

The module of affine descent classes

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The module of affine descents

Some background

Solomon's descent algebra

Affine structures

The affine descent module

Descents

$W = \langle s_\alpha : \alpha \in \Delta \rangle$ finite Coxeter group

$\Delta = \{\alpha_1, \dots, \alpha_n\}$ (nodes of Dynkin diagram)

$D(w) := \{1 \leq i \leq n : w\alpha_i < 0\}$

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Type A is “cyclic” descents:

w	$D(w)$	$\tilde{D}(w)$
123	$\emptyset$	$\{0\}$
132	$\{2\}$	$\{0, 2\}$
213	$\{1\}$	$\{0, 1\}$
231	$\{2\}$	$\{2\}$
312	$\{1\}$	$\{1\}$
321	$\{1, 2\}$	$\{1, 2\}$

Descent algebras

$$X_J := \sum_{D(w) \subseteq J} w \quad (\text{in } \mathbb{Z}[W])$$

Solomon (1976): $\text{Sol}(W) := \text{Span}\{X_J : J \subseteq [n]\}$ is a subalgebra

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Solomon (1976): $\text{Sol}(W) := \text{Span}\{X_J : J \subseteq [n]\}$ is a subalgebra

$$\tilde{X}_J := \sum_{\tilde{D}(w) \subseteq J} w \quad (\text{in } \mathbb{Z}[W])$$

$$\tilde{x}_k := \sum_{|J|=k} \tilde{X}_J \quad (\text{in } \mathbb{Z}[W])$$

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Cellini (1995): $\text{Span}\{\tilde{x}_k : k \in [n]\}$ is a commutative subalgebra
 $\text{Span}\{\tilde{X}_J\}$ is NOT a subalgebra. However...

Main result

$$\widetilde{\text{Sol}}(W) := \text{Span} \left\{ \widetilde{X}_J = \sum_{\widetilde{D}(w) \subseteq J} w : J \subseteq [0, n] \right\}$$

Theorem (Aguiar-P.)

$\widetilde{\text{Sol}}(W)$ is a left module over Solomon's descent algebra

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- We call $\widetilde{\text{Sol}}(W)$ the “module of affine descents”

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Theorem (Aguiar-P.)

$\widetilde{\text{Sol}}(W)$ is a left module over Solomon's descent algebra

- We call $\widetilde{\text{Sol}}(W)$ the “module of affine descents”
- Equivalent to one of Moszkowski's “reflection modules”

The geometric approach

Tits (1976), Bidigare (1997):

$$\text{Coxeter complex} \longleftrightarrow \text{Sol}(W)$$

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Our approach:

