Reflection length in affine Coxeter groups

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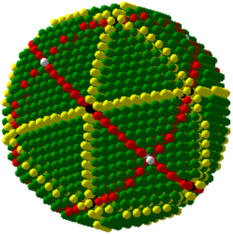
George Washington University, UC–Santa Barbara, DePaul University, Karlruhe Institute of Technology

FPSAC 2018



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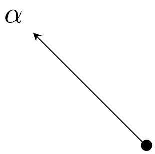
Real hyperplane reflections



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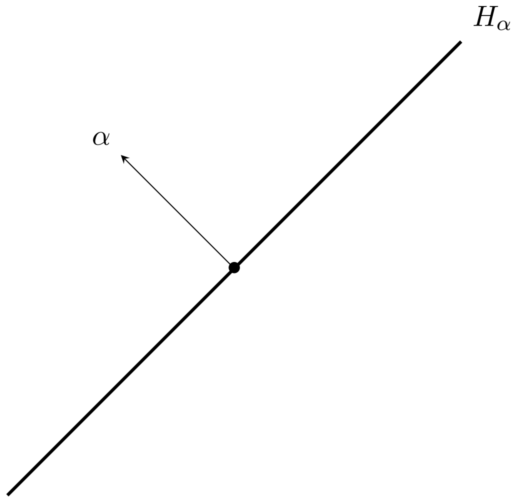

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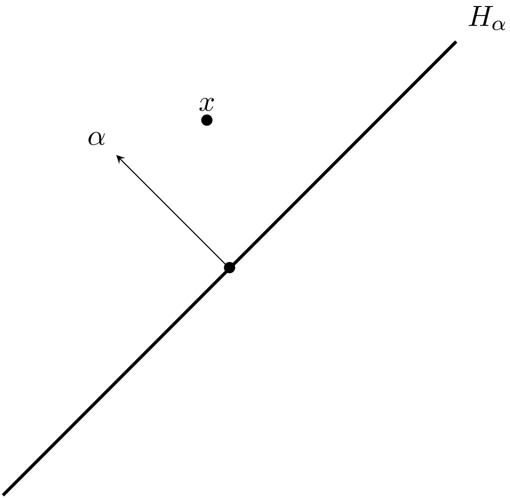

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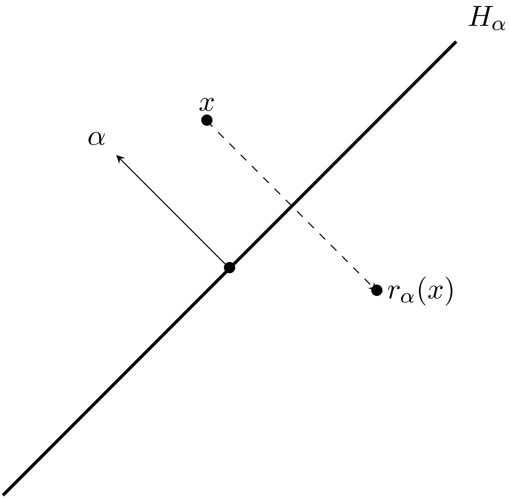

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(for more, see books by Humphreys, or by Björner and Brenti)

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$$\ell_S(w) = \textsc{inv}(w)$$
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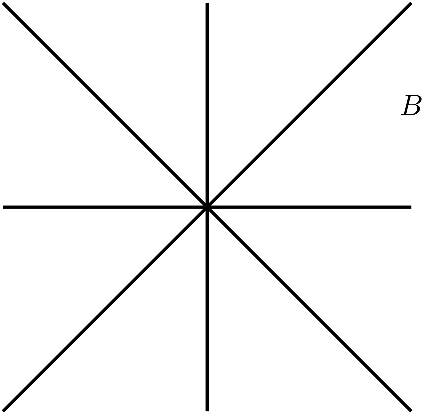
Coxeter arrangement

Picking a base region identifies open cells with group elements



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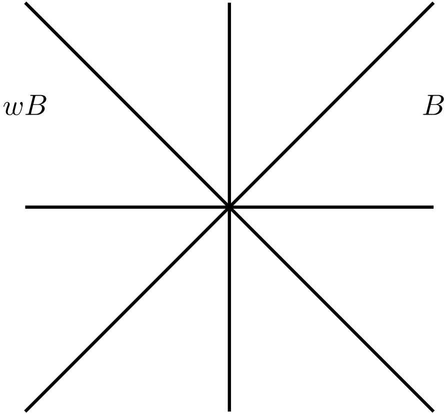

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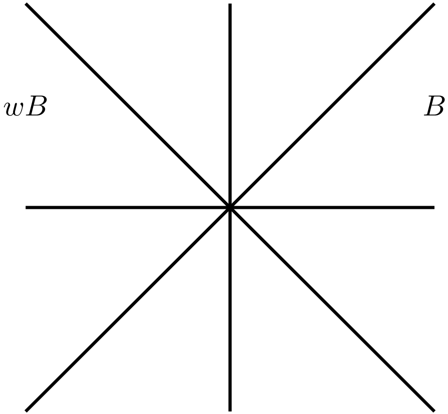

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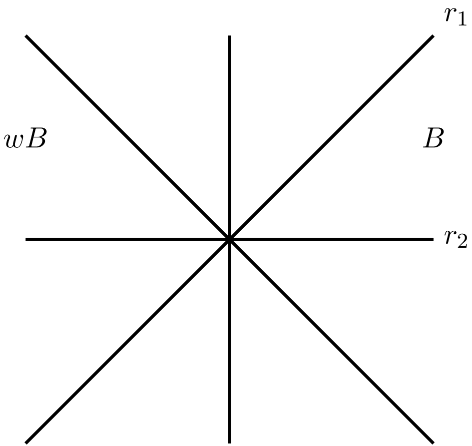

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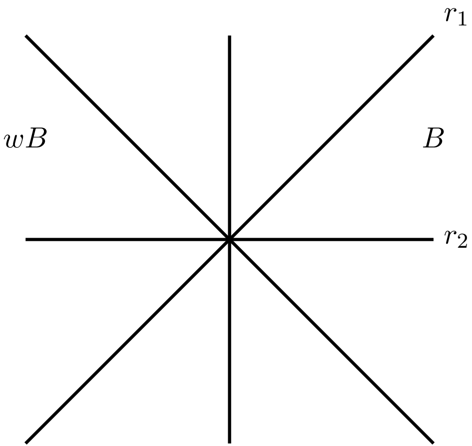

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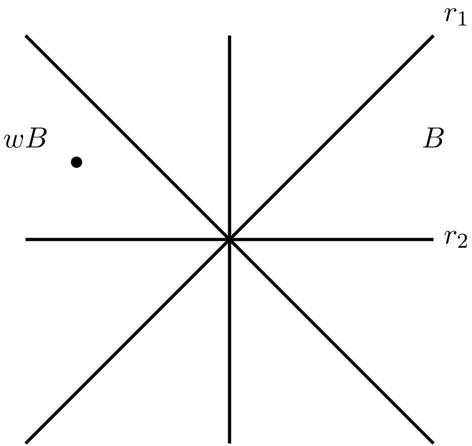

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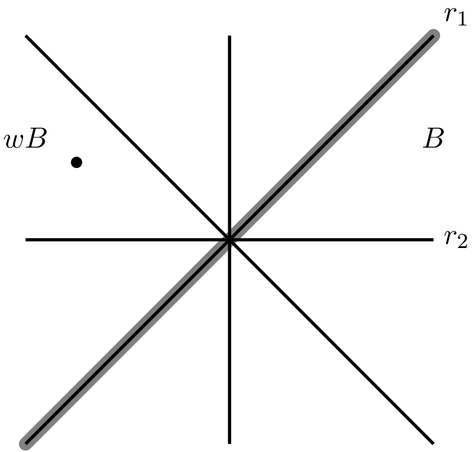

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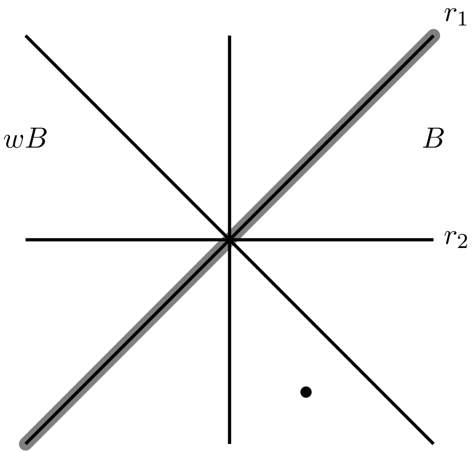

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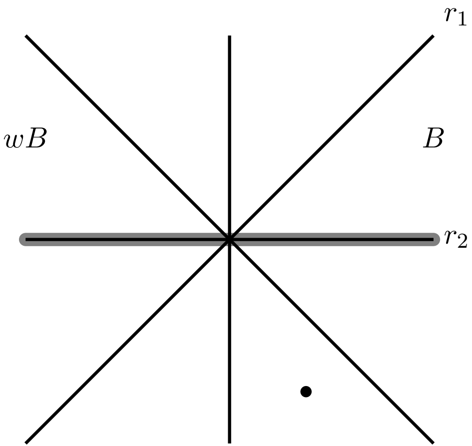

