A funny thing happened on the way to Steppenwolf Theatre. . .

from lattice paths to polytopes and Hopf algebras

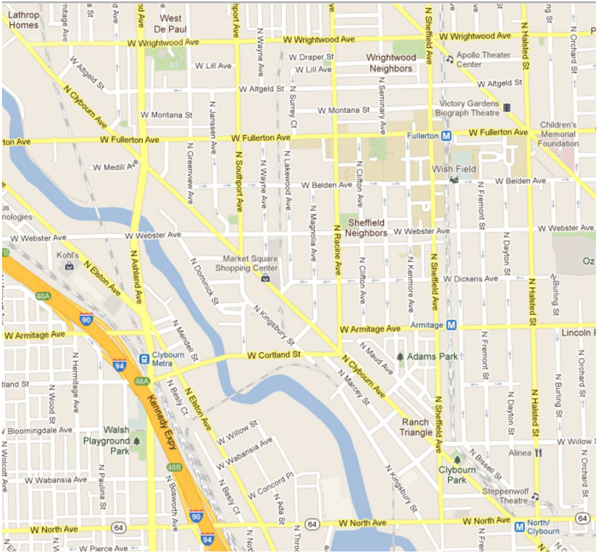
T. Kyle Petersen

DePaul University
Department of Mathematical Sciences

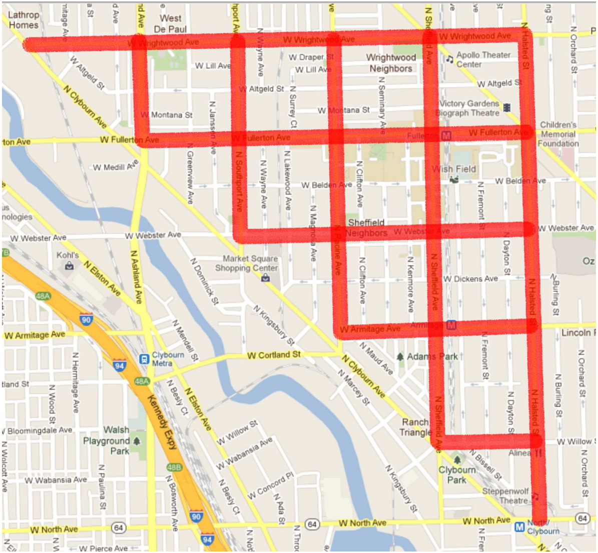
MathFest
Madison, WI
August 2012

Combinatorics, geometry, and an algebra of paths

The problem


Lathrop Homes
West De Paul
W Wrightwood Ave
W Wrightwood Ave
W Wrightwood Ave
W Draper St
W Lill Ave
W Lill Ave
Wrightwood Neighbors
Apollo Theater Center
W Altgeld St
W Altgeld St
W Altgeld St
N Clybourn Ave
W Montana St
W Montana St
Victory Gardens Biograph Theatre
Children's Memorial Foundation
W Fullerton Ave
W Fullerton Ave
Fullerton
W Fullerton Ave
W Medill Ave
Wish Field
W Belden Ave
W Belden Ave
N Greenview Ave
N Janssen Ave
N Southport Ave
N Wayne Ave
N Lakewood Ave
N Surrey Ct
N Seminary Ave
N Clifton Ave
N Fremont St
N Halsted St
N Orchard St
N Sheffield Ave
Sheffield Neighbors
W Webster Ave
W Webster Ave
W Webster Ave
W Webster Ave
N Magnolia Ave
N Racine Ave
Kohl's
N Elston Ave
N Ashland Ave
Market Square Shopping Center
N Dominick St
W Dickens Ave
N Kenmore Ave
N Dayton St
N Burling St
N Kingsbury St
W Armitage Ave
W Armitage Ave
Armitage
Lincoln
Clybourn Metra
W Cortland St
N Mendell St
Adams Park
N Maud Ave
N Clybourn Ave
N Marcey St
N Hermitage Ave
N Wood St
N Besly Ct
N Elston Ave
Kennedy Expy
W Bloomingdale Ave
Walsh Playground Park
W Willow St
W Wabansia Ave
Ranch Triangle
N Bissell St
Alinea
W Willow
Clybourn Park
Steppenwolf Theatre
W Wabansia Ave
N Paulina St
N Wolcott Ave
N Bosworth Ave
W Concord Pl
N Ada St
N Kingsbury St
W North Ave
W North Ave
W North Ave
North/ Clybourn
W Pierce Ave
90
94
48A
48B
64

The problem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Counting "Clybourn paths"

Let C_n denote the number of paths of length $2n$

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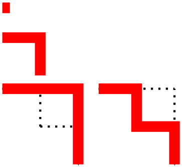
$C_2 = 2$:



Counting “Clybourn paths”

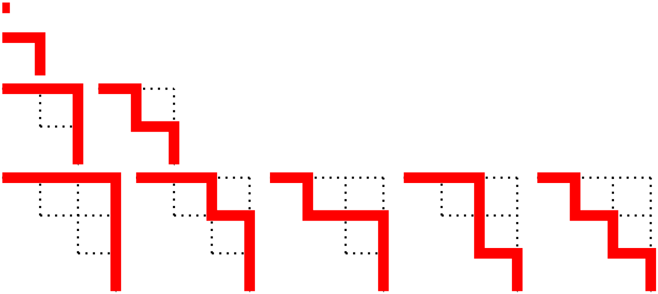
Let C_n denote the number of paths of length $2n$

$C_0 = 1$:

$C_1 = 1$:

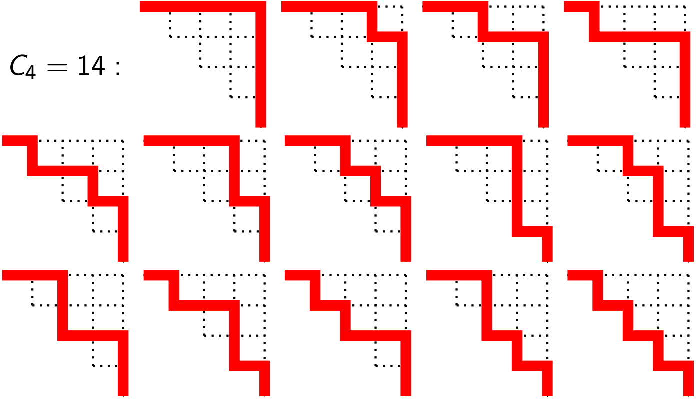
$C_2 = 2$:

$C_3 = 5$:



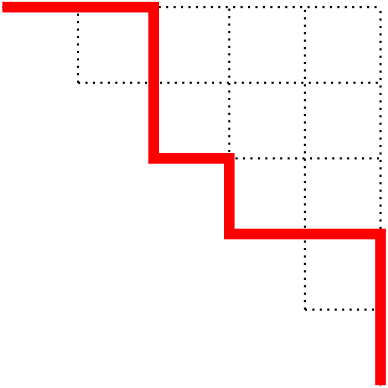
Counting “Clybourn paths”

$C_4 = 14$:



Counting “Clybourn paths”

C_5?



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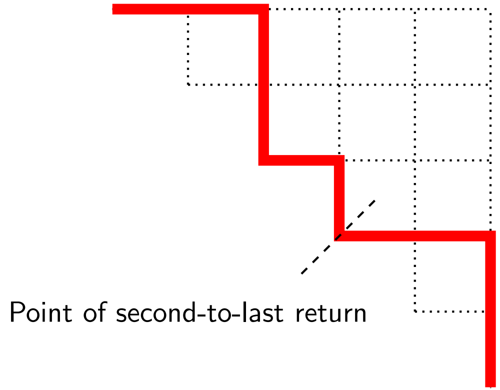

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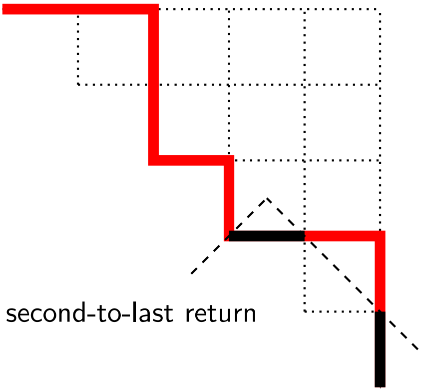
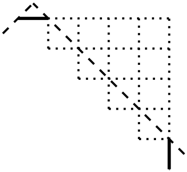


Point of second-to-last return

Counting "Clybourn paths"

$C_5 =$

Counting "Clybourn paths"

$C_5 =$ 

Counting "Clybourn paths"

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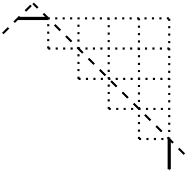
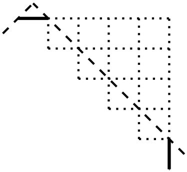


$C_0 \cdot C_4$

Counting "Clybourn paths"

$C_5 =$  $+$

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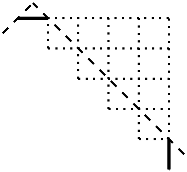

$C_0 \cdot C_4$

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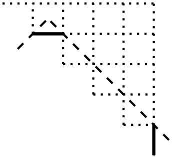
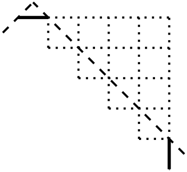


$C_1 \cdot C_3$

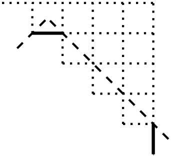
Counting "Clybourn paths"

$C_5 =$



$C_0 \cdot C_4$

$+$

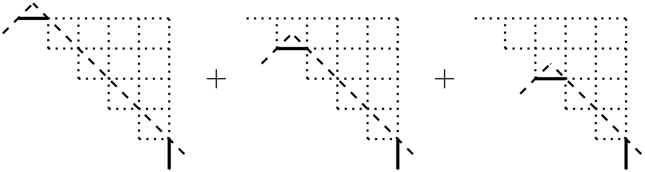


$C_1 \cdot C_3$

$+$

Counting "Clybourn paths"

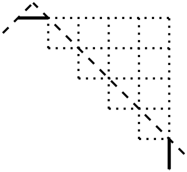
$C_5 =$



$C_0 \cdot C_4 \quad + \quad C_1 \cdot C_3 \quad + \quad C_2 \cdot C_2$

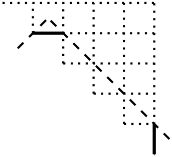
Counting "Clybourn paths"

$C_5 =$



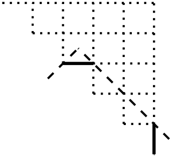
$C_0 \cdot C_4$

$+$



$C_1 \cdot C_3$

$+$

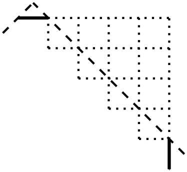


$C_2 \cdot C_2$

$+$

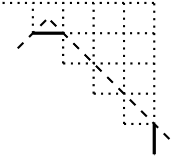
Counting “Clybourn paths”