$$\text{Steinberg torus} \longleftrightarrow \widetilde{\text{Sol}}(W)$$

The module of affine descents

Some background

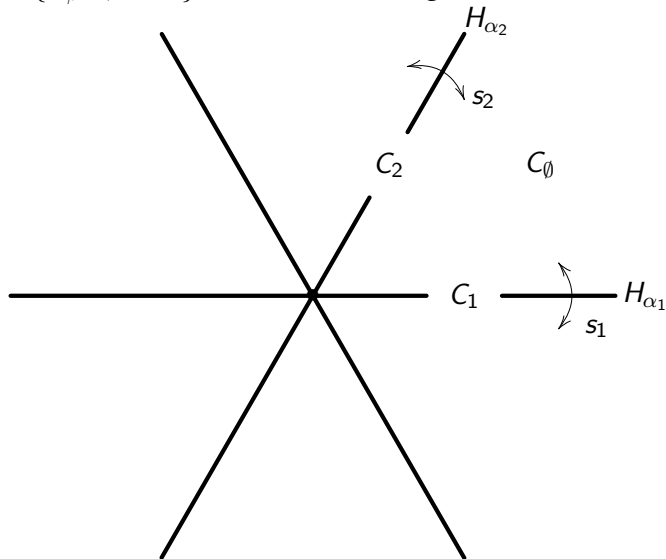
Solomon's descent algebra

Affine structures

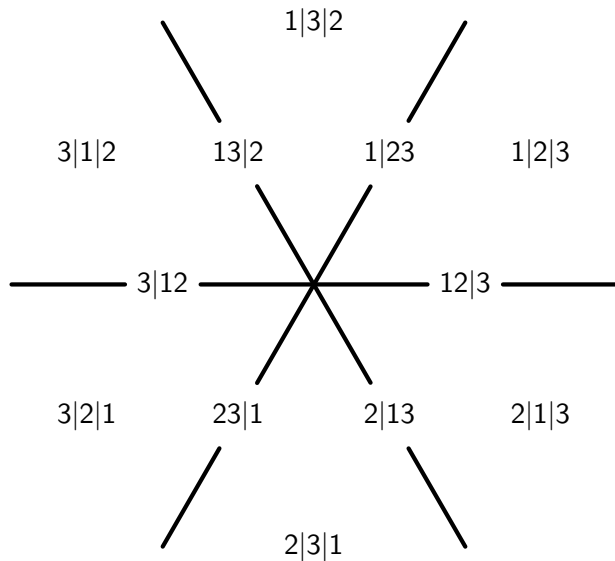
The affine descent module

The Coxeter arrangement

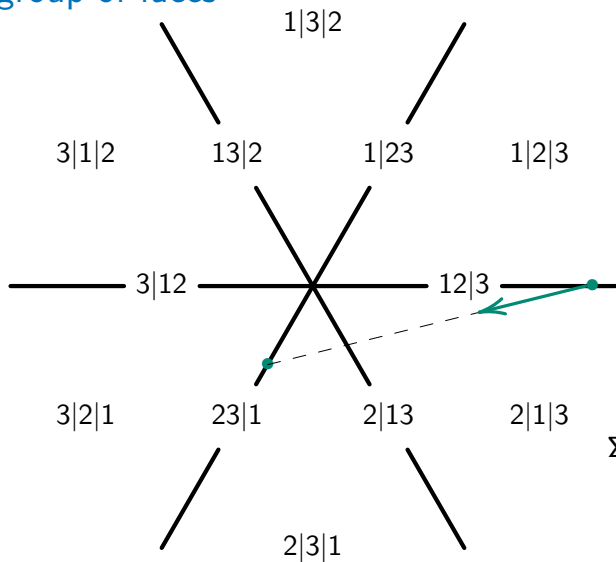
$\Sigma = \{H_\beta : \beta \in \Phi\}$, the *Coxeter arrangement*



Semigroup of faces



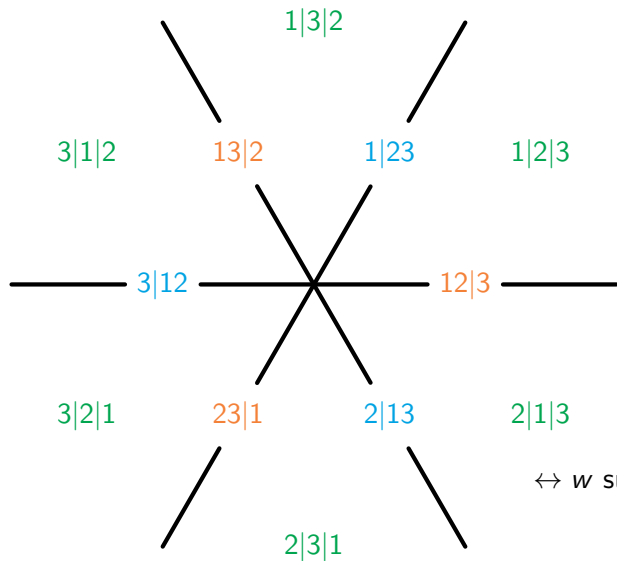
Semigroup of faces



Σ is a semigroup

$$12|3 \cdot 23|1 = 2|1|3$$

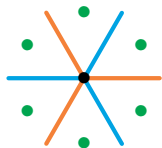
W -orbits



faces of color J

$\leftrightarrow w$ such that $D(w) \subseteq J$

The algebra Σ^W of invariants



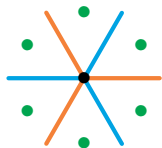
$$\sigma_{\emptyset} = 123 \text{ (origin)}$$

$$\sigma_1 = 1|23 + 2|13 + 3|12$$

$$\sigma_2 = 12|3 + 13|2 + 23|1$$

$$\sigma_{12} = 1|2|3 + 1|3|2 + 2|1|3 + 2|3|1 + 3|1|2 + 3|2|1$$

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W -orbits closed under products of faces:

$$\begin{aligned} \sigma_1 \sigma_2 &= 1|2|3 + 1|3|2 + 1|23| + \\ &\quad 2|1|3 + 2|3|1 + 2|13| + \\ &\quad 3|1|2 + 3|2|1 + 3|12| \\ &= \sigma_{12} + \sigma_1 \end{aligned}$$