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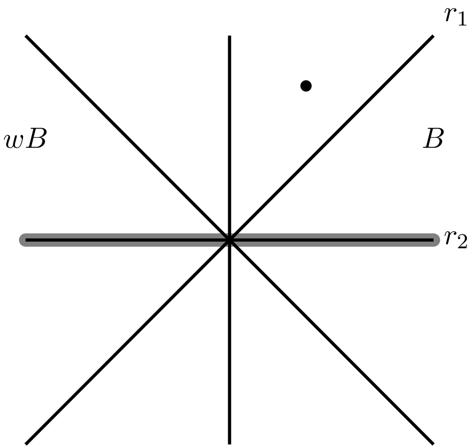

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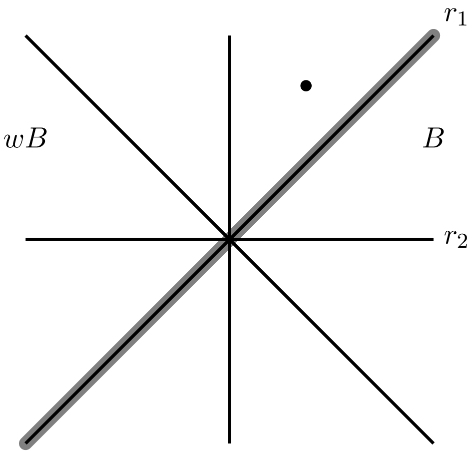

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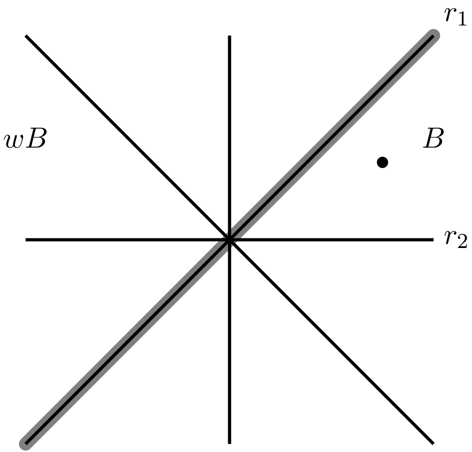

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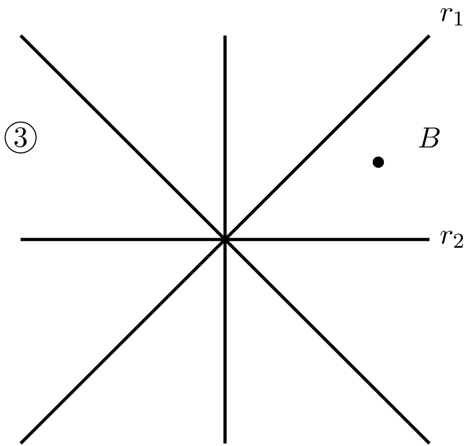

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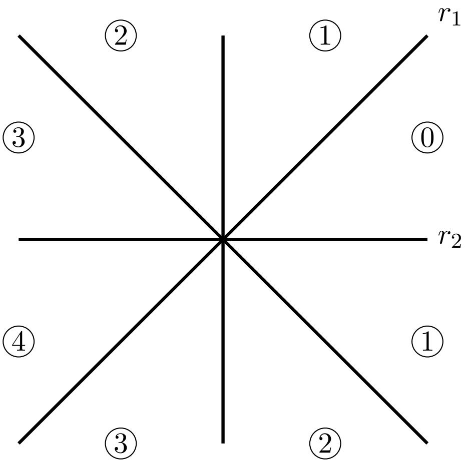

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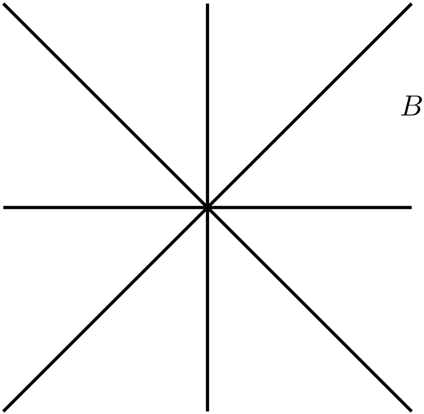
Use any reflections



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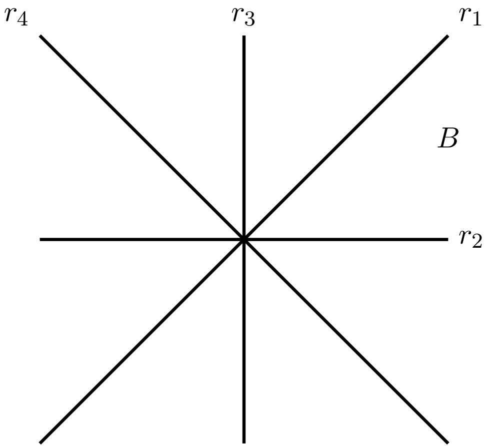

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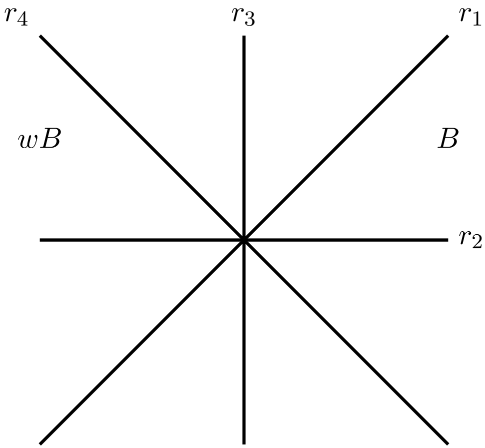

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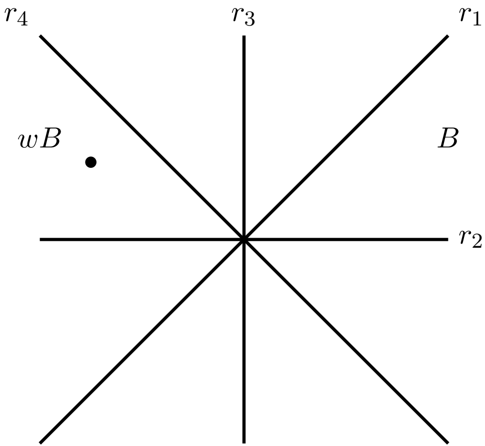

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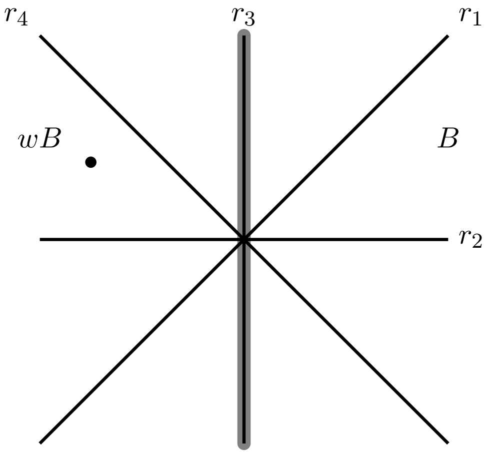

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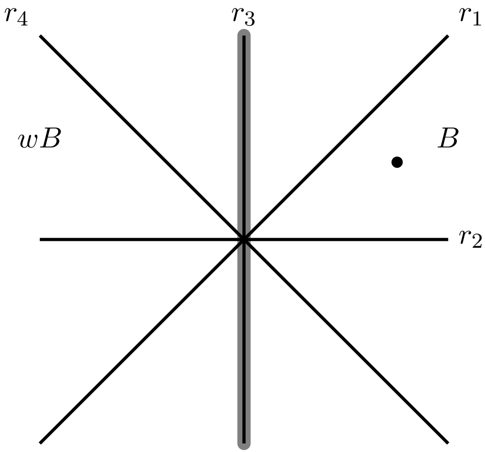

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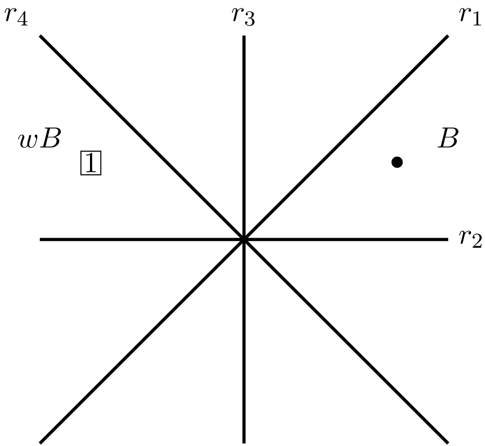

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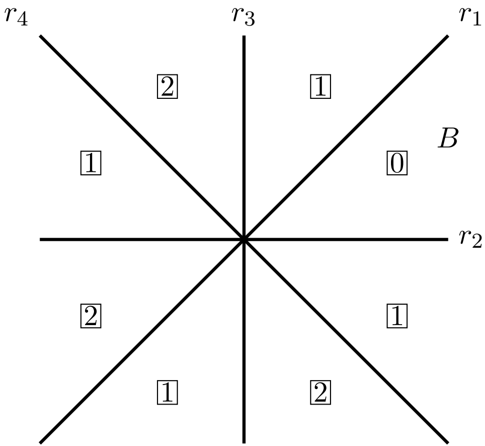
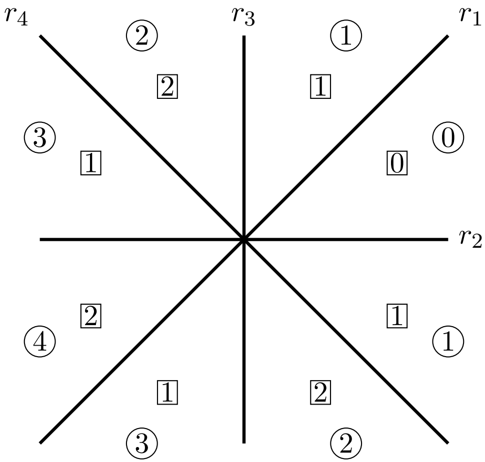


Reflection Length

Use any reflections, compare with length



Nice results for reflection length for finite W

Theorem (Shephard and Todd, 1954)

$$\sum_{w \in W} t^{\ell_R(w)} = \prod_{i=1}^{n} (1 + e_i t),$$

where the e_i are the exponents of W

Theorem (Carter, 1972)

For any $w \in W$, $\ell_R(w) = \dim(w)$, where $\dim(w)$ is the dimension of the smallest span of roots that contains $\mathrm{I\scriptstyle M}(w - 1)$

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What about infinite Coxeter groups?