$C_5 =$



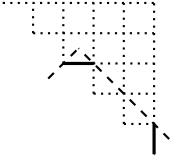
$C_0 \cdot C_4$

$+$



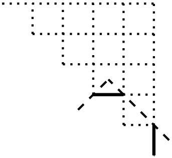
$C_1 \cdot C_3$

$+$



$C_2 \cdot C_2$

$+$



$C_3 \cdot C_1$

Counting “Clybourn paths”

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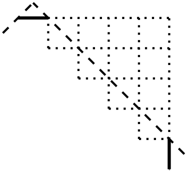
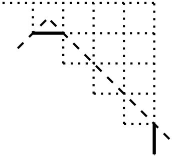


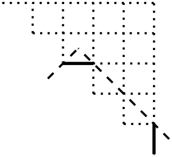
$C_0 \cdot C_4$

$+$



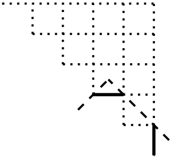
$C_1 \cdot C_3$

$+$



$C_2 \cdot C_2$

$+$



$C_3 \cdot C_1$

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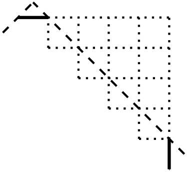
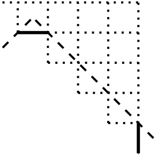


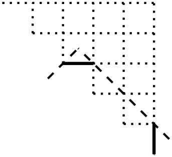
$C_0 \cdot C_4$

$+$



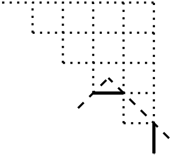
$C_1 \cdot C_3$

$+$



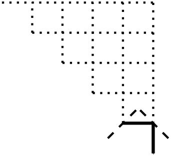
$C_2 \cdot C_2$

$+$



$C_3 \cdot C_1$

$+$



$C_4 \cdot C_0$

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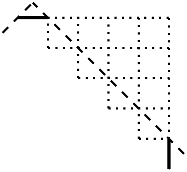
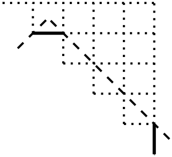


$C_0 \cdot C_4$

$+$



$C_1 \cdot C_3$

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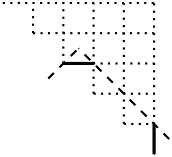

$C_2 \cdot C_2$

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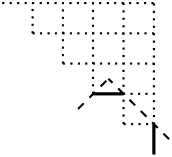

$C_3 \cdot C_1$

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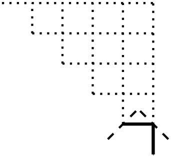

$C_4 \cdot C_0$

$$= 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = \mathbf{42}$$

Catalan numbers

$$C_0 = 1, \quad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

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$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

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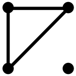
- noncrossing partitions: 

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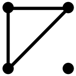
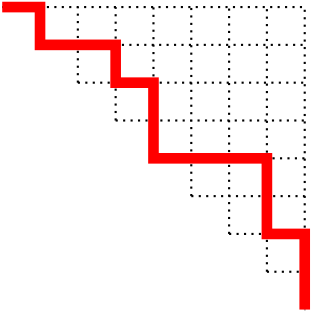
- noncrossing partitions: 
- and about 200 other sets of things...

Combinatorics, geometry, and an algebra of paths

1. Combinatorics: Walking to Steppenwolf
2. Geometry: Tamari poset/associahedron
3. Algebra: Loday-Ronco

A local transformation



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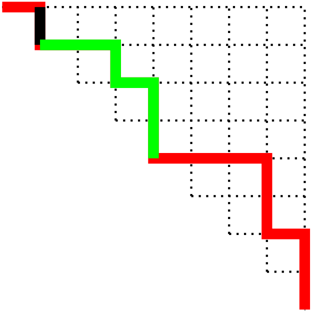

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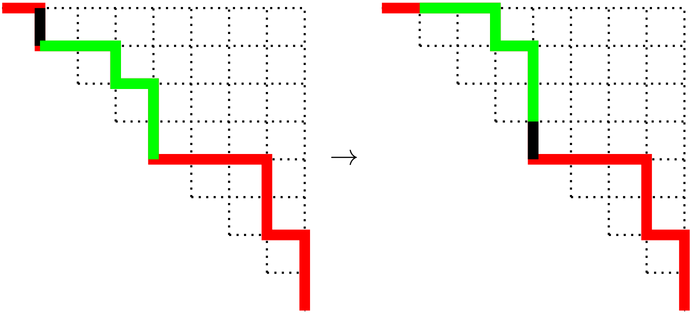

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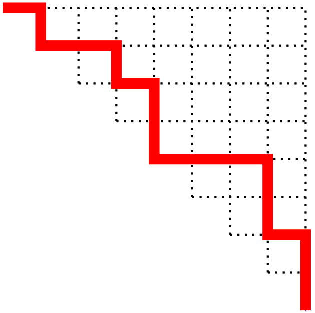

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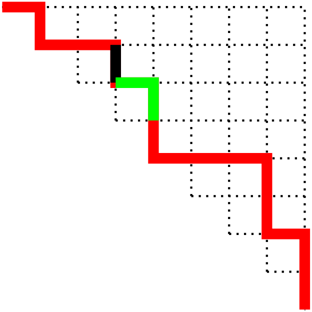

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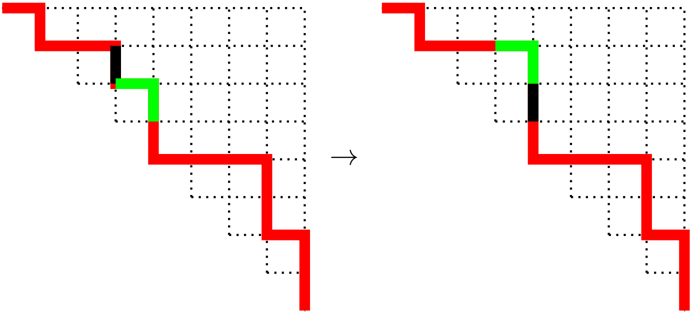

