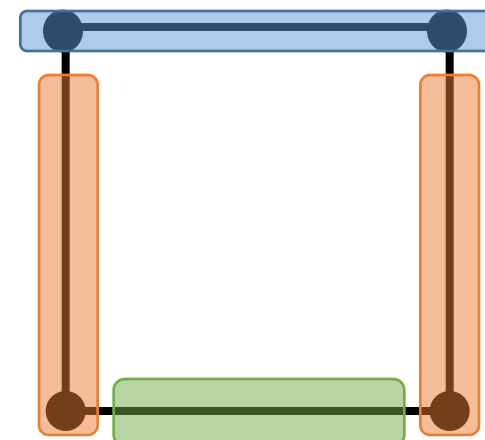
4

6

4

1

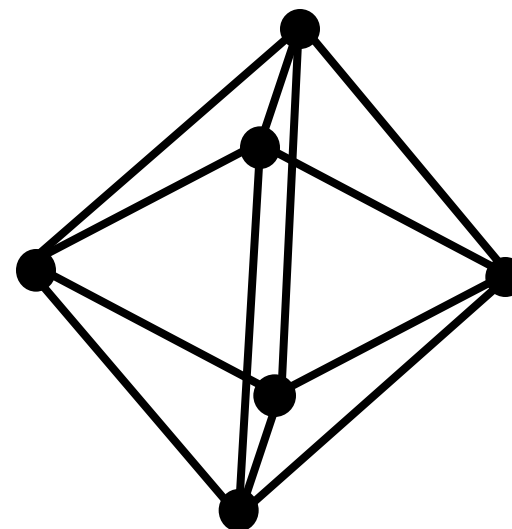
16



Pascal's Triangle

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	

Sums
1
2
4
8
16



Pascal's Triangle

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
					16

Sums

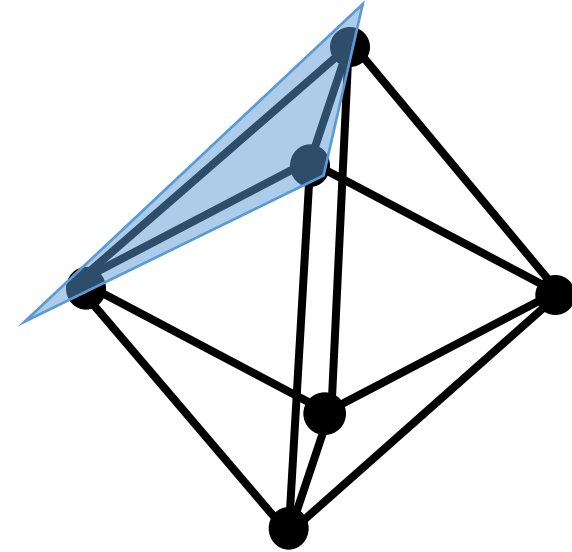
1

2

4

8

16



Pascal's Triangle

1

1

1

1

1

Sums

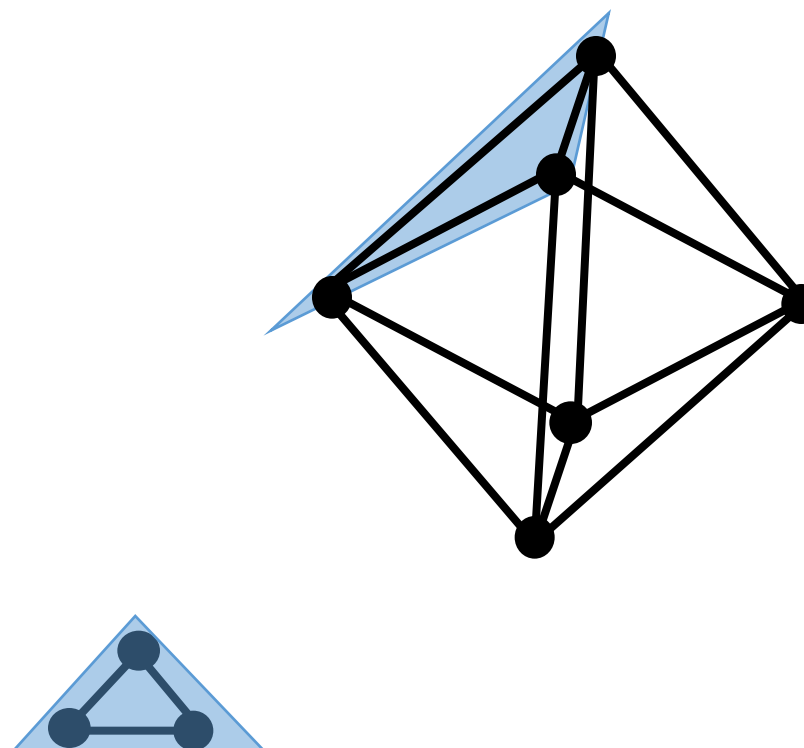
1

2

4

8

16



Pascal's Triangle

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	6	15	10	4	1
1	7	21	14	7	1
1	8	28	16	8	1
1	9	36	24	9	1
1	10	45	35	10	1
1	11	55	46	11	1
1	12	66	58	12	1
1	13	78	71	13	1
1	14	91	84	14	1
1	15	105	98	15	1
1	16	120	112	16	1
1	17	136	128	17	1
1	18	153	144	18	1
1	19	171	166	19	1
1	20	190	189	20	1
1	21	210	210	21	1
1	22	231	242	22	1
1	23	253	276	23	1
1	24	276	312	24	1
1	25	300	350	25	1
1	26	325	390	26	1
1	27	351	432	27	1
1	28	378	476	28	1
1	29	406	522	29	1
1	30	435	570	30	1
1	31	465	620	31	1
1	32	496	672	32	1
1	33	528	726	33	1
1	34	561	782	34	1
1	35	595	840	35	1
1	36	630	900	36	1
1	37	666	962	37	1
1	38	703	1026	38	1
1	39	741	1092	39	1
1	40	780	1160	40	1
1	41	820	1230	41	1
1	42	861	1302	42	1
1	43	903	1376	43	1
1	44	946	1452	44	1
1	45	990	1530	45	1
1	46	1035	1610	46	1
1	47	1081	1692	47	1
1	48	1128	1776	48	1
1	49	1176	1862	49	1
1	50	1225	1950	50	1
1	51	1275	2040	51	1
1	52	1326	2132	52	1
1	53	1378	2226	53	1
1	54	1431	2322	54	1
1	55	1485	2420	55	1
1	56	1540	2520	56	1
1	57	1596	2622	57	1
1	58	1653	2726	58	1
1	59	1711	2832	59	1
1	60	1770	2940	60	1
1	61	1830	3050	61	1
1	62	1891	3162	62	1
1	63	1953	3276	63	1
1	64	2016	3392	64	1
1	65	2080	3510	65	1
1	66	2145	3630	66	1
1	67	2211	3752	67	1
1	68	2278	3876	68	1
1	69	2346	4002	69	1
1	70	2415	4130	70	1
1	71	2485	4260	71	1
1	72	2556	4392	72	1
1	73	2628	4526	73	1
1	74	2701	4662	74	1
1	75	2775	4800	75	1
1	76	2850	4940	76	1
1	77	2926	5082	77	1
1	78	3003	5226	78	1
1	79	3081	5372	79	1
1	80	3160	5520	80	1
1	81	3240	5670	81	1
1	82	3321	5822	82	1
1	83	3403	5976	83	1
1	84	3486	6132	84	1
1	85	3570	6290	85	1
1	86	3655	6450	86	1
1	87	3741	6612	87	1
1	88	3828	6776	88	1
1	89	3916	6942	89	1
1	90	4005	7110	90	1
1	91	4095	7280	91	1
1	92	4186	7452	92	1
1	93	4278	7626	93	1
1	94	4371	7802	94	1
1	95	4465	7980	95	1
1	96	4560	8160	96	1
1	97	4656	8342	97	1
1	98	4753	8526	98	1
1	99	4851	8712	99	1
1	100	4950	8900	100	1