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This work generalizes Carter's result to the affine setting.

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W is the (infinite) group of Euclidean isometries generated by R (or minimally, S_0)

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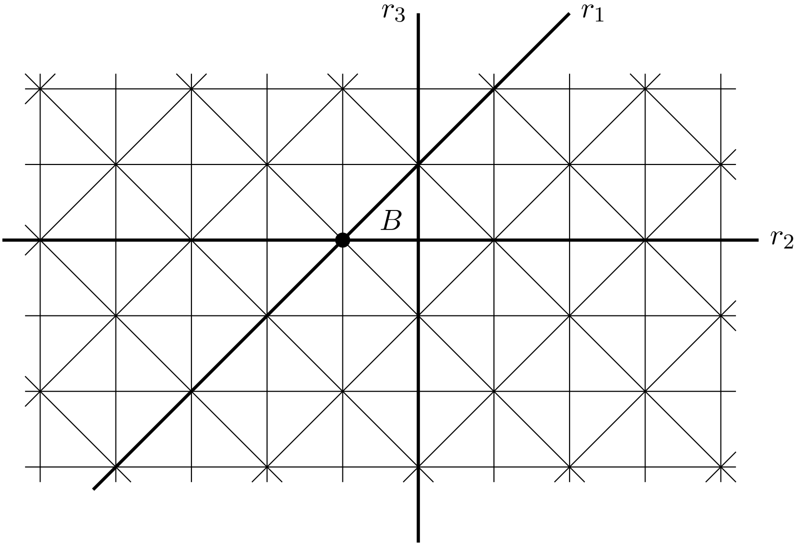

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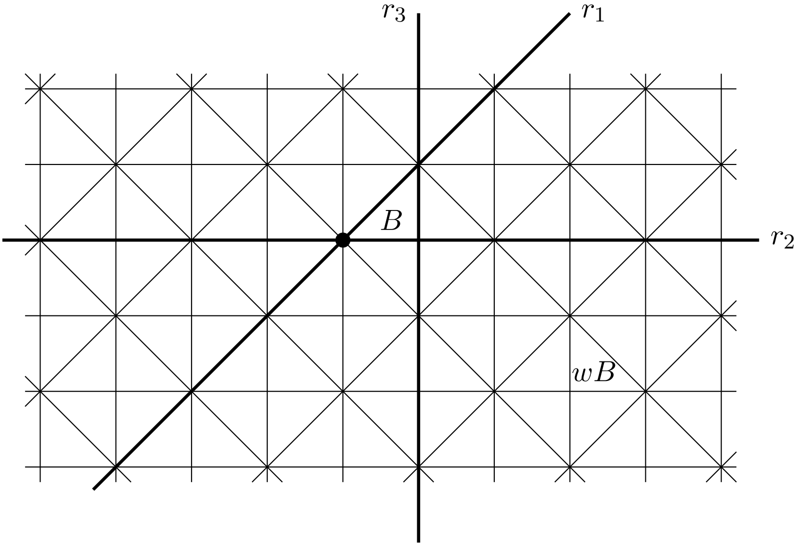

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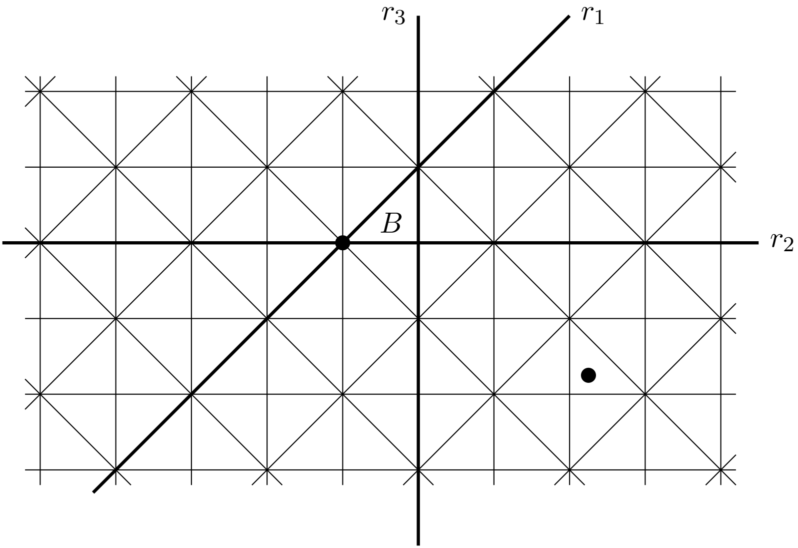

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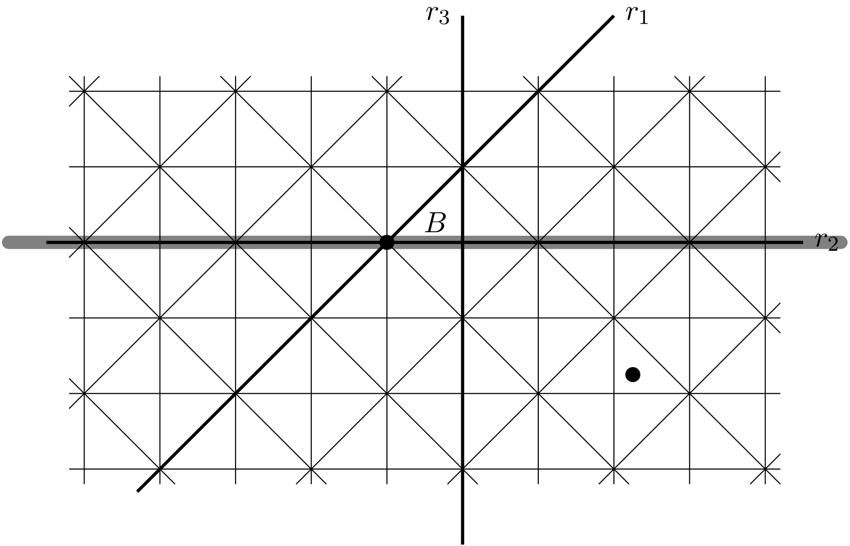

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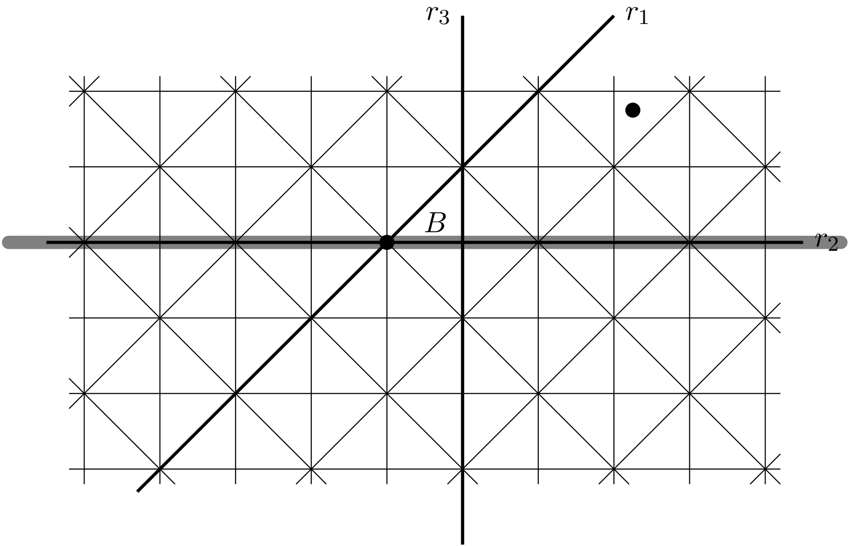

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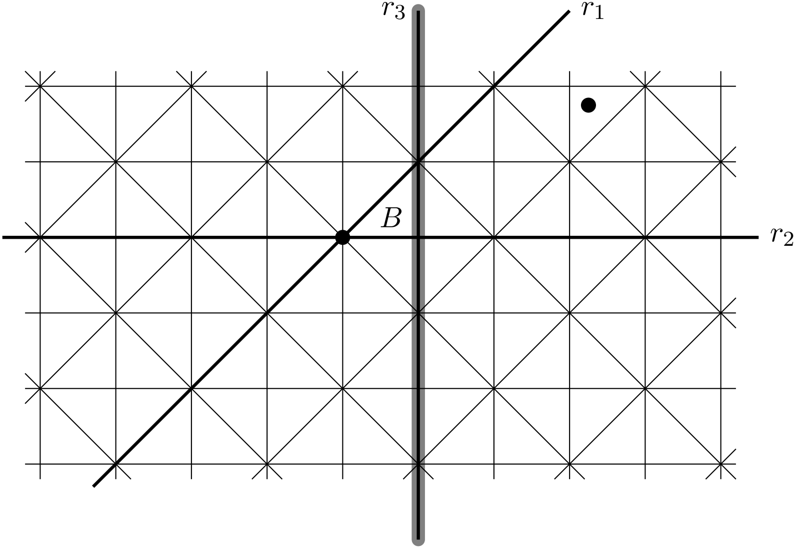