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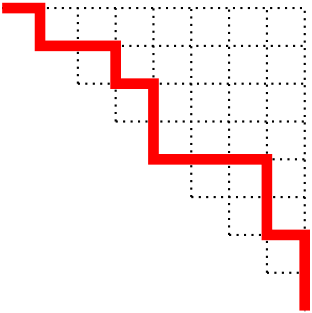

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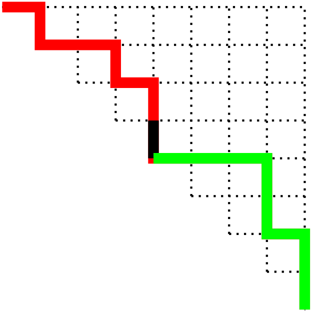

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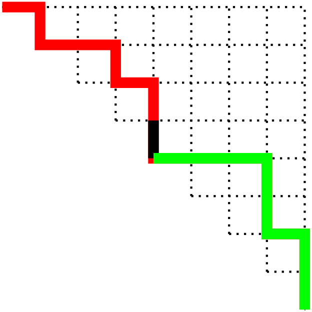

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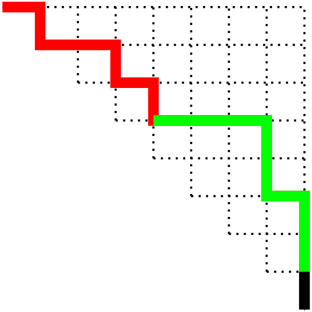

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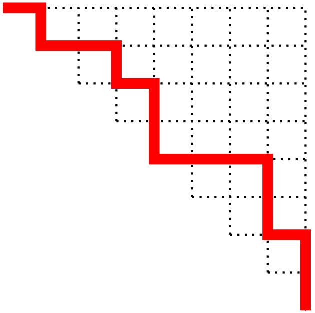

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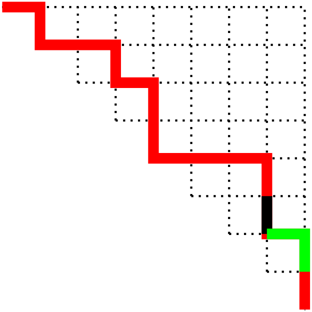

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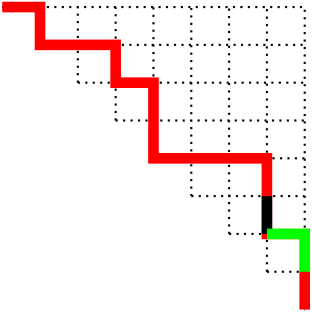
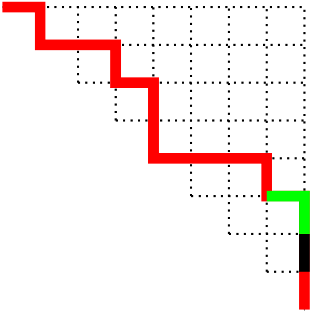


$\rightarrow$



A local transformation

$n = 0$

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$n = 0$ $n = 1$

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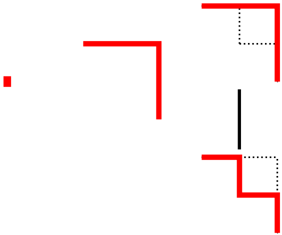
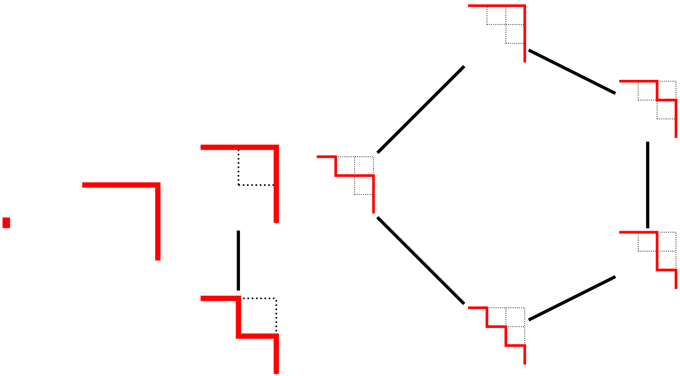