Sums

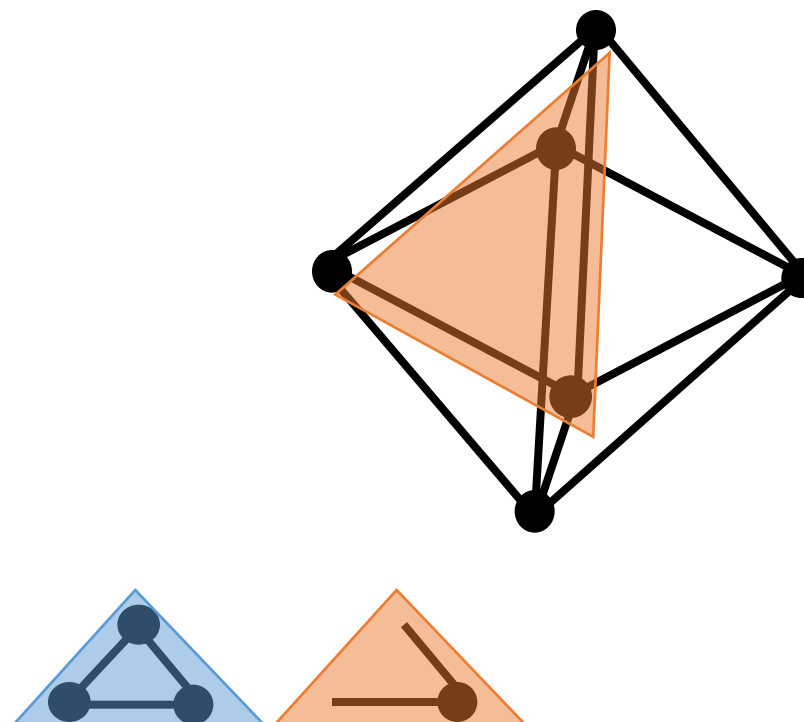
1

2

4

8

16



Pascal's Triangle

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	

Sums

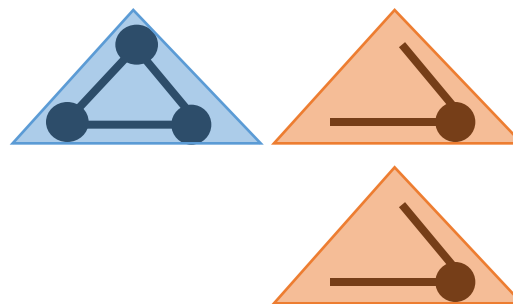
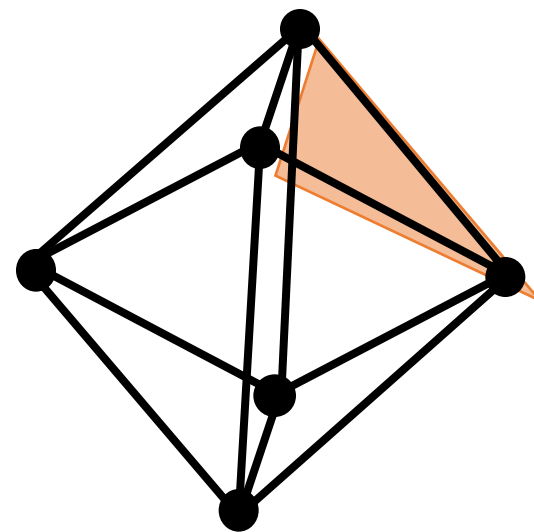
1

2

4

8

16



Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Sums

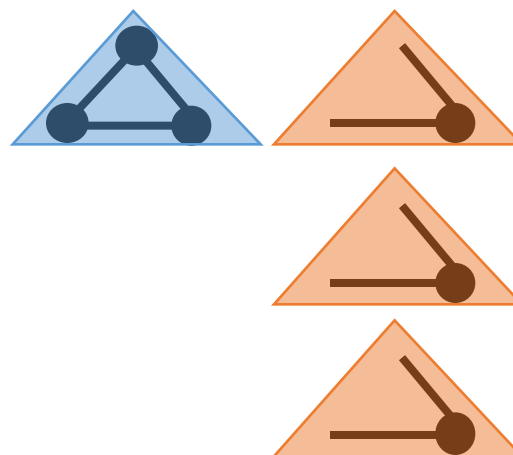
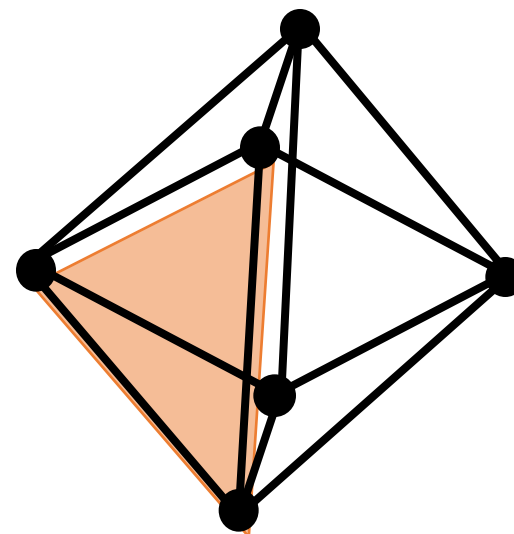
1