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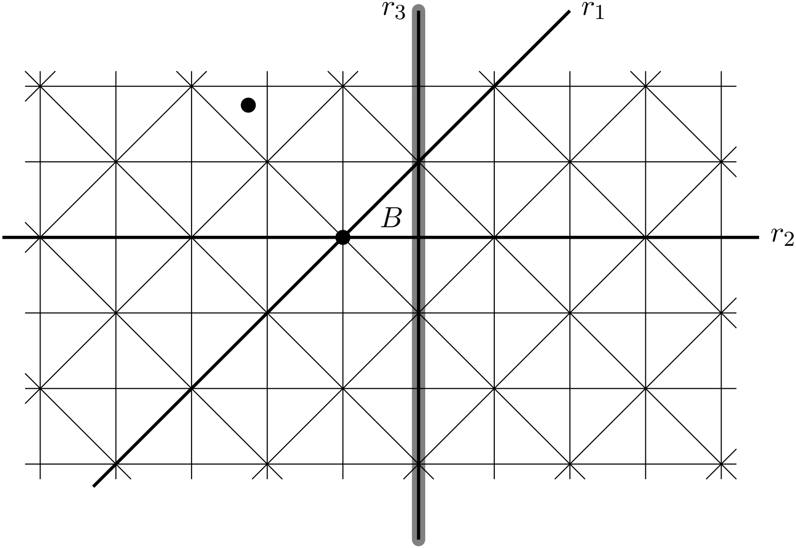

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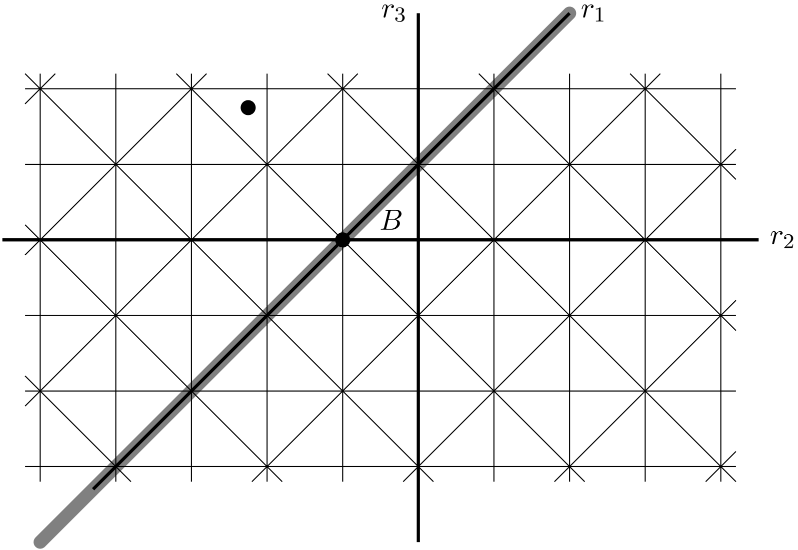

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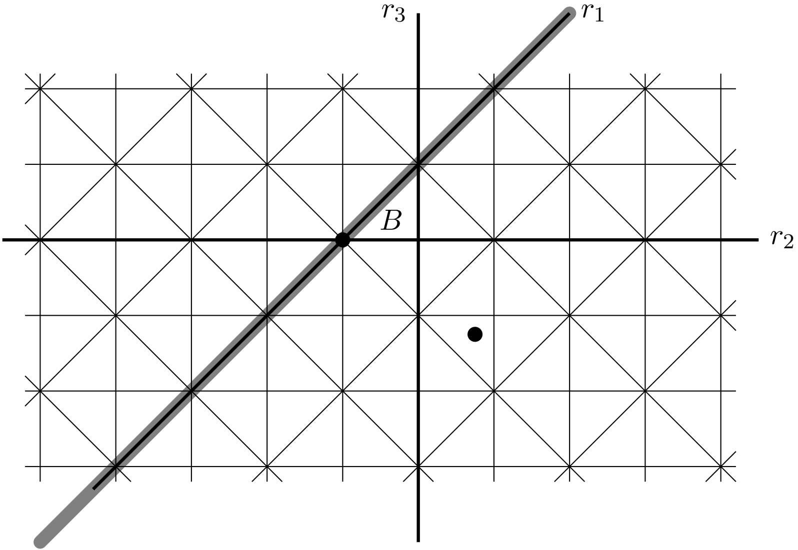

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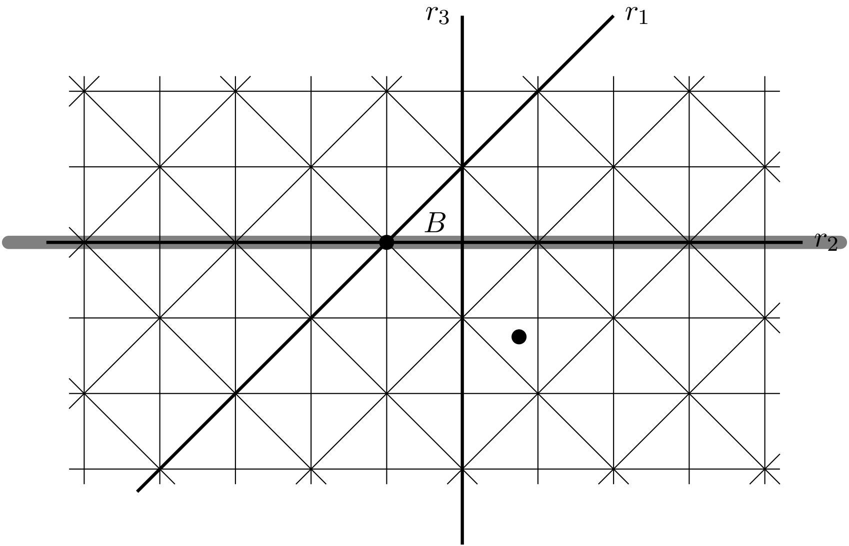

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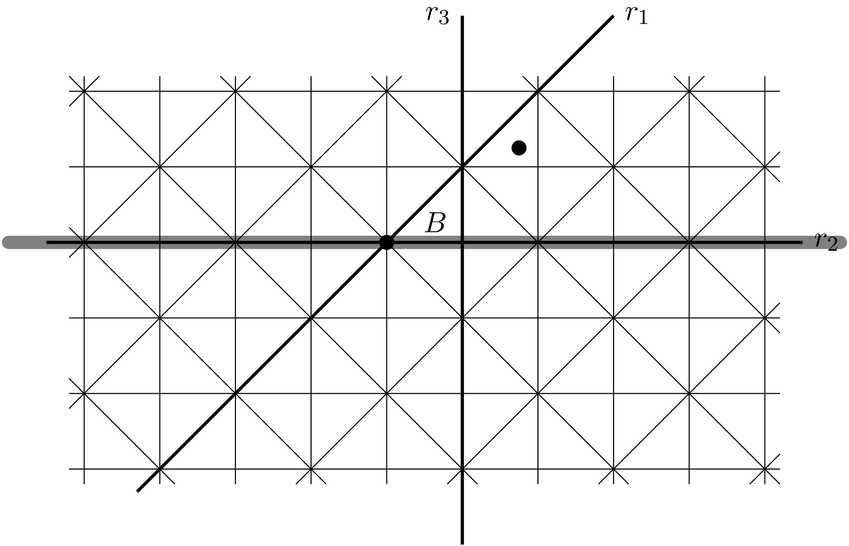

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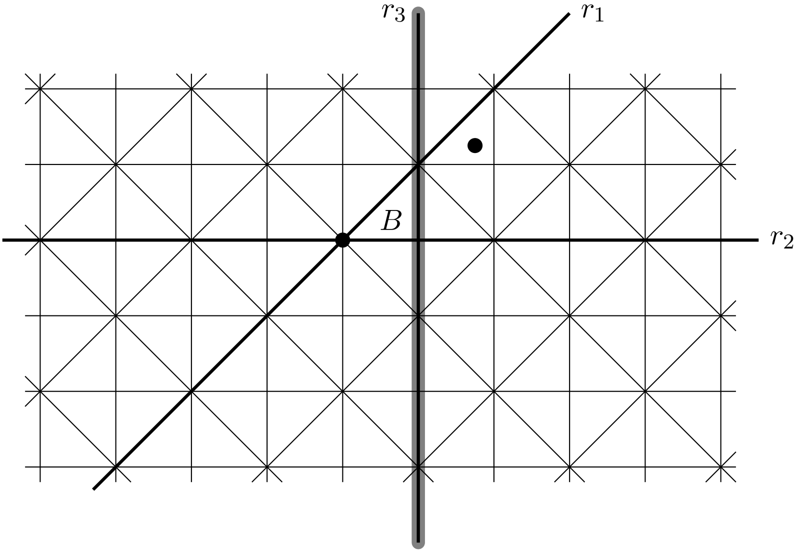