$n = 0$ $n = 1$ $n = 2$

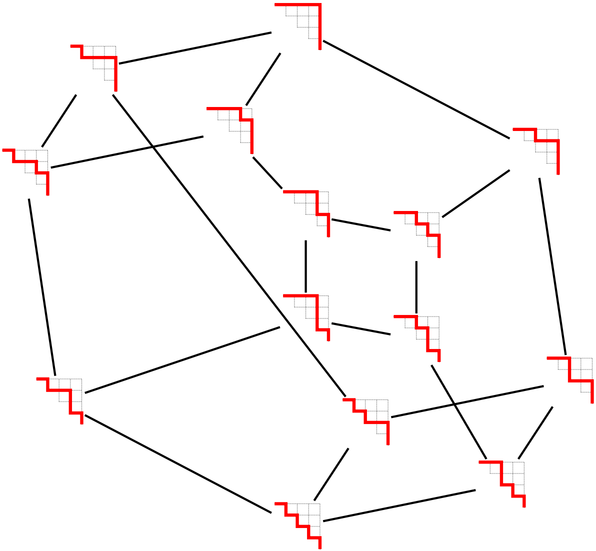
A local transformation



$n = 0$ $n = 1$ $n = 2$ $n = 3$

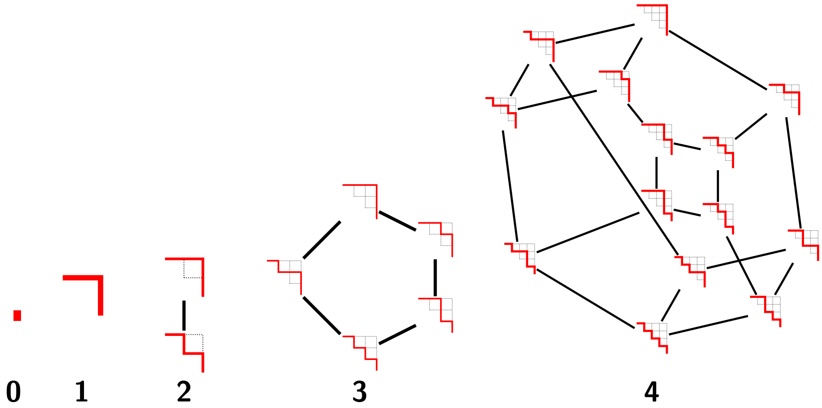
Tamari poset (Associahedron)

$n = 4$



Combinatorics, geometry, and an algebra of paths

Jean-Louis Loday: "the integers as molecules"

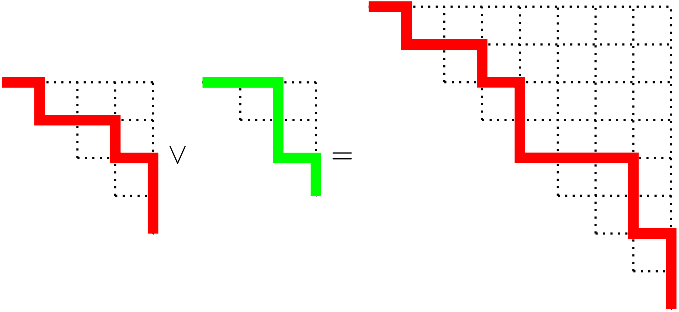


A meditation

We can do arithmetic at the molecular level. Can we do it at the atomic level?

The wedge operation

$p \vee q$:



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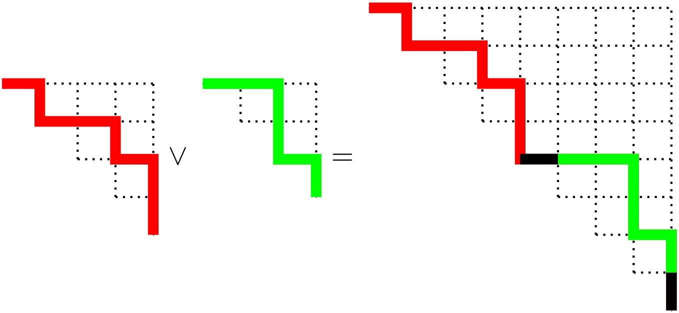

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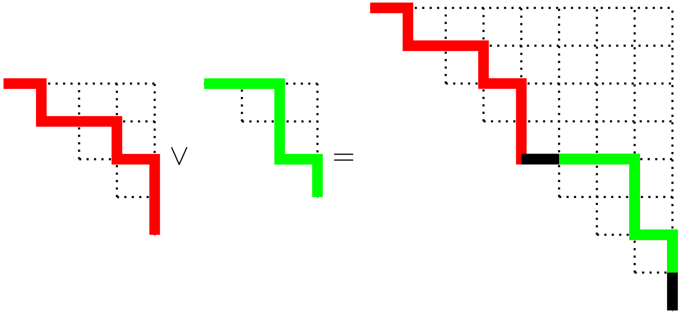
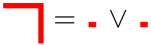


(Recall the role the wedge played in the Catalan identity)

Decomposing paths

$p = p^{\ell} \vee p^{r}$ (unique decomposition)



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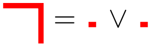

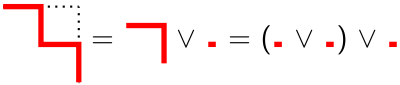

Decomposing paths

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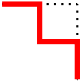
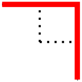
 $= \bullet \vee \bullet$

 $=$  $\vee \bullet = (\bullet \vee \bullet) \vee \bullet$

 $= \bullet \vee$  $= \bullet \vee (\bullet \vee \bullet)$

Defining addition

Goal: define addition for paths,

$$p + q,$$

in a way that generalizes addition for integers

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- $p + \mathbf{0} = \mathbf{0} + p = p$
- for $p, q \neq \mathbf{0}$, $p + q = (p \dashv q) \cup (p \vdash q)$
- now recursively,

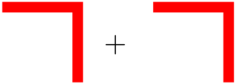
$$p \dashv q = p^{\ell} \vee (p^{r} + q) \quad \text{and} \quad p \vdash q = (p + q^{\ell}) \vee q^{r}$$

Why $1 + 1 = 2$

Why $1 + 1 = 2$

1

Why $1 + 1 = 2$



$\mathbf{1}$ $\mathbf{1}$

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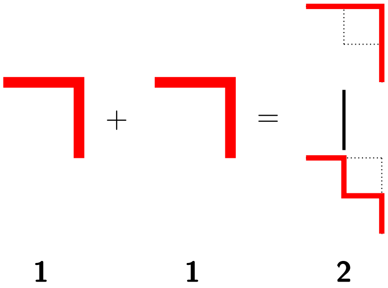


Why $1 + 1 = 2$

Split the plus sign: $+ \quad = \quad \dashv \cup \vdash$

$$\urcorner + \urcorner = (\urcorner \dashv \urcorner) \cup (\urcorner \vdash \urcorner)$$