2

4

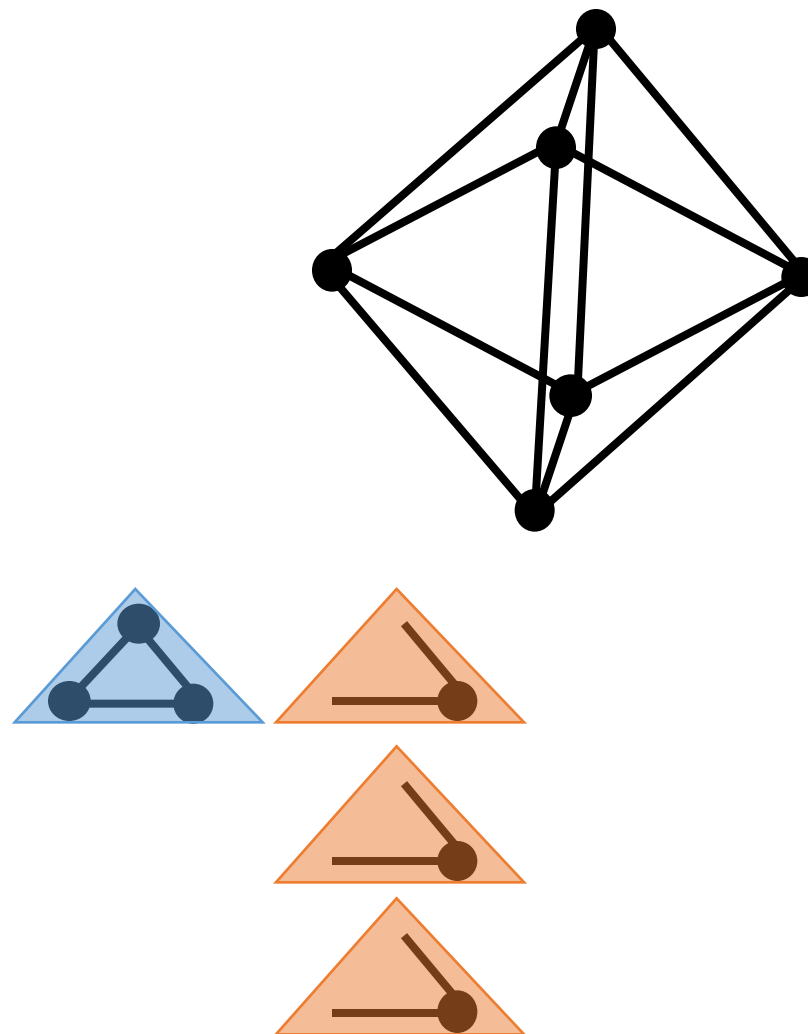
8

16



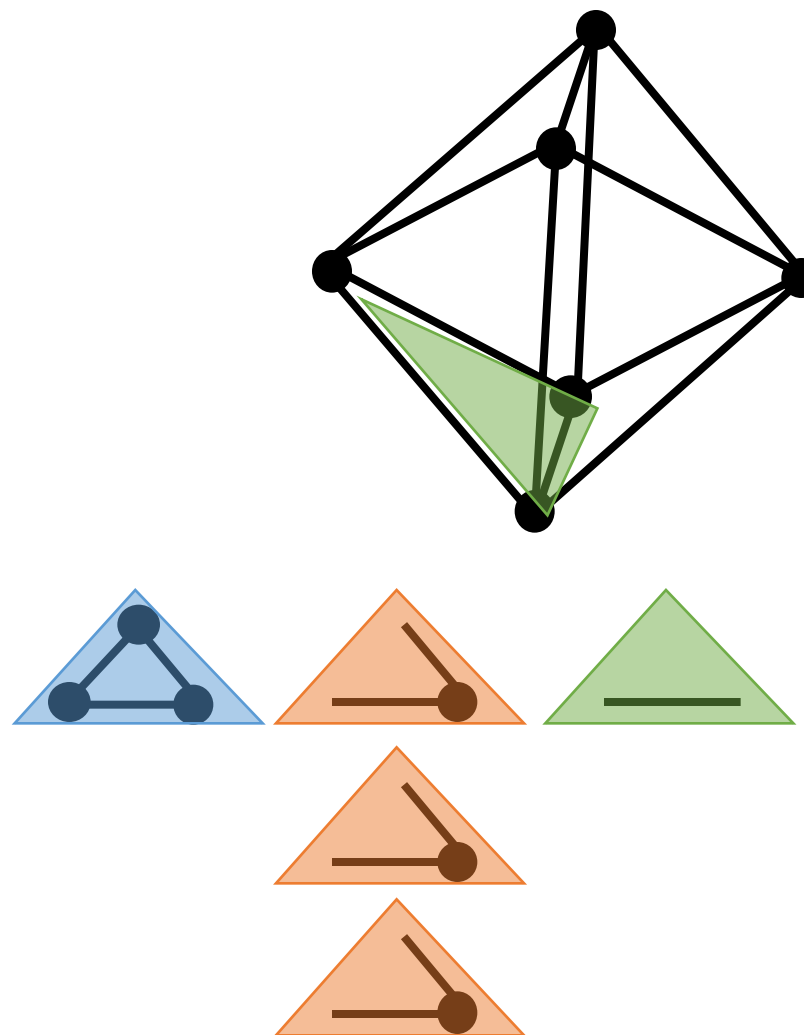
Pascal's Triangle

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
Sums					
1					
2					
4					
8					
16					



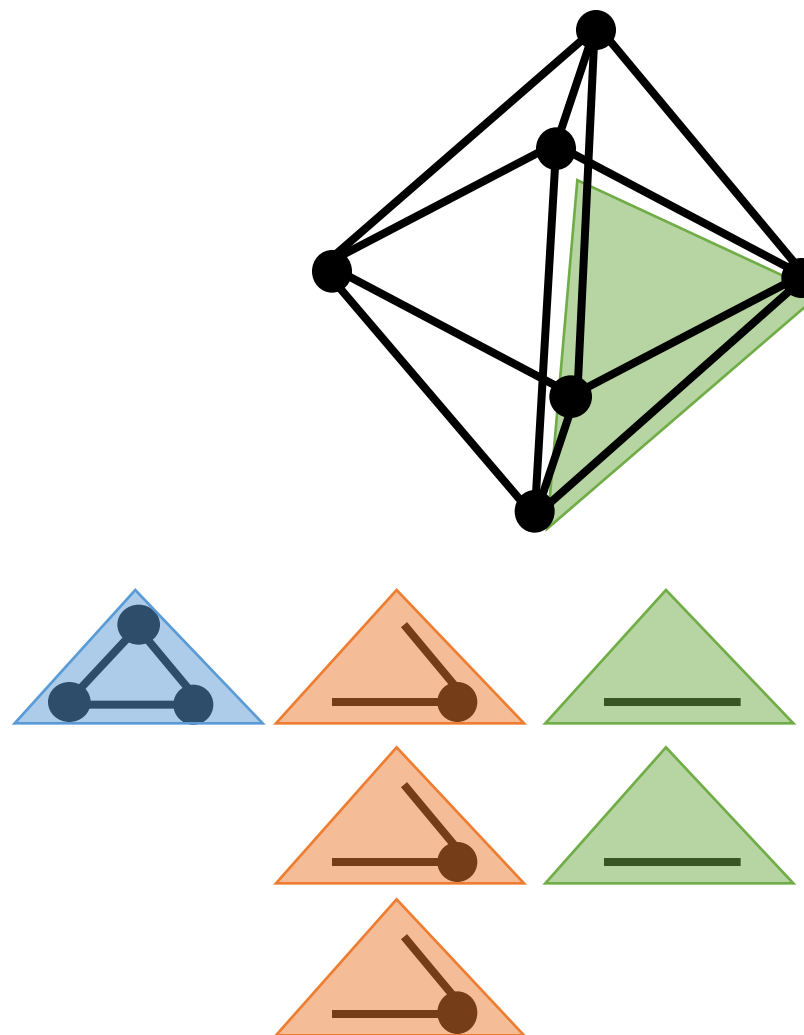
Pascal's Triangle

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
Sums					
1					
2					
4					
8					
16					



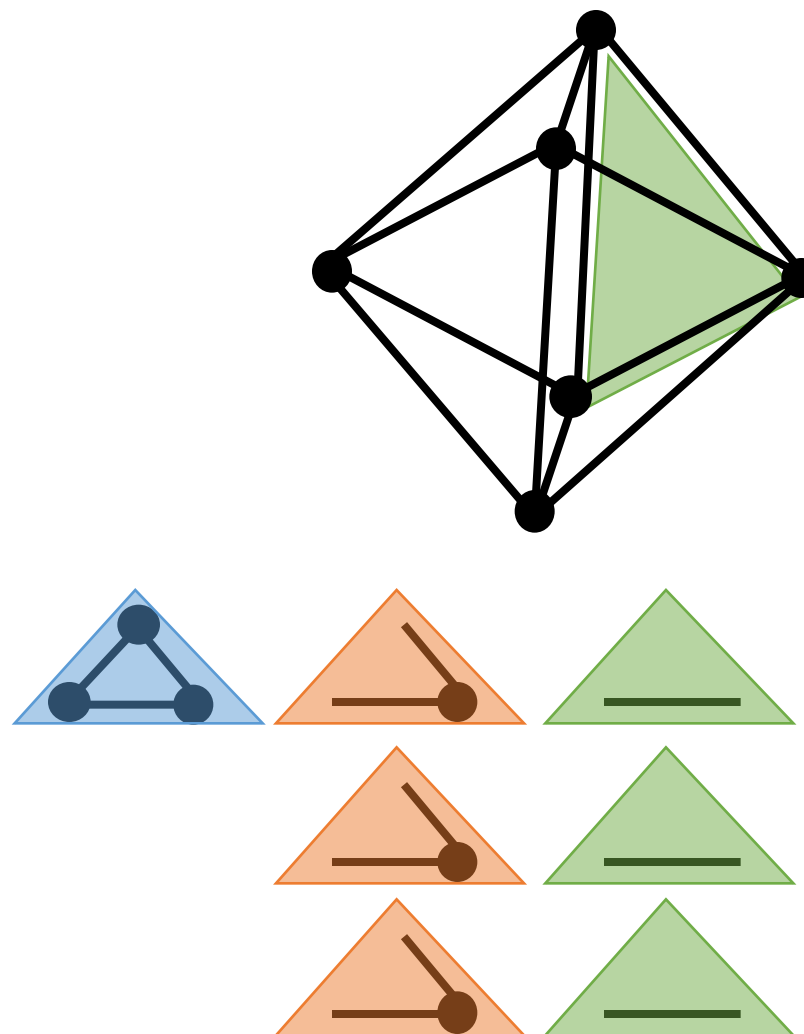
Pascal's Triangle

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
Sums					
1					
2					
4					
8					
16					