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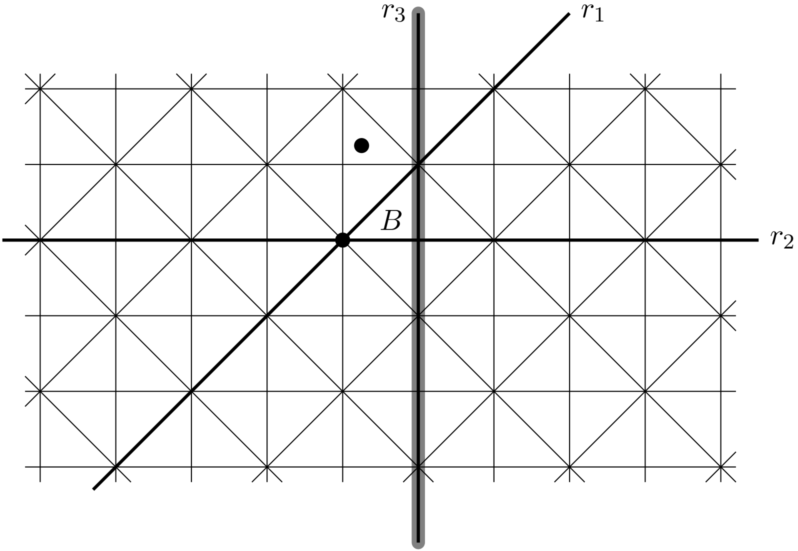

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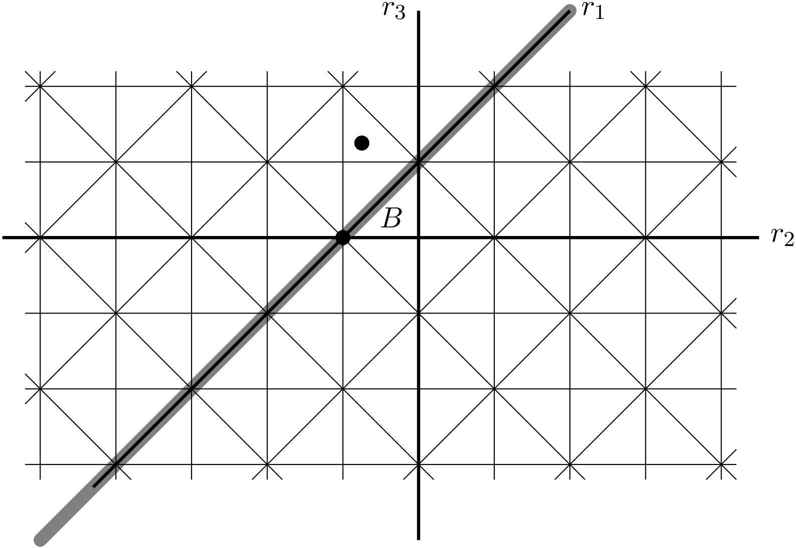

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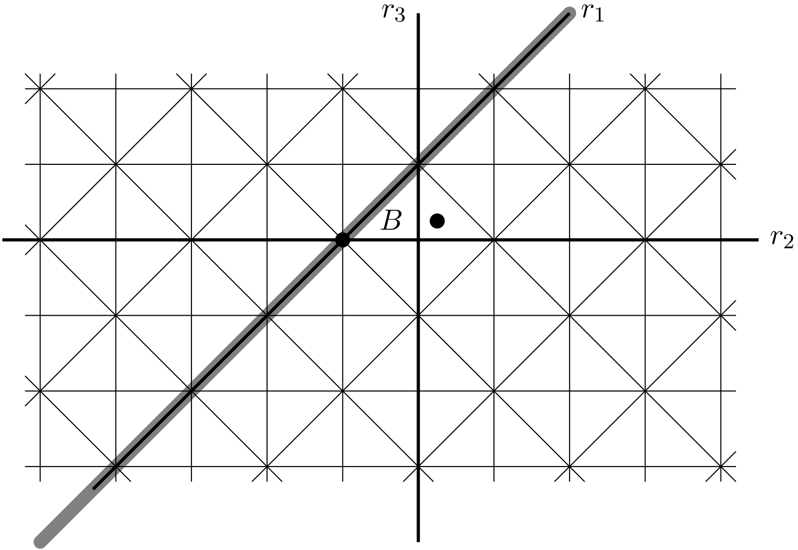

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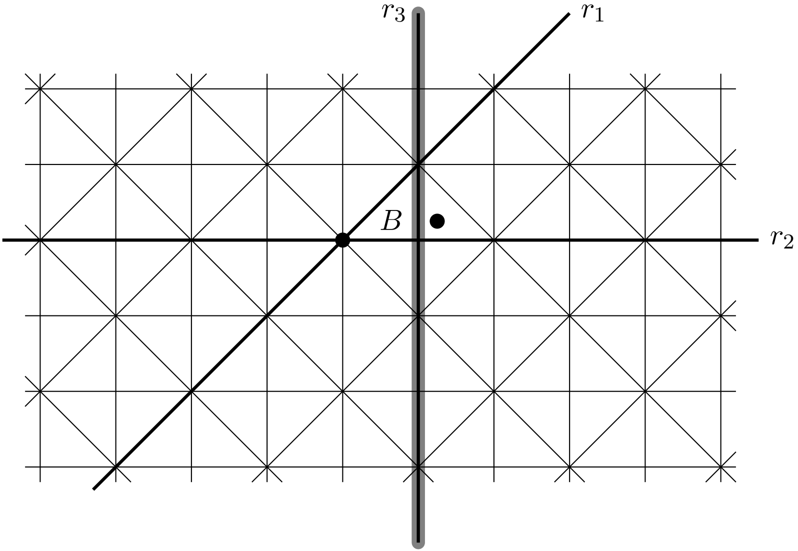

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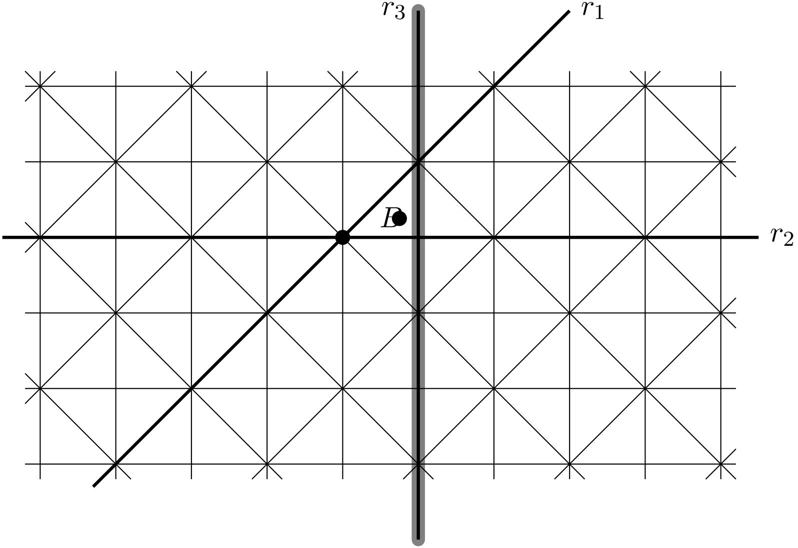

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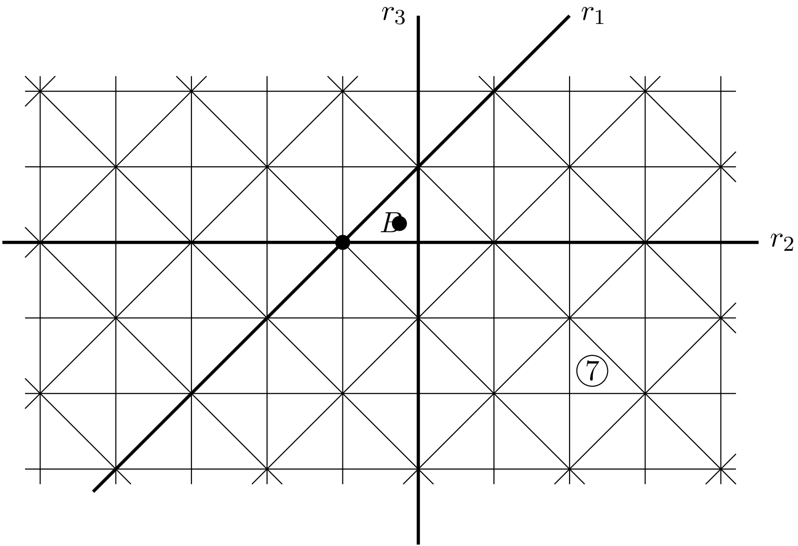