Solomon's descent algebra

Theorem (Bidigare, Tits, - see also Brown)

Σ^W is (anti-)isomorphic to Solomon's descent algebra via:

$$\sigma_J \leftrightarrow X_J := \sum_{D(w) \subseteq J} w \quad (J \subseteq [n])$$

$\sigma_\emptyset = 123$	$X_\emptyset = 123$
$\sigma_1 = 1 23 + 2 13 + 3 12$	$X_1 = 123 + 213 + 312$
$\sigma_2 = 12 3 + 13 2 + 23 1$	$X_2 = 123 + 132 + 231$
$\sigma_{12} = \begin{array}{l} 1 2 3 + 1 3 2 + 2 1 3 \\ + 2 3 1 + 3 1 2 + 3 2 1 \end{array}$	$X_{12} = \begin{array}{l} 123 + 132 + 213 \\ + 231 + 312 + 321 \end{array}$

Solomon's descent algebra

$$\sigma_J \sigma_K = \sum_L c_{J,K}^L \sigma_L \quad \sigma_1 \sigma_2 = \begin{array}{l} 1|2|3 + 1|3|2 + 1|23 + \\ 2|1|3 + 2|3|1 + 2|13 + \\ 3|1|2 + 3|2|1 + 3|12 \end{array}$$

$$= \sigma_{12} + \sigma_1$$

$$X_K X_J = \sum_L c_{J,K}^L X_L \quad X_2 X_1 = \begin{array}{l} 123 + 132 + 231 + \\ 213 + 312 + 321 + \\ 312 + 213 + 123 \end{array}$$

$$= X_{12} + X_1$$

The module of affine descents

Some background

Solomon's descent algebra

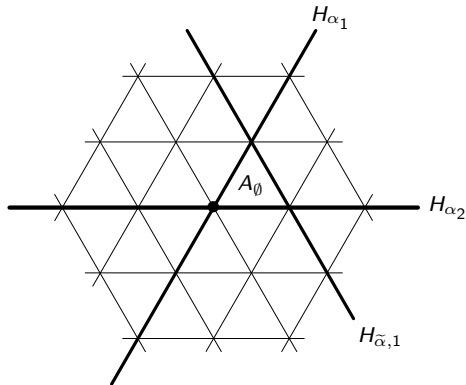
Affine structures

The affine descent module

The affine arrangement

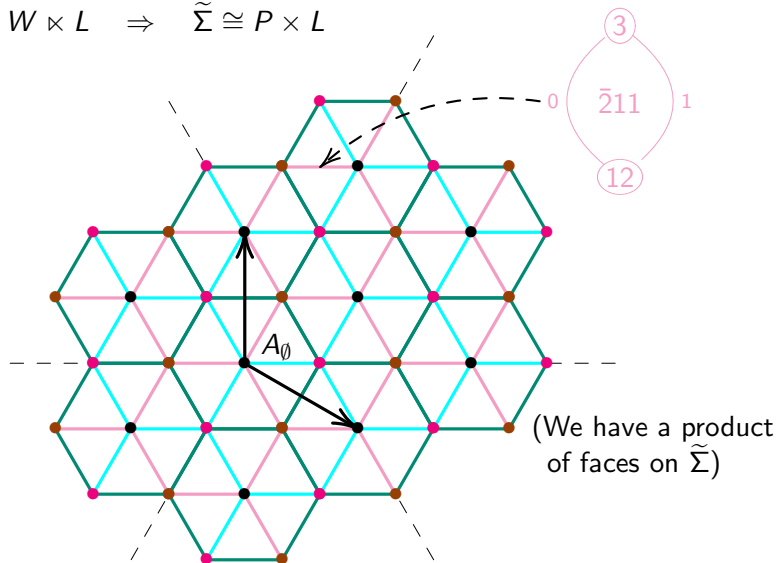
$$\widetilde{W} = \langle s_\alpha : \alpha \in \Delta_0 \rangle$$