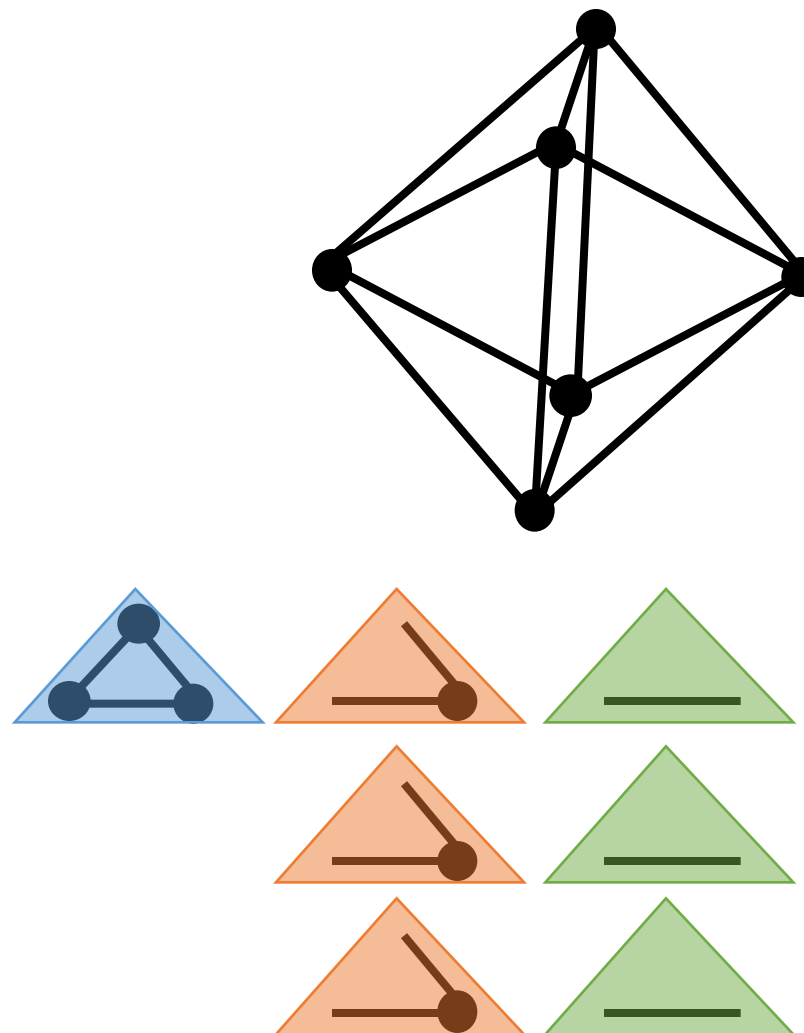
Pascal's Triangle

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
Sums					
1					
2					
4					
8					
16					



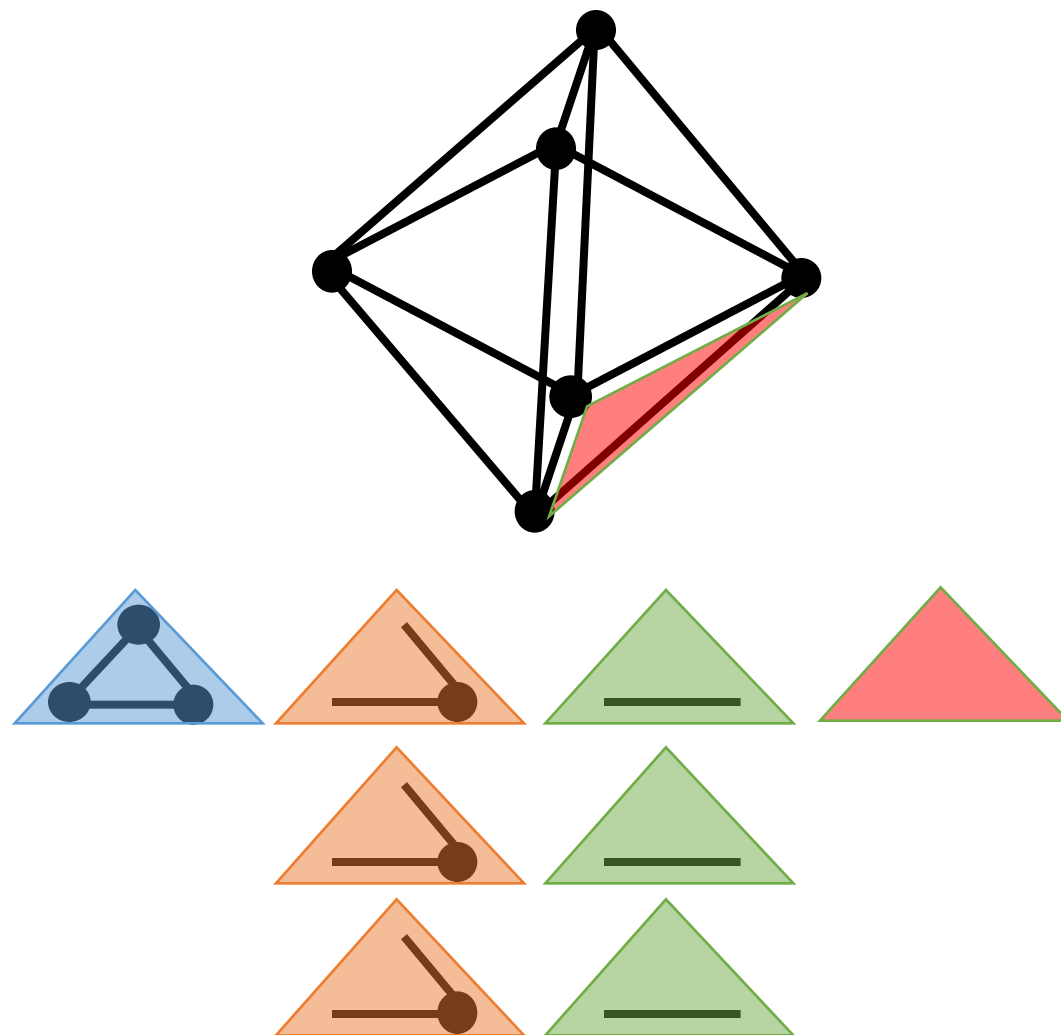
Pascal's Triangle

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
Sums					
1					
2					
4					
8					
16					



Pascal's Triangle

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
Sums					
1					
2					
4					
8					
16					