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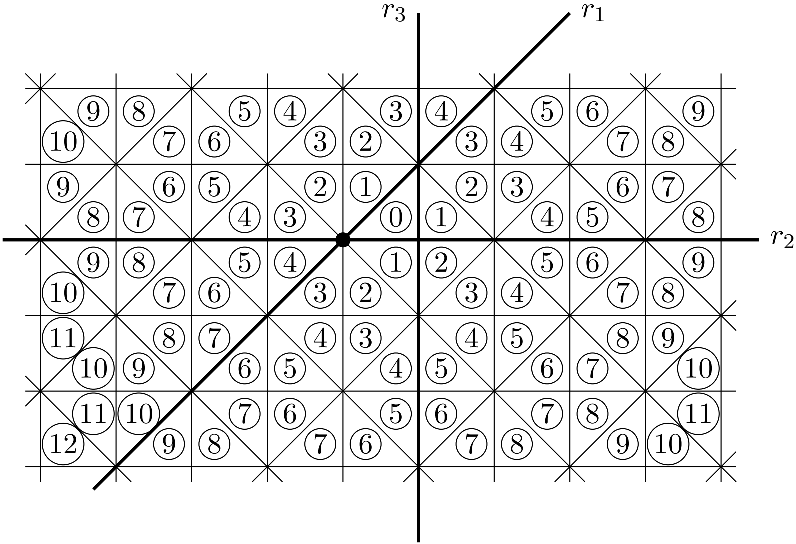
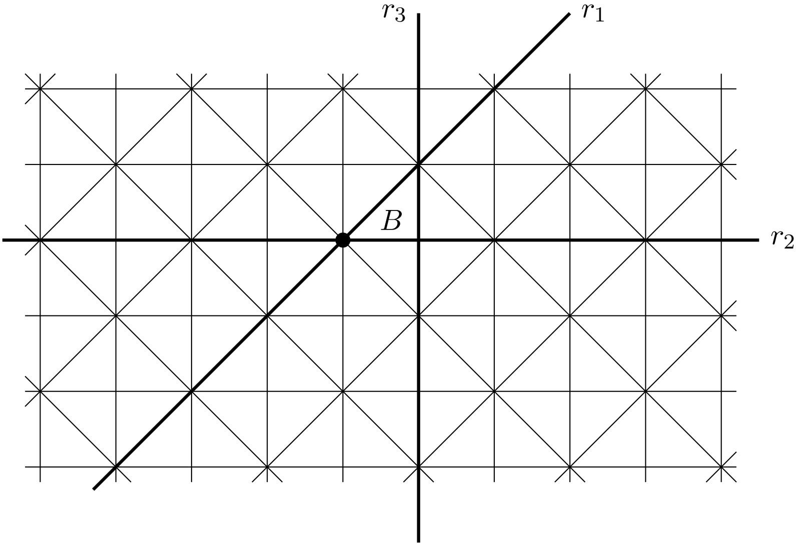
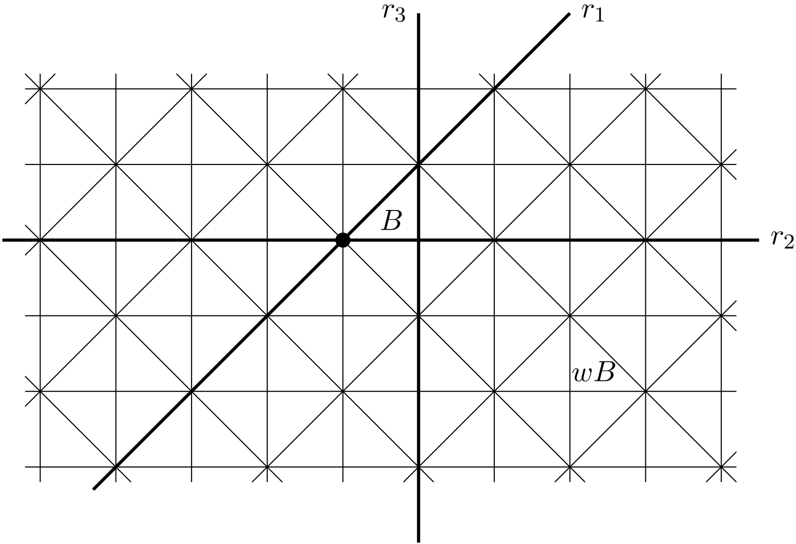


Reflection Length



Reflection Length



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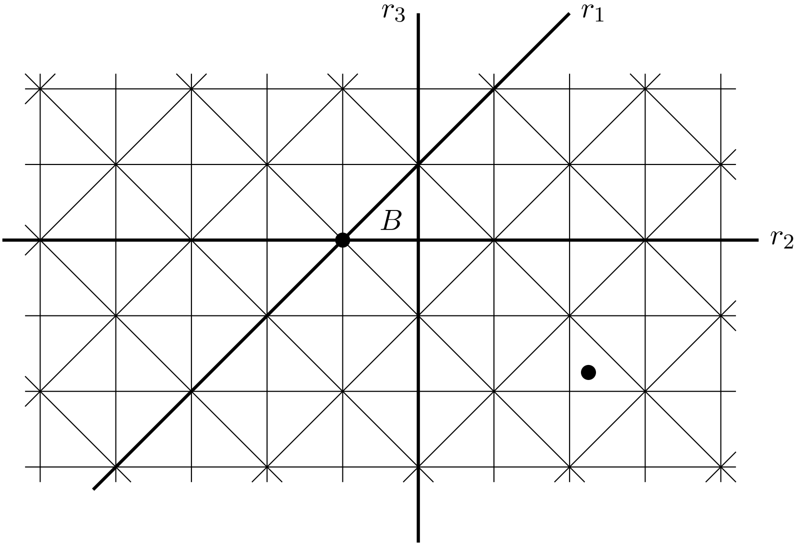
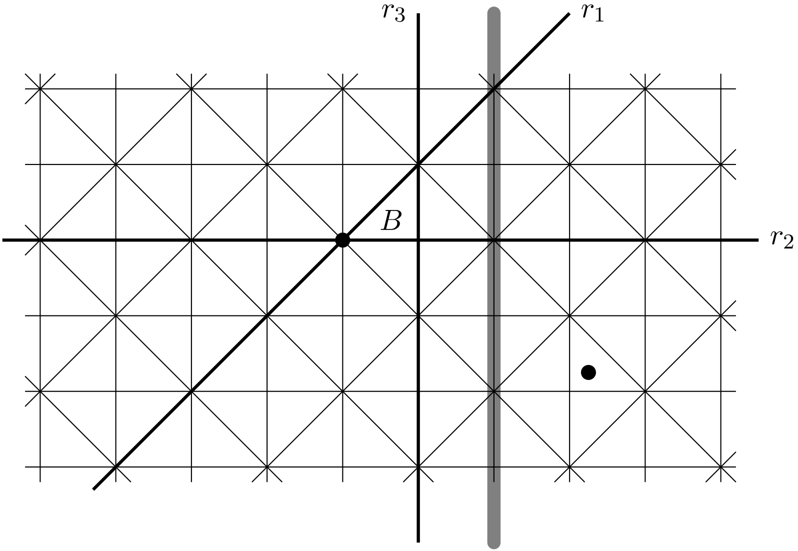
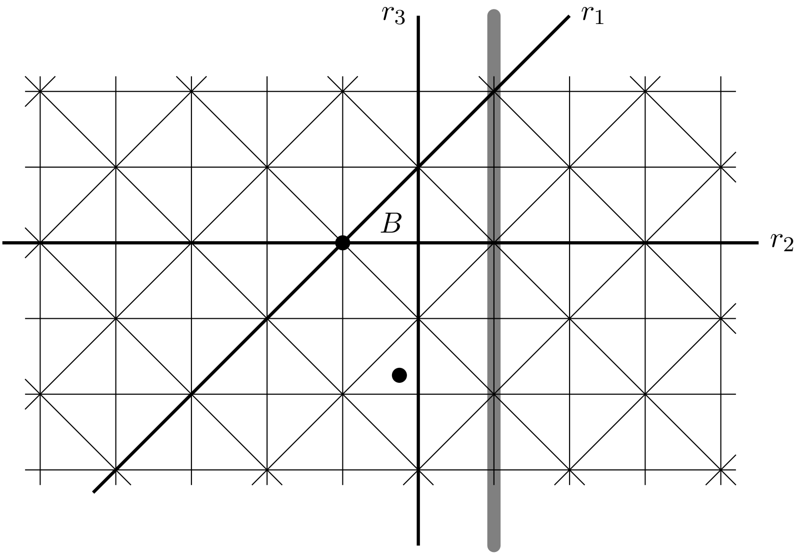