Why $1 + 1 = 2$

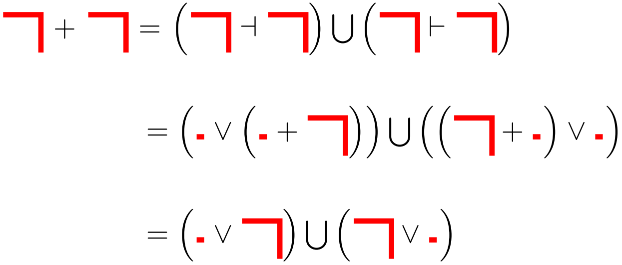
Split the plus sign: $+ \quad = \quad \dashv \cup \vdash$

$$\neg + \neg = (\neg \dashv \neg) \cup (\neg \vdash \neg)$$

$$= (\bullet \vee (\bullet + \neg)) \cup ((\neg + \bullet) \vee \bullet)$$

Why $1 + 1 = 2$

Split the plus sign: $+ \quad = \quad \dashv \cup \vdash$



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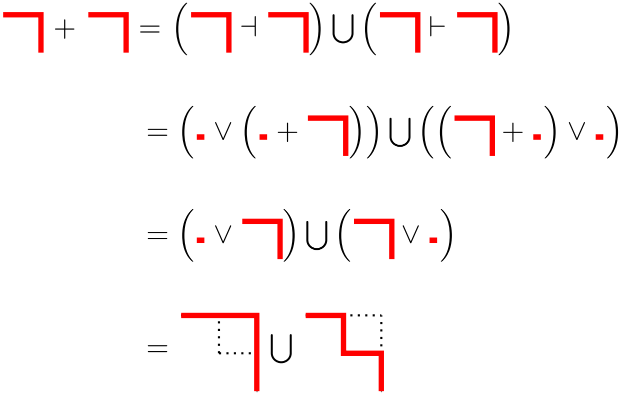

Why $1 + 2 = 3$

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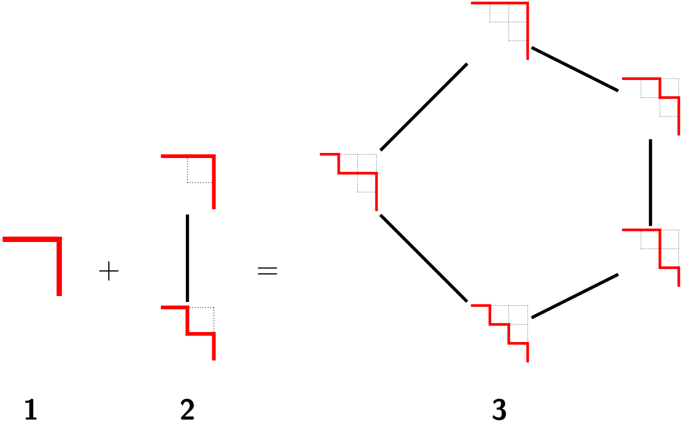
Want:



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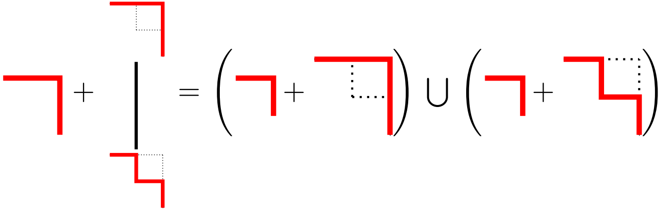

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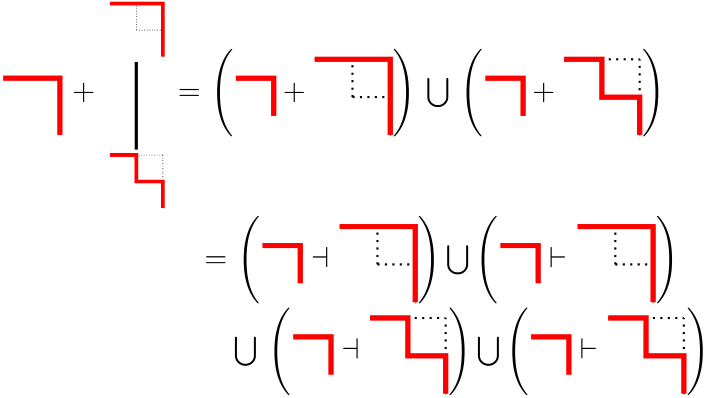

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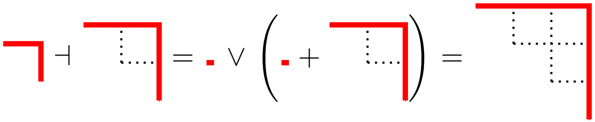

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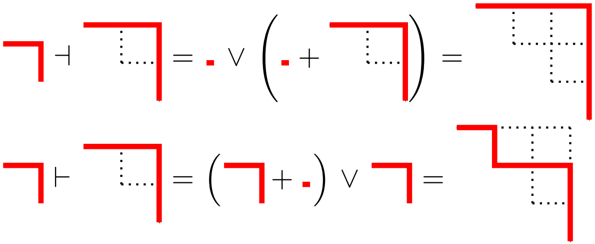

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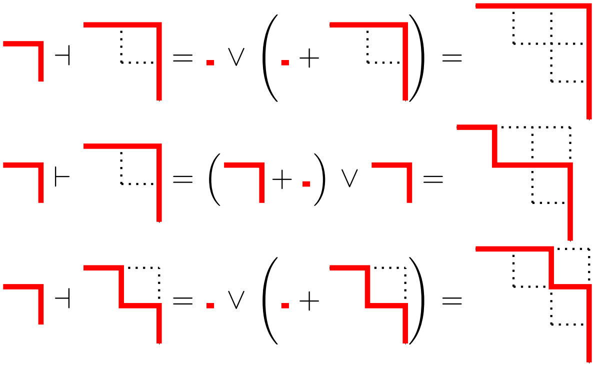