$\widetilde{\Sigma}(A_2) :$



The affine arrangement

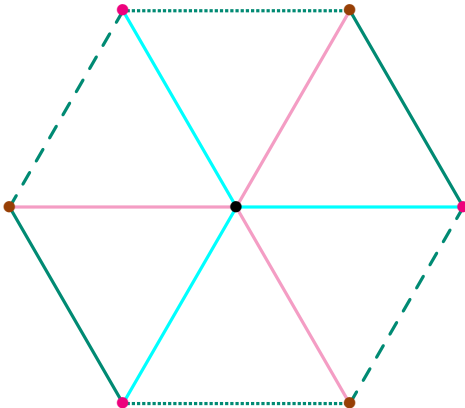
$$\widetilde{W} \cong W \ltimes L \Rightarrow \widetilde{\Sigma} \cong P \times L$$



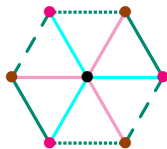
The Steinberg torus

$$\Sigma_T \cong \tilde{\Sigma}/L$$

(Product is ill-defined)



W-orbits in the Steinberg torus



$$\tilde{\sigma}_0 = \overset{0}{\circlearrowleft} (123), \quad \tilde{\sigma}_1 = \overset{1}{\circlearrowleft} (123), \quad \tilde{\sigma}_2 = \overset{2}{\circlearrowleft} (123)$$

$$\tilde{\sigma}_{01} = \overset{1}{\circlearrowleft} (1) \overset{1}{\circlearrowleft} (23) + \overset{2}{\circlearrowleft} (2) \overset{1}{\circlearrowleft} (13) + \overset{3}{\circlearrowleft} (3) \overset{1}{\circlearrowleft} (12)$$

$$\tilde{\sigma}_{02} = \overset{1}{\circlearrowleft} (1) \overset{0}{\circlearrowleft} (23) + \overset{2}{\circlearrowleft} (2) \overset{0}{\circlearrowleft} (13) + \overset{3}{\circlearrowleft} (3) \overset{0}{\circlearrowleft} (12)$$

$$\tilde{\sigma}_{12} = \overset{1}{\circlearrowleft} (1) \overset{2}{\circlearrowleft} (23) + \overset{2}{\circlearrowleft} (2) \overset{2}{\circlearrowleft} (13) + \overset{3}{\circlearrowleft} (3) \overset{2}{\circlearrowleft} (12)$$

$$\tilde{\sigma}_{012} = \overset{0}{\circlearrowleft} (1) \overset{1}{\circlearrowleft} (3) \overset{2}{\circlearrowleft} (2) + \overset{0}{\circlearrowleft} (1) \overset{1}{\circlearrowleft} (2) \overset{2}{\circlearrowleft} (3) + \overset{0}{\circlearrowleft} (2) \overset{1}{\circlearrowleft} (3) \overset{1}{\circlearrowleft} (1) + \overset{0}{\circlearrowleft} (2) \overset{1}{\circlearrowleft} (1) \overset{1}{\circlearrowleft} (3) + \overset{0}{\circlearrowleft} (3) \overset{1}{\circlearrowleft} (2) \overset{1}{\circlearrowleft} (1) + \overset{0}{\circlearrowleft} (3) \overset{1}{\circlearrowleft} (1) \overset{1}{\circlearrowleft} (2)$$

The Steinberg torus

Theorem (Dilks-P.-Stembridge (2009))

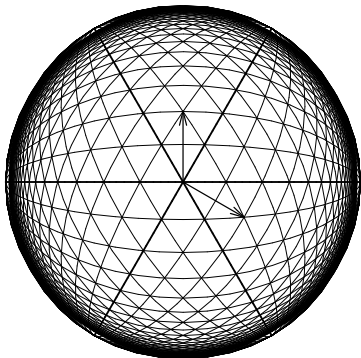
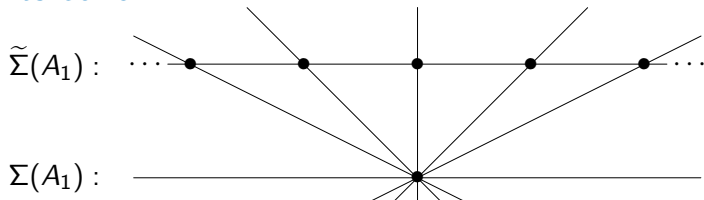
Affine descent sets characterize faces of the Steinberg torus:

$$\{\text{faces with color-set } J\} \leftrightarrow \{w \in W : \tilde{D}(w) \subseteq J\},$$