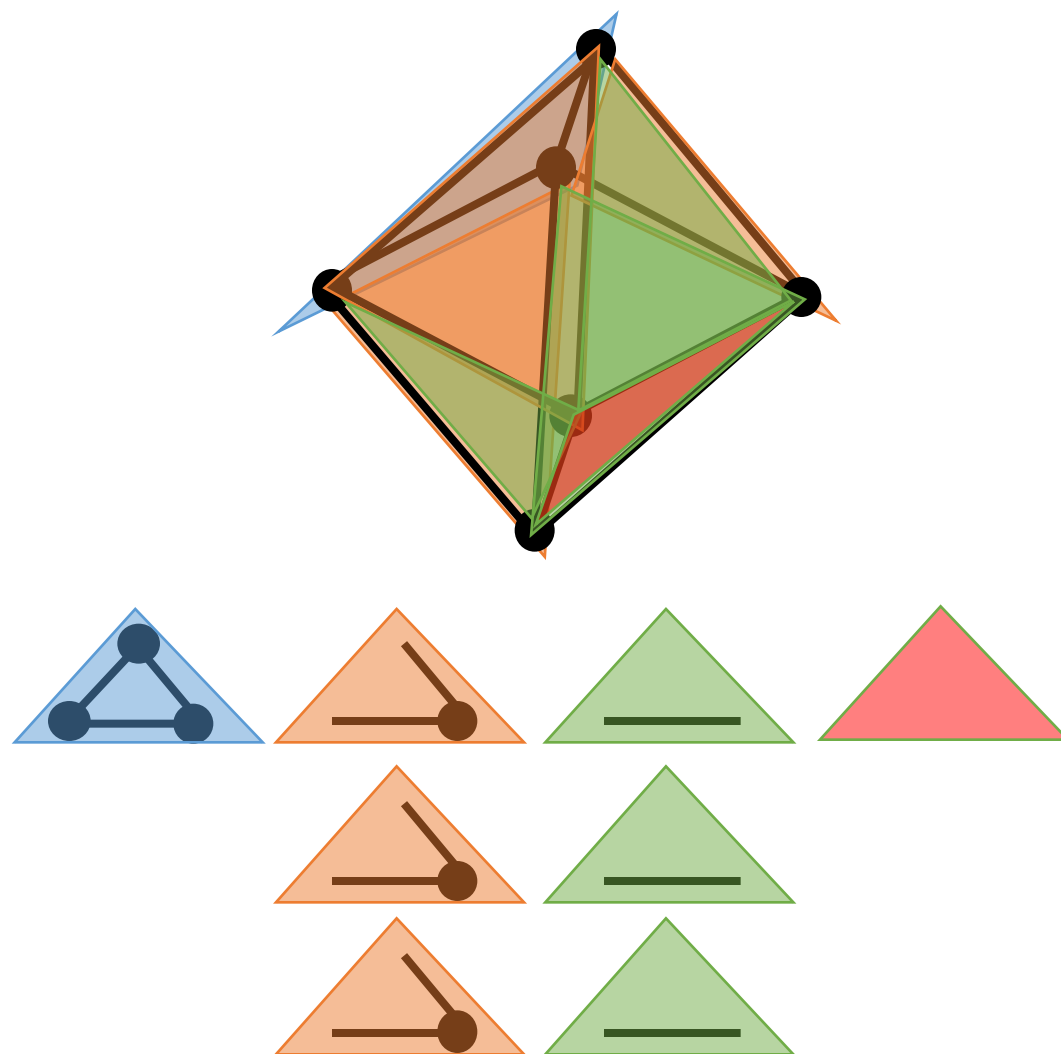


Pascal's Triangle

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

Sums

1
2
4
8
16

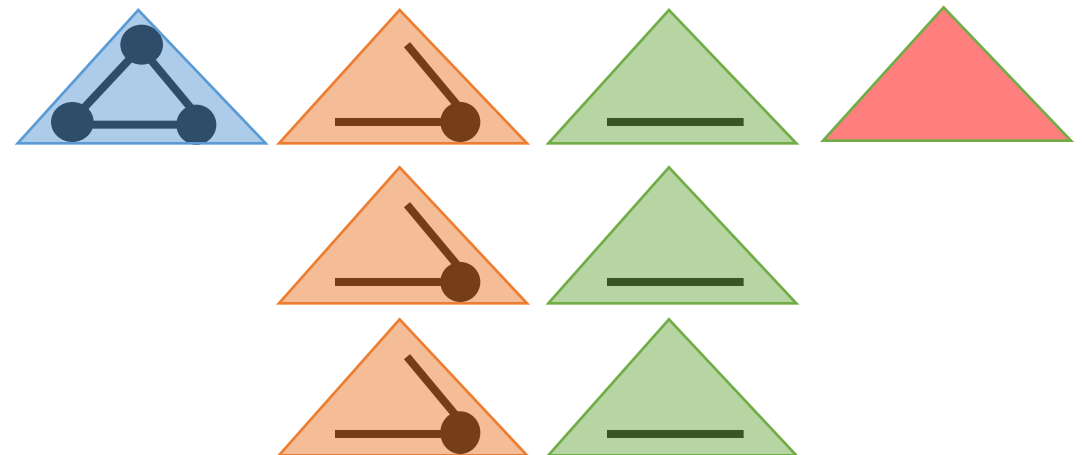
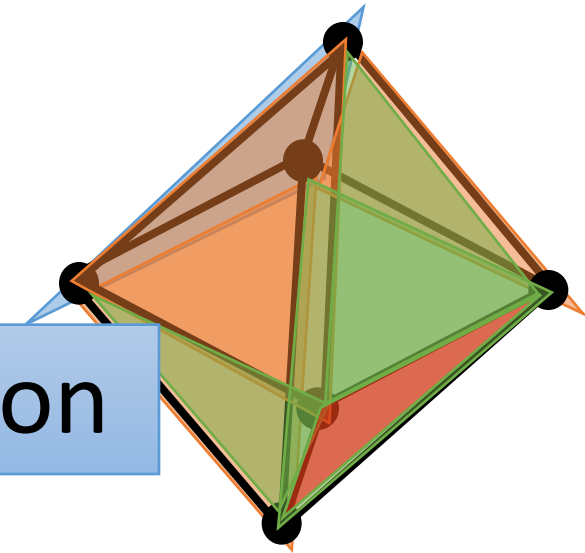