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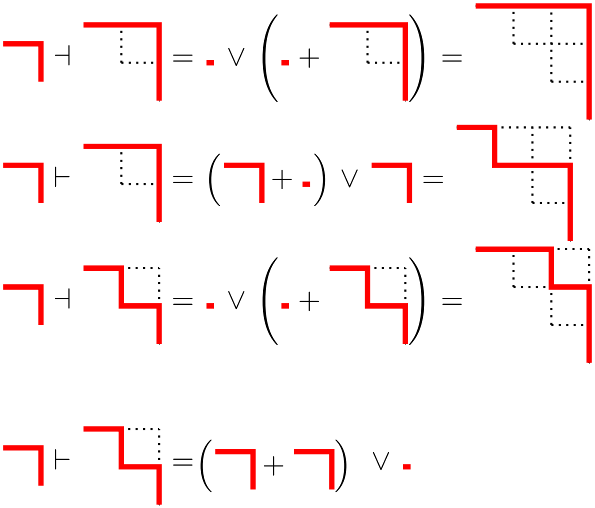

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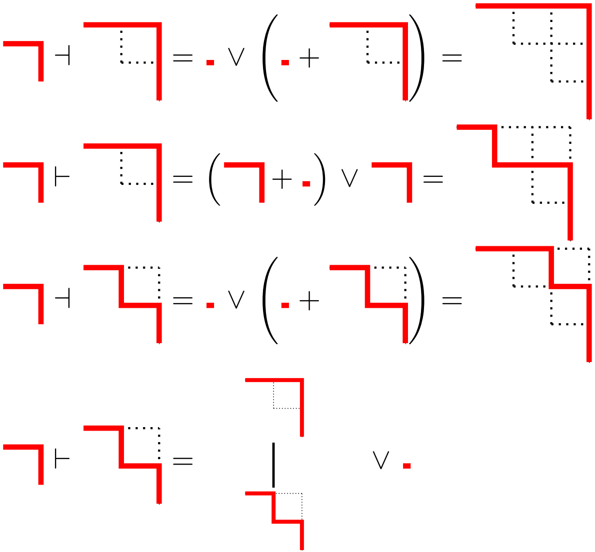

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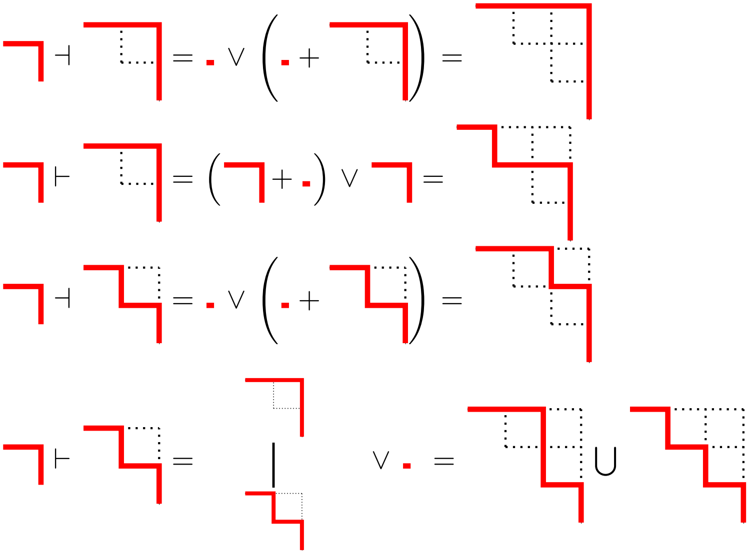


Surprise!

Theorem

If $a + b = c$ (as nonnegative integers), then

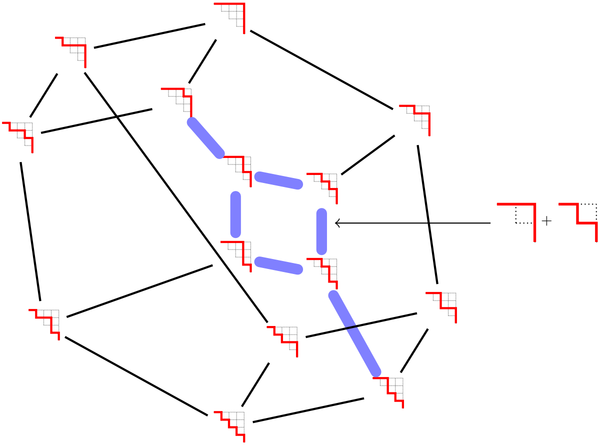
$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$

(as Tamari lattices)

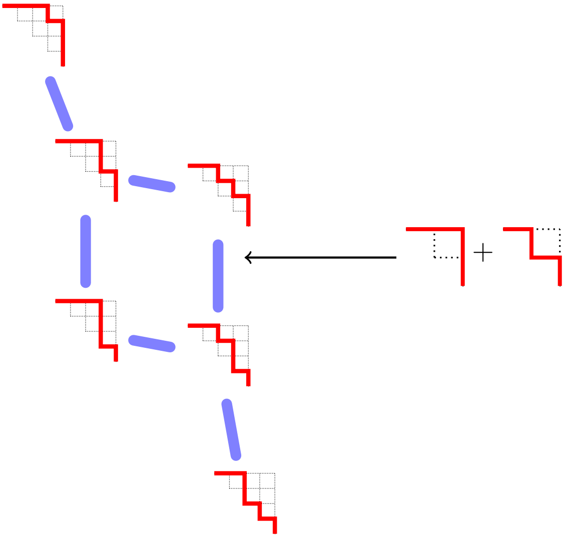
A better way to add

Can we add two paths more simply?