i.e.,

$$\tilde{\sigma}_J \leftrightarrow \tilde{X}_J$$

The Tits cone



The module of affine descents

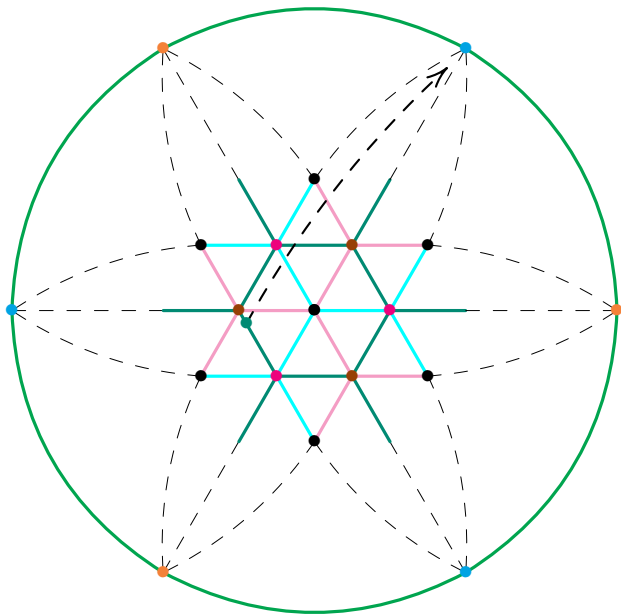
Some background

Solomon's descent algebra

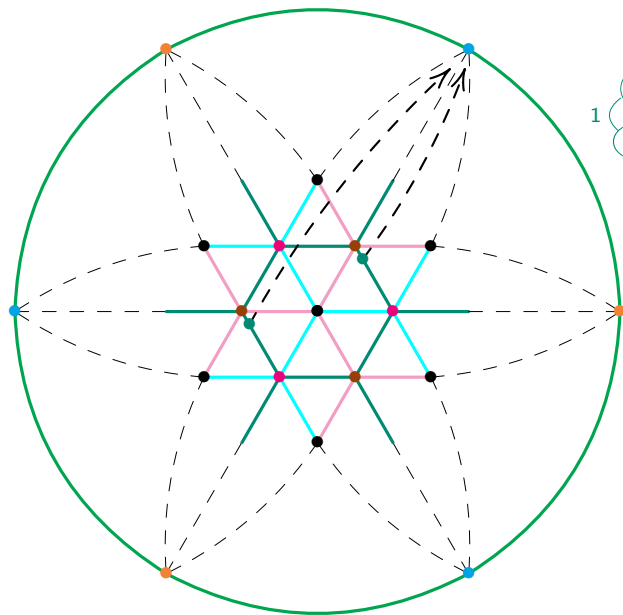
Affine structures

The affine descent module

The product $\tilde{\Sigma} \times \Sigma \rightarrow \tilde{\Sigma}$



The product $\Sigma_T \times \Sigma \rightarrow \Sigma_T$



$$1 \begin{pmatrix} 2 \\ 13 \end{pmatrix} 2 \cdot 1|23$$

$$= \begin{matrix} 0 & 3 & 1 \\ 1 & & 2 \\ & 2 & \end{matrix}$$

The semigroup Σ
acts on Σ_T

W -invariants

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acts on

$$(\Sigma_T)^W \cong \widetilde{\text{Sol}}(W)$$

$$\tilde{\sigma}_J \sigma_K = \sum_L c_{J,K}^L \tilde{\sigma}_L \leftrightarrow X_K \tilde{X}_J = \sum_L c_{J,K}^L \tilde{X}_L$$

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$$\tilde{\sigma}_J \sigma_K = \sum_L c_{J,K}^L \tilde{\sigma}_L \leftrightarrow X_K \tilde{X}_J = \sum_L c_{J,K}^L \tilde{X}_L$$

$$2 \begin{array}{c} \textcircled{1} \\ \textcircled{23} \end{array} 0 + 2 \begin{array}{c} \textcircled{2} \\ \textcircled{13} \end{array} 0 + 2 \begin{array}{c} \textcircled{3} \\ \textcircled{12} \end{array} 0 \leftrightarrow 231 + 132 + 123$$

Main result

Theorem (Aguilar-P.)

For any irreducible (finite) Weyl group

1. $\widetilde{\Sigma}(W)$ is a right module over $\Sigma(W)$
2. $\Sigma_T(W)$ is a right module over $\Sigma(W)$
3. $\widetilde{\text{Sol}}(W)$ is a left module over $\text{Sol}(W)$