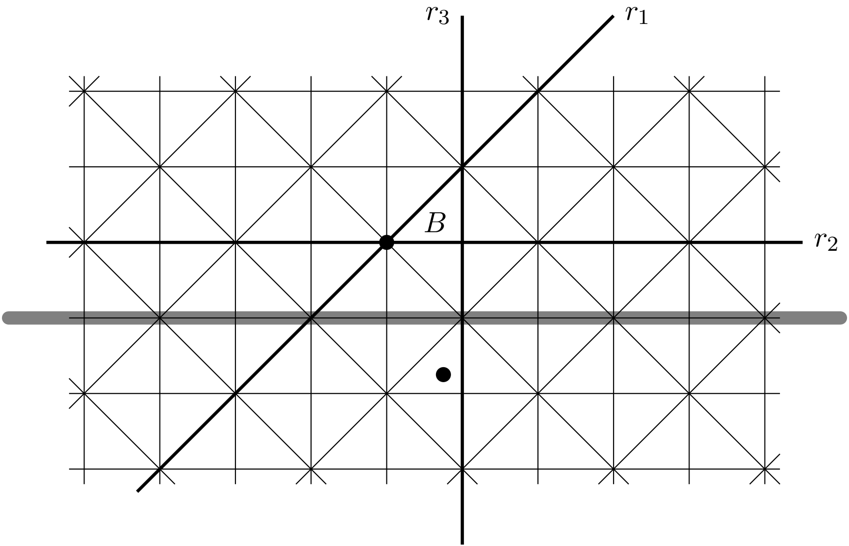
Reflection Length



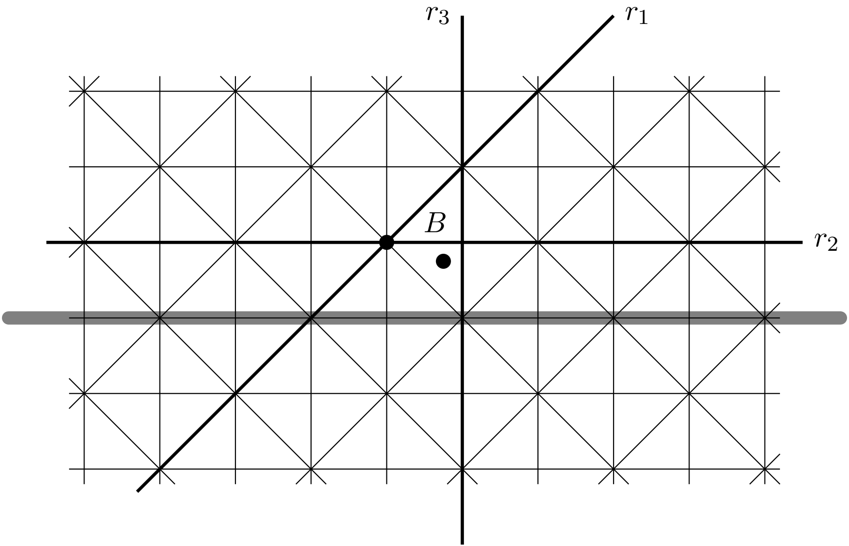
Reflection Length



Reflection Length



Reflection Length



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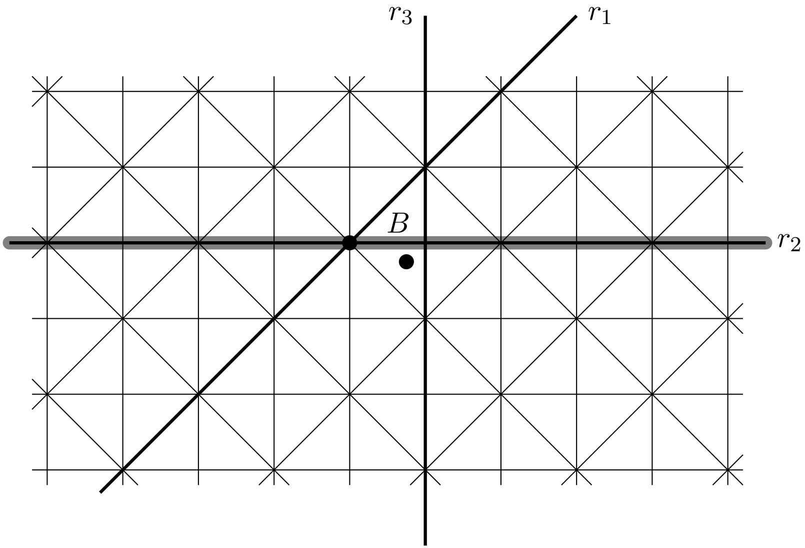
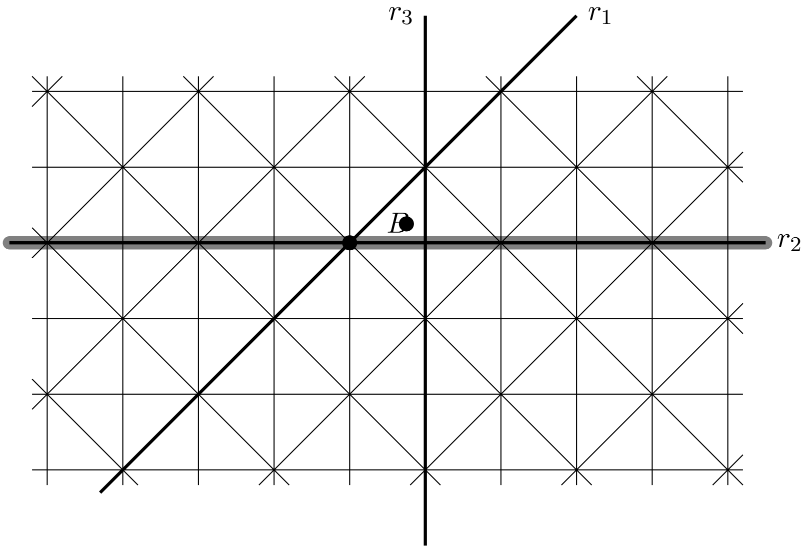
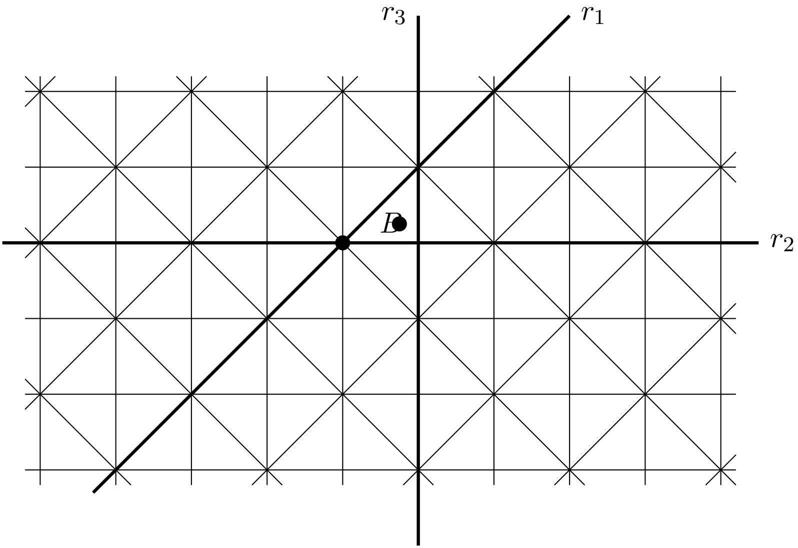


Reflection Length



Reflection Length



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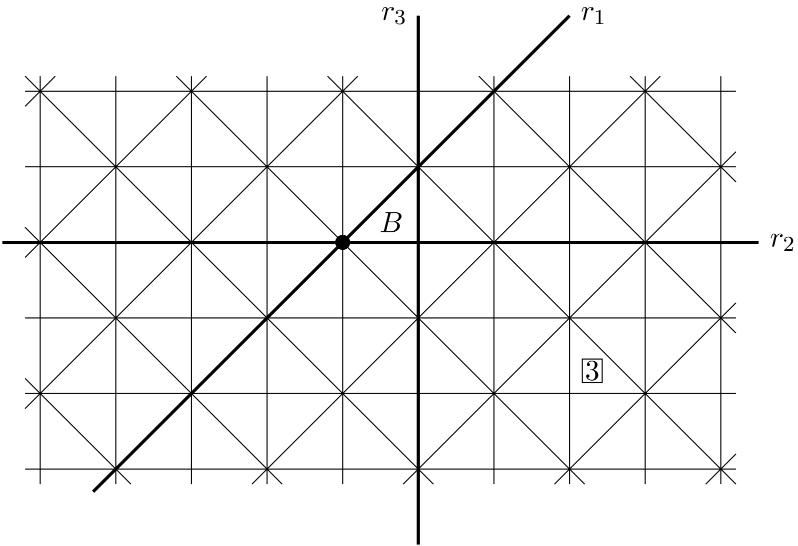
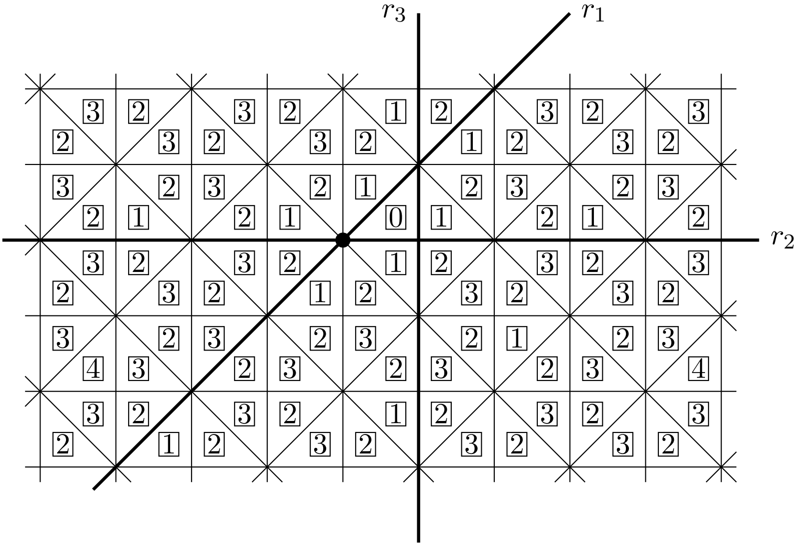
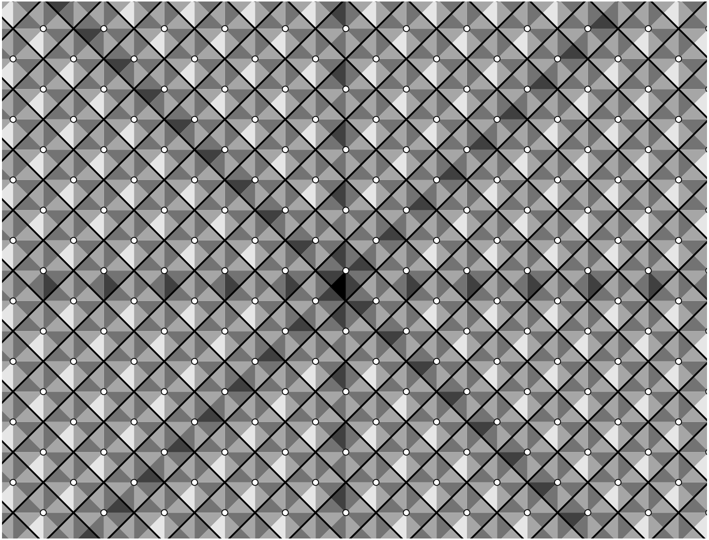