Pascal's Triangle

Sums

1					1
1	1				2
1	2				
1	3	3	1		8
1	4	6	4	1	16

“Shelling” the octahedron



Euler's Triangle

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

Euler's Triangle

Sums

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

Euler's Triangle

Sums

1

1

1

1

1

4

1

1

11

11

1

1

26

66

26

1

Euler's Triangle

Sums

1

1

1

1

2

1

4

1

1

11

11

1

1

26

66

26

1

Euler's Triangle

Sums

1

1

1

1

2

1

4

1

6

1

11

11

1

1

26

66

26

1

Euler's Triangle

Sums

1					1
1	1				2
1	4	1			6
1	11	11	1		24
1	26	66	26	1	

Euler's Triangle

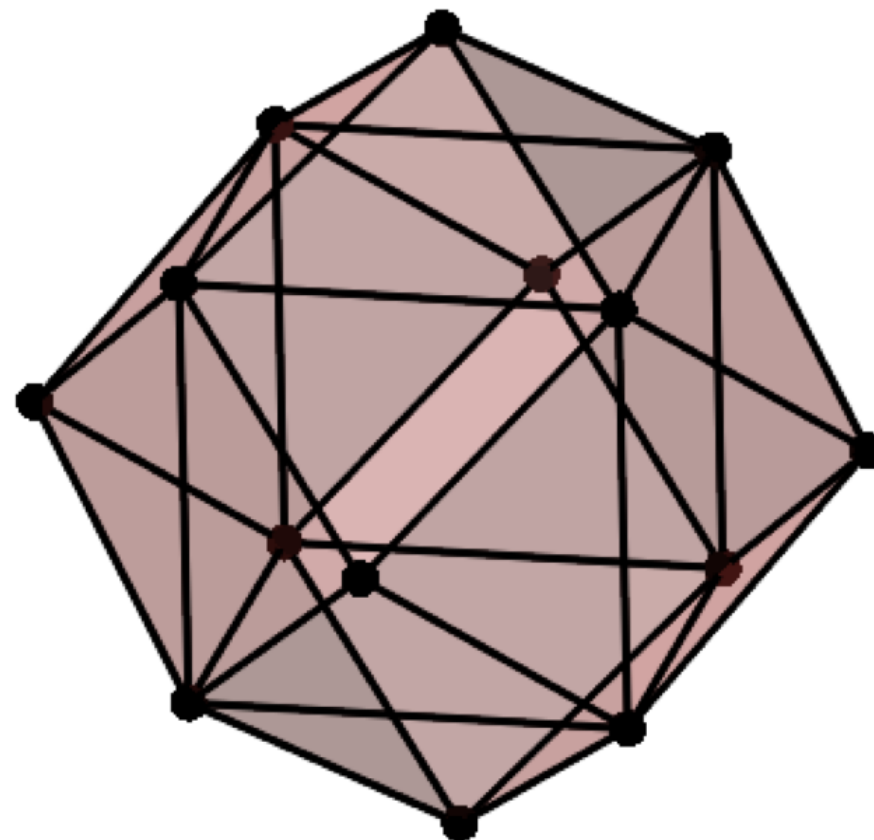
Sums

1					1
1	1				2
1	4	1			6
1	11	11	1		24
1	26	66	26	1	120

Euler's Triangle

1					
1	1				
1	4	1			
1	11	11	1		
1	26	66	26	1	

Sums
1
2
6
24
120



Euler's Triangle

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

Sums

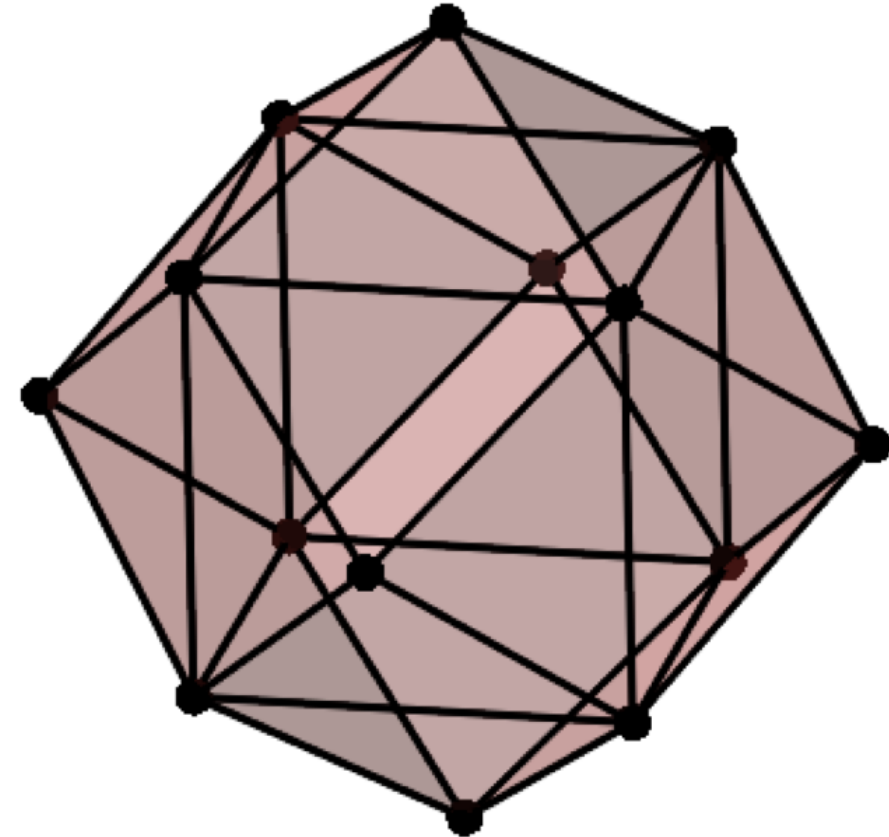
1

2

6

24

120



“Shelling” the Coxeter complex

Narayana's Triangle

1

1 1

1 3 1

1 6 6 1

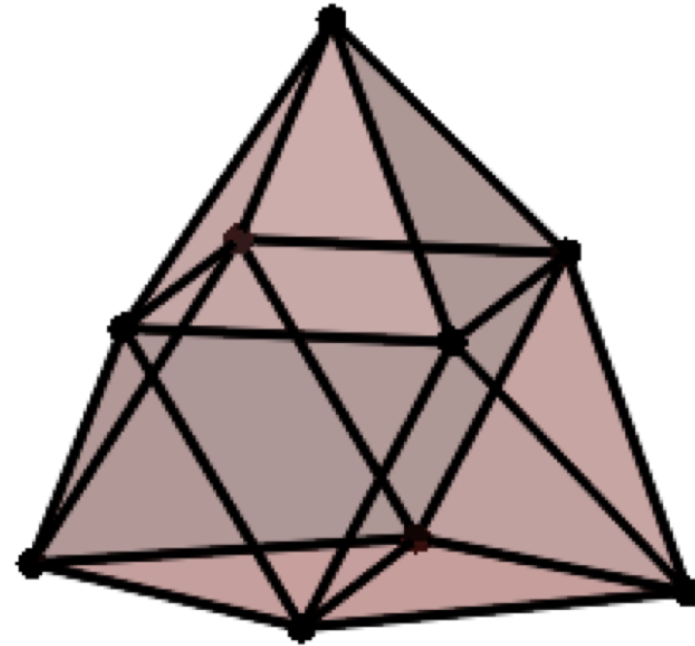
1 10 20 10 1

Narayana's Triangle

1					1
1	1				2
1	3	1			5
1	6	6	1		14
1	10	20	10	1	42

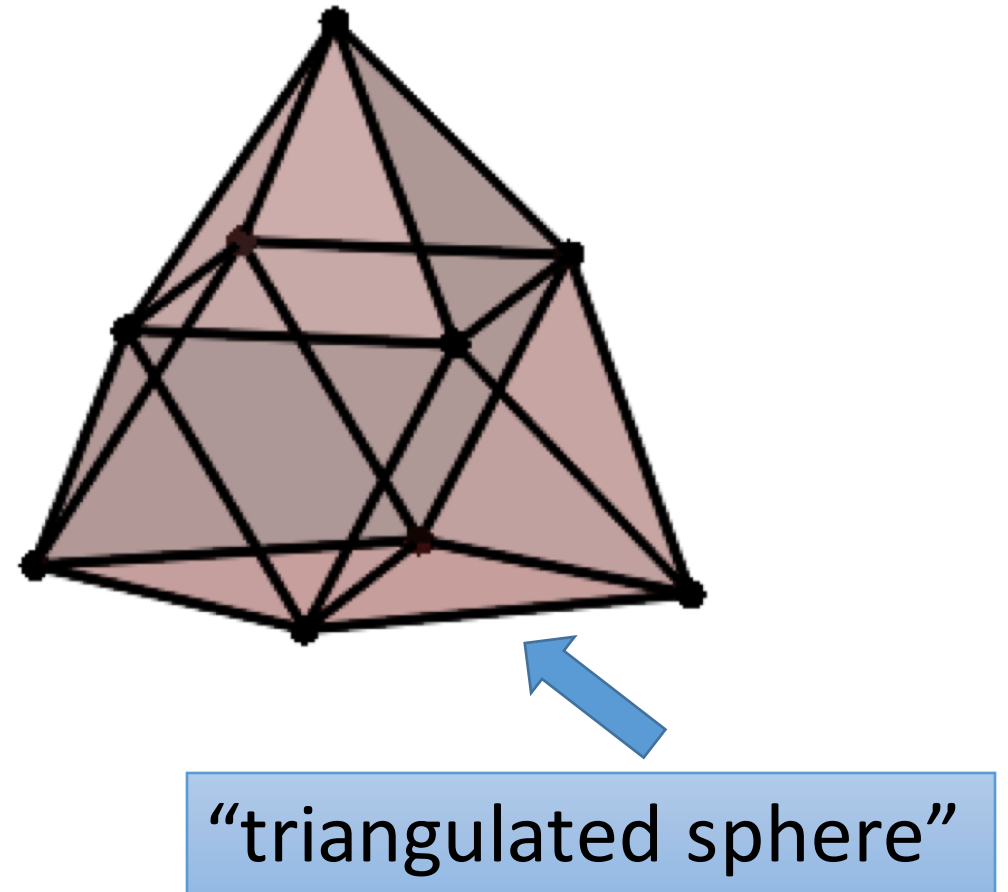
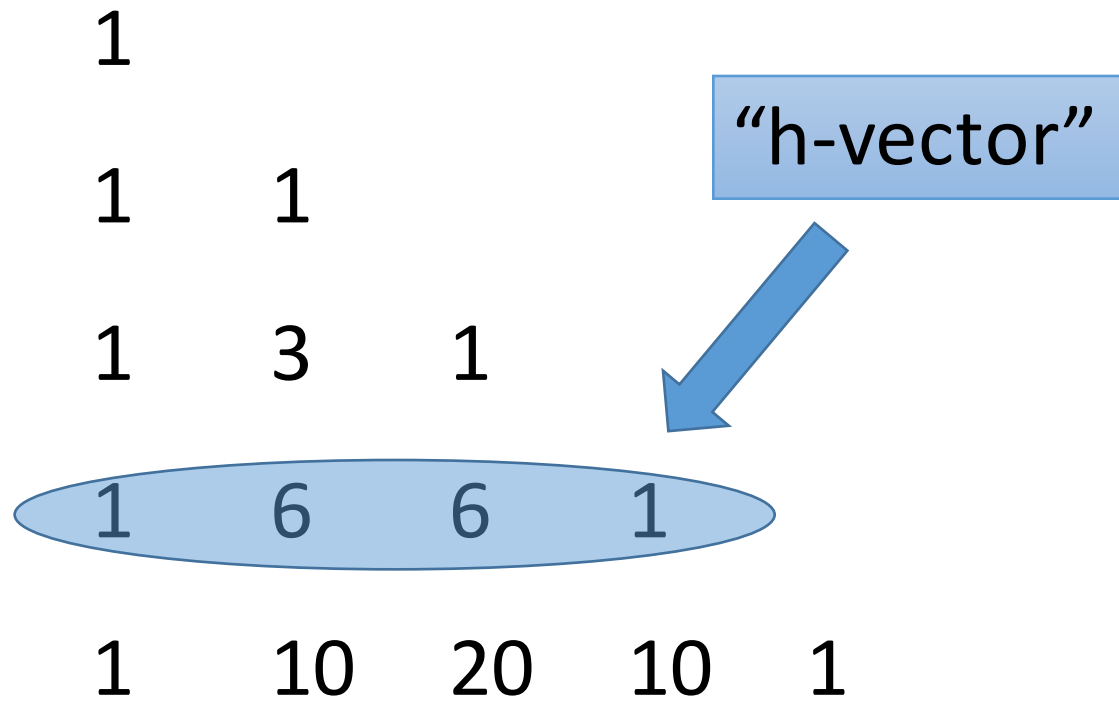
Narayana's Triangle