Adding paths and the Tamari poset



Adding paths and the Tamari poset



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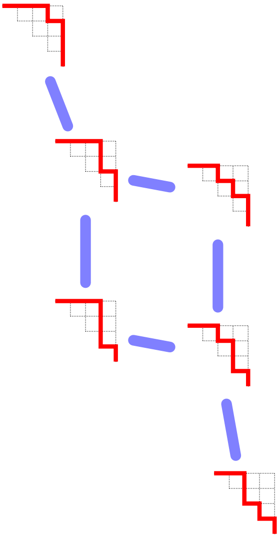

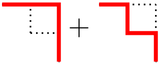

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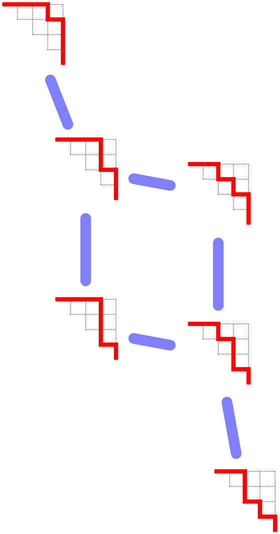

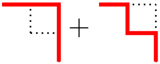

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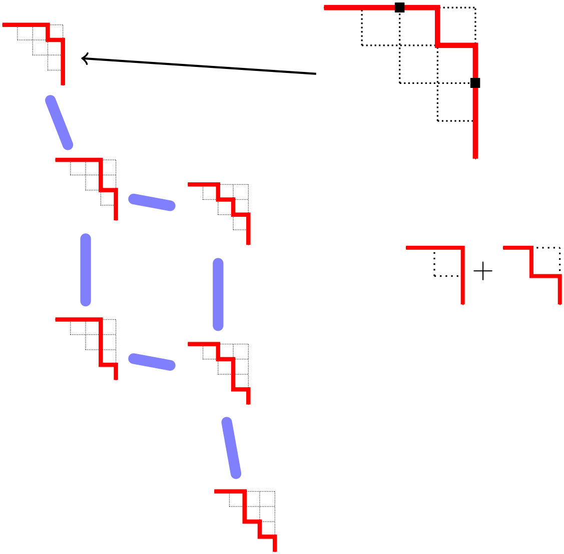

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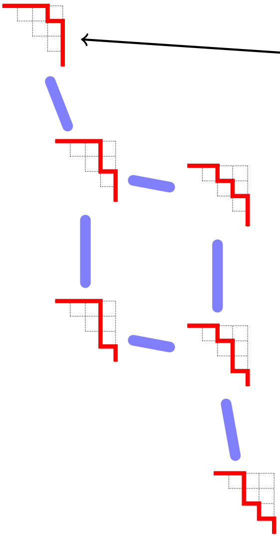
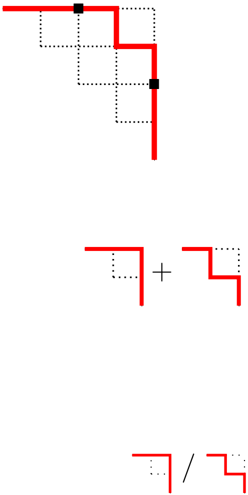
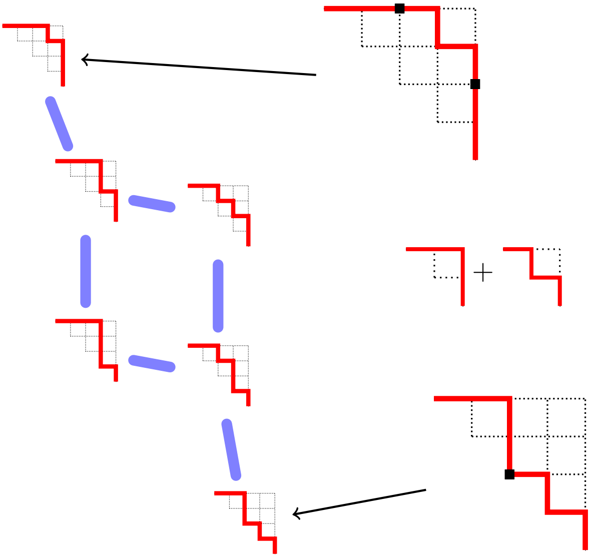




Adding paths and the Tamari poset



Adding paths and the Tamari poset

Theorem (Loday-Ronco)

For any paths p and q,

$$p + q = \bigcup_{p/q \leq r \leq p \backslash q} r$$

Adding paths and the Tamari poset

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There is also a multiplication that refines ordinary integer multiplication. . .

Adding paths and the Tamari poset

Theorem (Loday-Ronco)

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There is also a multiplication that refines ordinary integer multiplication. . . the Loday-Ronco algebra is the "free dendriform algebra on one generator" and a "combinatorial Hopf algebra"

Food for thought

We can now do arithmetic:

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- at the molecular level (whole numbers)

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- at the atomic level (paths)

Food for thought

We can now do arithmetic:

- at the molecular level (whole numbers)
- at the atomic level (paths)
- What about the subatomic level?

References

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- J.-L. Loday, *Arithmetree.* J. Algebra 258 (2002), 275–309.
- J.-L. Loday and M. Ronco, *Hopf algebra of the planar binary trees.* Adv. Math. 139 (1998), 293–309