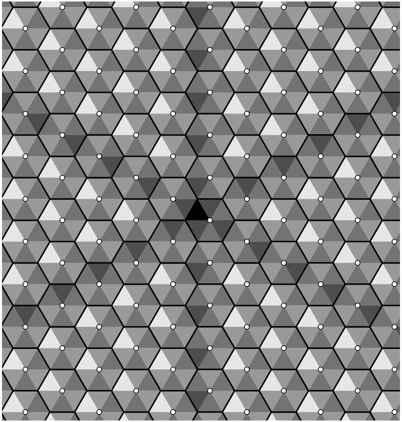
Reflection Length



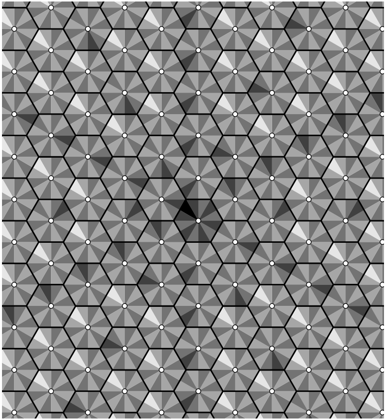
Reflection Length Wallpaper



Other reflection length wallpaper



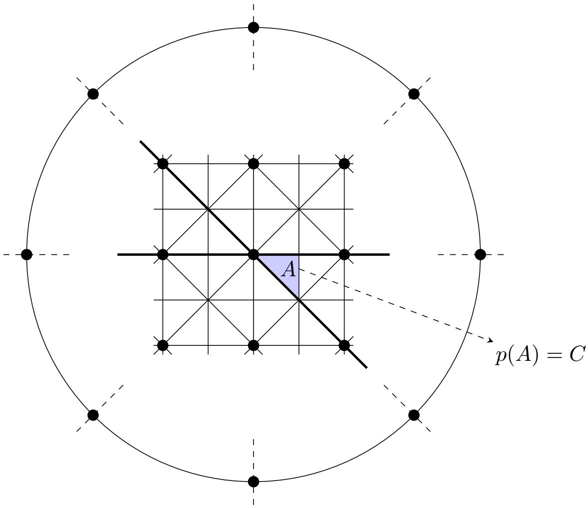
A_2



G_2

Reflection length in affine Coxeter groups

Projection onto the spherical subgroup



$$p : W \twoheadrightarrow W_0$$

$$r_{\alpha,j} \mapsto r_\alpha$$

Translations and Normal Forms

(inclusion) $i : W_0 \hookrightarrow W$, $r_\alpha \mapsto r_{\alpha,0}$
(projection) $p : W \twoheadrightarrow W_0$, $r_{\alpha,j} \mapsto r_\alpha$
(translations) $T = \textsc{Ker}(p) = \{t_\lambda : \lambda \in \Phi^\vee\}$
(quotient) $W_0 \cong W/T$
(semidirect product) $W \cong T \rtimes W_0$

Definition (Normal form)

Each $w \in W$ can be written as

$$w = t_\lambda u$$

for some $t_\lambda \in T$ and $u \in W_0$.

Translations and Elliptical elements

Choice of $i : W_0 \hookrightarrow W$ is like choosing the origin for V, and any $u \in W_0$ fixes this point

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More generally, say $u \in W$ is *elliptical* if it fixes a point (u not necessarily in W_0)

Definition (Translation-elliptic factorizations)

Each $w \in W$ can be written as

$$w = t_\lambda u$$

for some $t_\lambda \in T$ and elliptical element u.

Geometric interpretation of reflection length

Definition (Move-set)

$$\mathrm{Mov}(w) = \{\lambda : w(x) = x + \lambda \text{ for some } x \in V\}$$

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$\dim(w)$ *is the dimension of the smallest root subspace (span of roots) containing* $\mathrm{MOV}(w)$

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A straightforward consequence:

Proposition

If u *is elliptical, then* $\textsc{Mov}(u) = \textsc{Mov}(p(u))$ *and hence*

$$\dim(u) = \dim(p(u))$$

Geometric interpretation of reflection length

Corollary

If w is elliptical (i.e., fixes a point),

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If w is a translation,

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How to bring these two extremes together?

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(differential dimension) $d(w) = \dim(w) - \dim(p(w))$

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An element w is:

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Theorem (Statistics and geometry)

An element w is:

- *elliptical if and only if $d(w) = 0$*

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w	$\textsc{Mov}(w)$	d	e	ℓ_R
identity	the origin	0	0	0
reflection	a root line	0	1	1
rotation	the plane	0	2	2
translation	an affine point	1 or 2	0	2 or 4
glide reflection	an affine line	1	1	3

Main theorems

Theorem (Formula)

For any w in an affine Coxeter group,

$$\ell_R(w) = 2d(w) + e(w) = 2\dim(w) - \dim(p(w))$$

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Theorem (Factorization)

For any w in an affine Coxeter group, there exists a translation-elliptic factorization $w = t_\lambda u$ (not necessary a normal form) such that

$$\ell_R(w) = \ell_R(t_\lambda) + \ell_R(u)$$

Affine Symmetric Groups

Theorem (Symmetric Group Formula)

For any w in the affine Symmetric group $\widetilde{\mathfrak{S}}_n$ with normal form $t_\lambda \pi$,

$$d(w) = \mathrm{CYC}(\pi) - \nu(\lambda/\pi)$$
$$e(w) = n - \mathrm{CYC}(\pi)$$
$$\ell_R(w) = n - 2\nu(\lambda/\pi) + \mathrm{CYC}(\pi)$$

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Ex: $w = t_\lambda\pi$, with
$\lambda = (-2, -1, 3, 1, 1, -2, 0)$, $\pi = (1, 5, 7)(2, 4)(3)(6)$

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Ex: $w = t_\lambda \pi$, with
$\lambda = (-2, -1, 3, 1, 1, -2, 0)$, $\pi = (1, 5, 7)(2, 4)(3)(6)$
$\lambda/\pi = (-1, 0, 3, -2)$
$\nu(\lambda/\pi) = 2$
$\ell_R(w) = 7 - 2 \cdot 2 + 4 = 7$

Reflection length in affine Coxeter groups

Shepard and Todd's factorization

Theorem (Shephard and Todd, 1954)

For a spherical Coxeter group W_0,

$$\sum_{u \in W_0} t^{\ell_R(u)} = \prod_{i=1}^{n} (1 + e_i t),$$

where the e_i *are the exponents of* W_0

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where the e_i are the exponents of W_0

Definition (Local generating function)

For any λ, let

$$f_\lambda(s,t) = \sum_{u \in W_0} s^{d(t_\lambda u)} t^{e(t_\lambda u)}$$

Local generating functions

Local generating functions

Proposition (Properties of local generating functions)

Let λ be a coroot. Then,

- *(Origin) If $\lambda = 0$, $f_\lambda(s,t) = \prod_{i=1}^{n}(1+e_i t)$*

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Local generating functions

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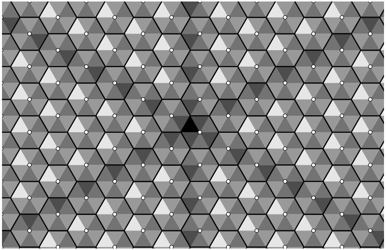
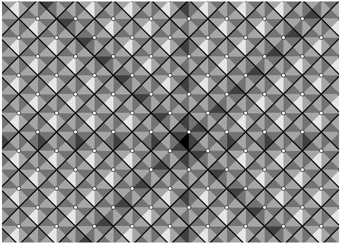
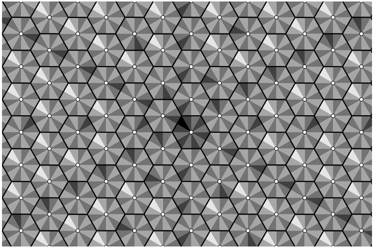
- *(Origin) If $\lambda = 0$, $f_\lambda(s,t) = \prod_{i=1}^n (1 + e_i t)$*
- *(Generic) If λ is generic, $f_\lambda(s,t) = \prod_{i=1}^n (s + e_i t)$*
- *(Permutations) If λ and λ' belong to the same W_0-orbit, $f_\lambda(s,t) = f_{\lambda'}(s,t)$.*

Local generating function wallpaper







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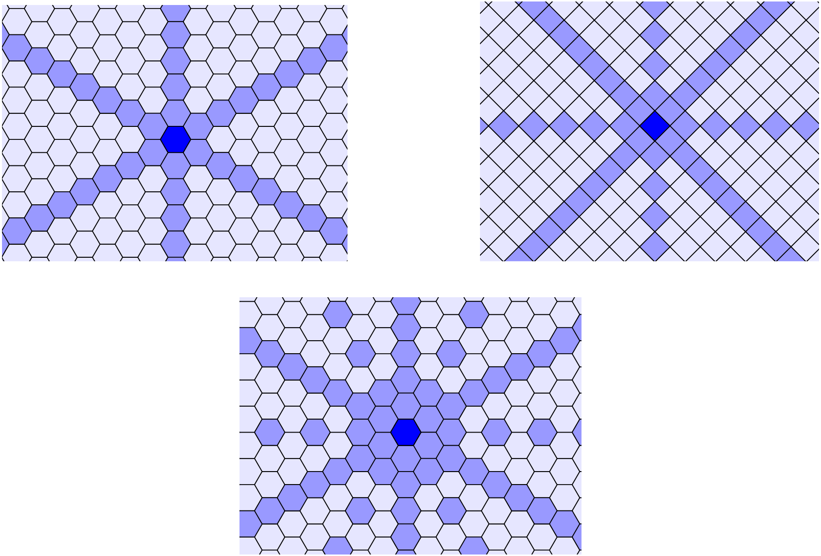

Affine A_3 local reflection length