1						1
1	1					2
1	3	1				5
1	6	6	1			14
1	10	20	10	1		42

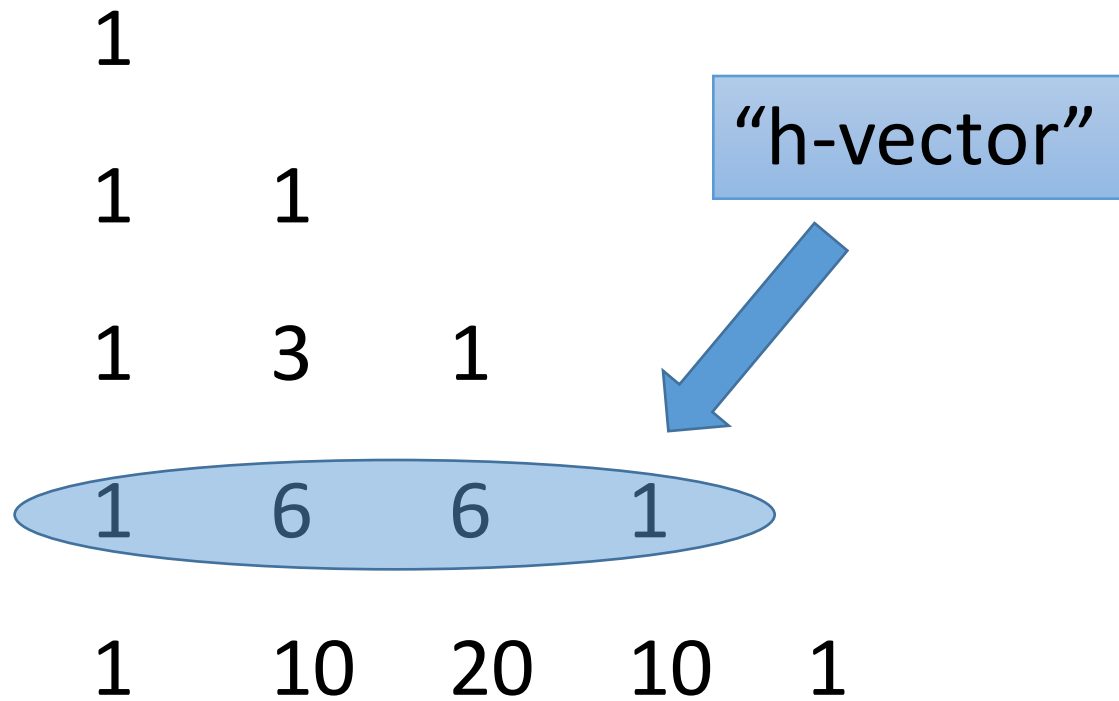


“Shelling” the noncrossing complex

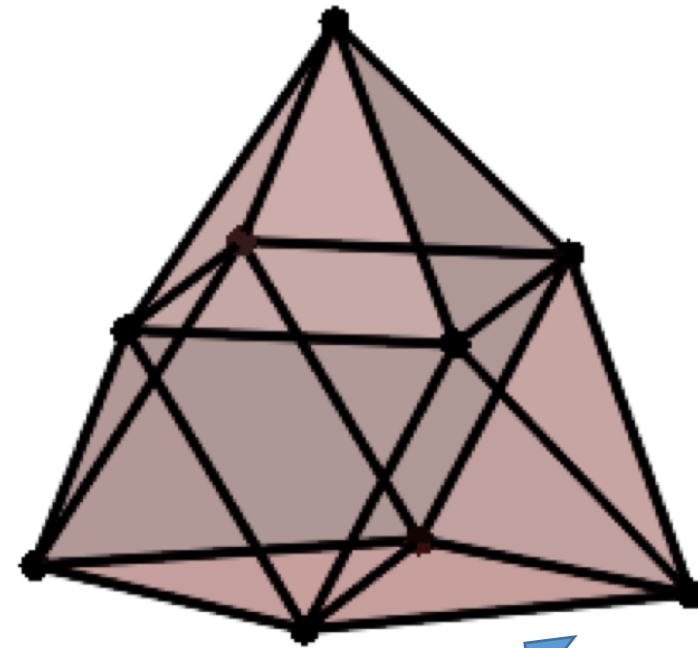
Combinatorial Topology



Combinatorial Topology



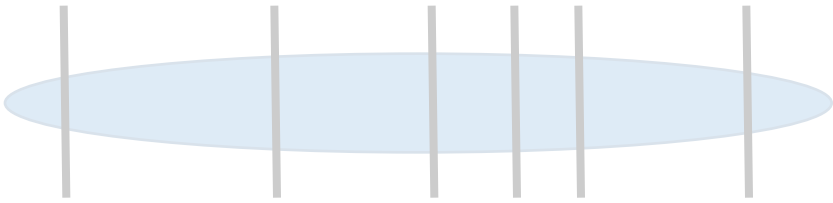
Big fish: Hopf conjecture, Charney-Davis conjecture, g-conjecture...



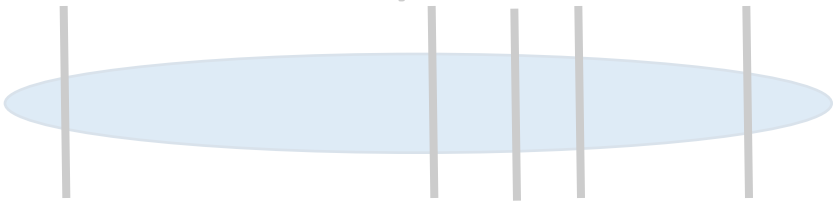
"triangulated sphere"

Back to Alice and Bob...

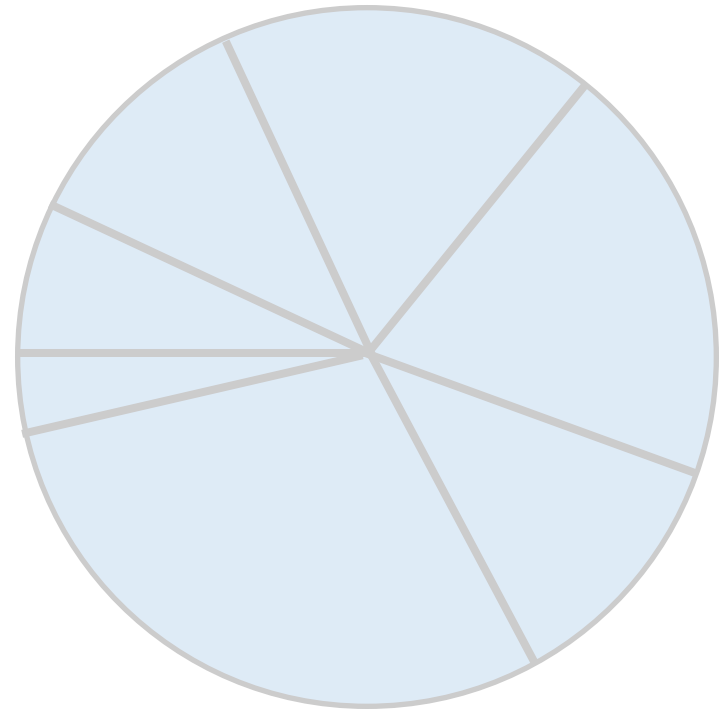
Version I: Submarine Sandwich



Version II: Submarine Sandwich
(even number)



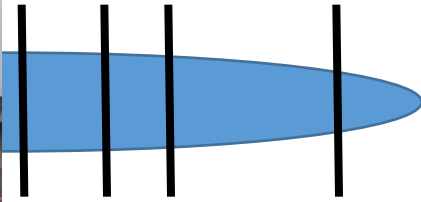
Version III: Pizza



Alice and Bob have lunch



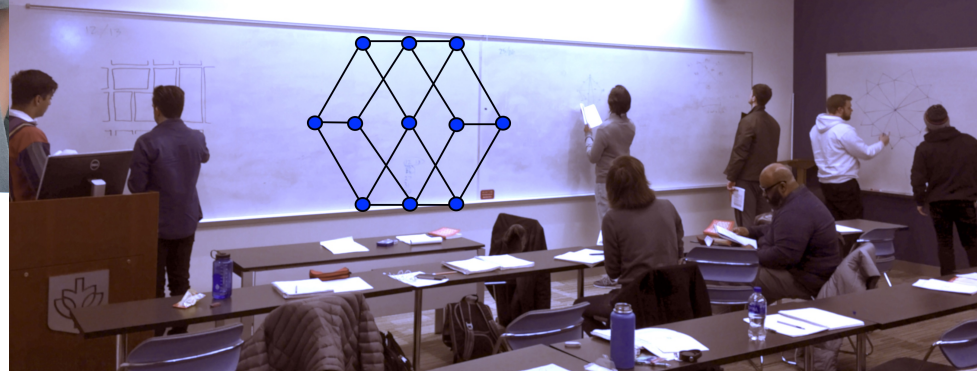
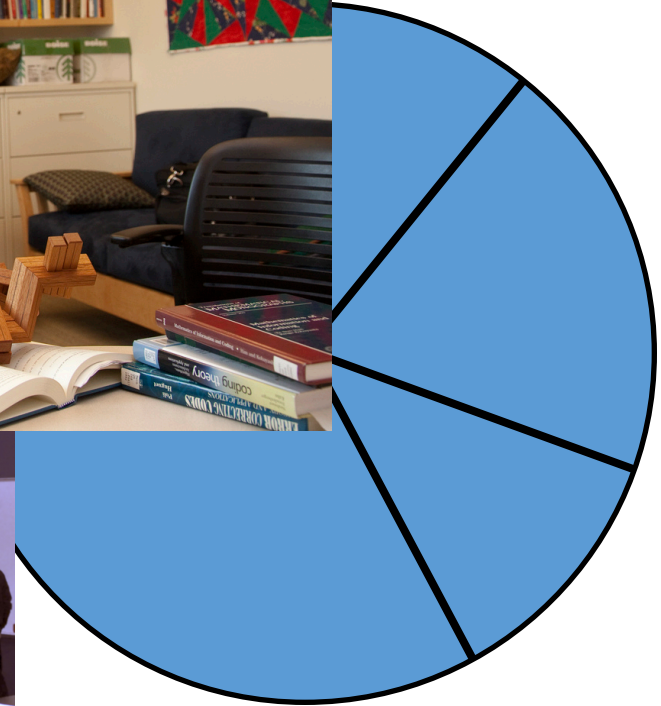
marine Sandv



marine Sandv

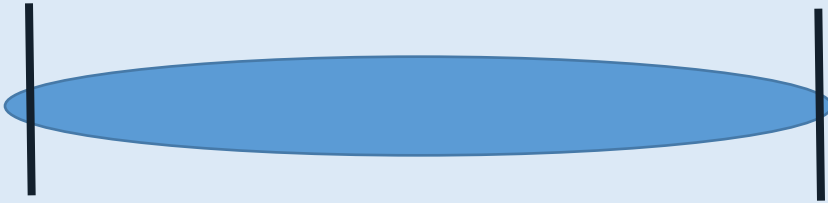


l: Pizza

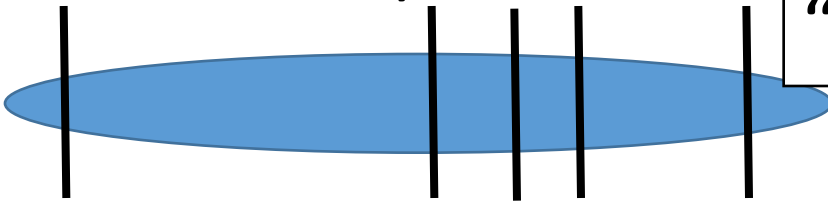


Alice and Bob have lunch

Version I: Submarine Sandwich



Version II: Submarine Sandwich
(even number)

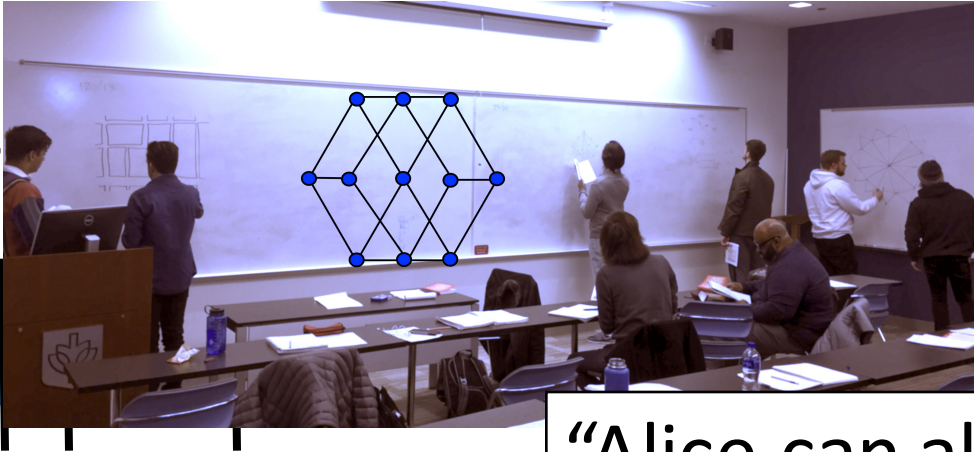
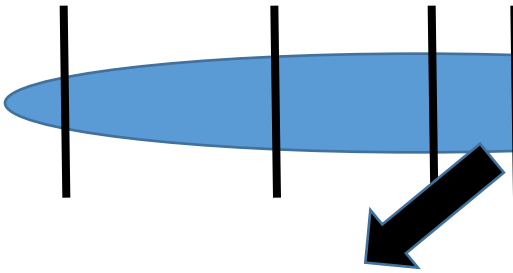


“Bob can get as much as he wants!”

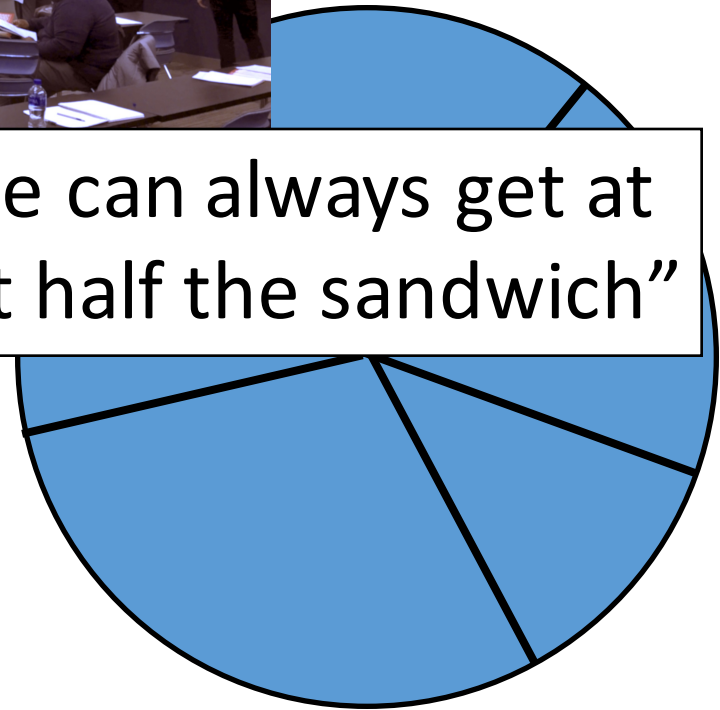


Alice and Bob have lunch

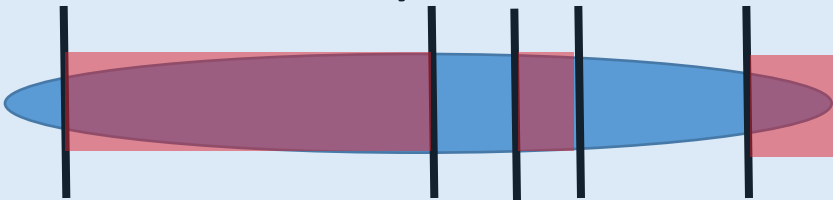
Version I: Submarine



Version III: Pizza



Version II: Submarine Sandwich
(even number)



“Alice can always get at least half the sandwich”

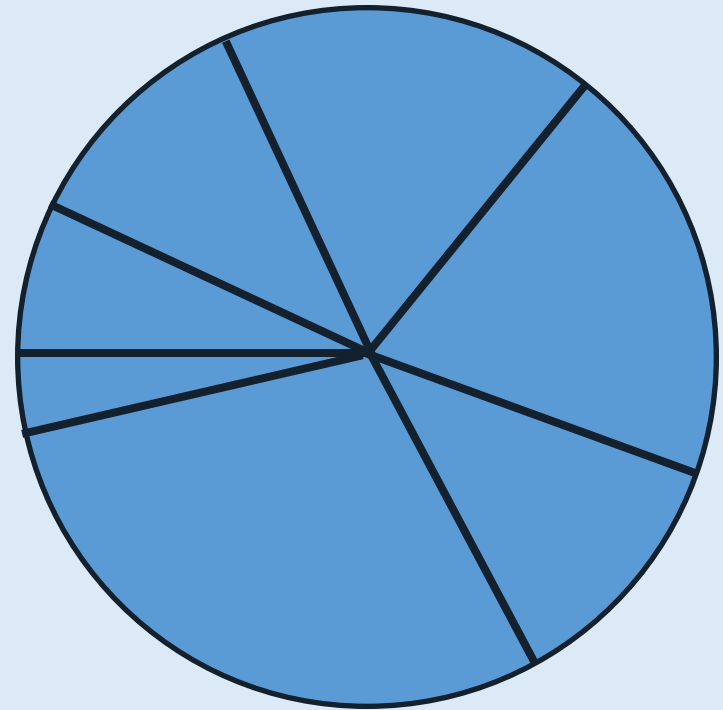
Alice and Bob have lunch

Version I: Sandwich

Version II: Sandwich
(even number)



Version III: Pizza



Alice and Bob have lunch



“Bob can get $\frac{5}{9}$ of the pizza”
(...??!!?)

