Napkin problems

Kyle Petersen

DePaul University
Department of Mathematical Sciences

Jim Propp's $2^{\binom{4}{2}}$th birthday conference

Napkin Problems

I had one foot out the door. . .

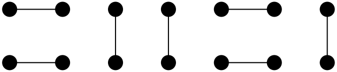
(Fall 2002)

I had one foot out the door. . .

(Fall 2002)

I had one foot out the door. . .

(Fall 2002)



(REACH, dimers)

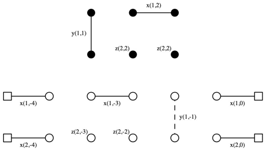
Only a year or two later. . .

A Reciprocity Theorem for Monomer-Dimer Coverings

Nick Anzalone[1] and John Baldwin[2] and Ilya Bronshtein[3] and T. Kyle Petersen[4]

[1]*University of Massachussetts-Boston, Boston, MA, USA, njanzalo@cs.umb.edu*
[2]*Harvard University, Cambridge, MA, USA, jbaldwin@fas.harvard.edu*
[3]*Brandeis University, Waltham, MA, USA, ilyabron@brandeis.edu*
[4]*Department of Mathematics, Brandeis University, Waltham, MA, USA, tkpeters@brandeis.edu*

received 14 February 2003, revised 14th April 2003, accepted .

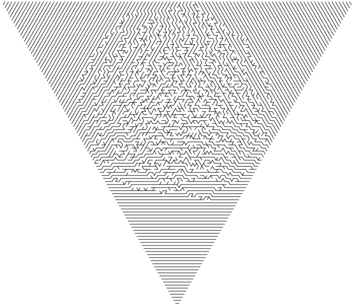


An arctic circle theorem for Groves

T. Kyle Petersen[a], David Speyer[b]

[a]*Department of Mathematics, Brandeis University, Waltham, MA, 02454, USA*
[b]*Department of Mathematics, University of California Berkeley, Berkeley, CA, 94720, USA*

Received 13 July 2004
Available online 18 January 2005



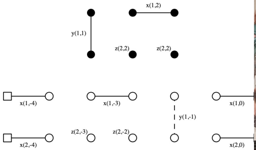
Only a year or two later. . .

A Reciprocity Theorem for Monomer-Din
Coverings

Nick Anzalone[1] and John Baldwin[2] and Ilya Bronshtein[3] and
Petersen[4]

[1]*University of Massachussetts-Boston, Boston, MA, USA, njanzalo@cs.umb.edu*
[2]*Harvard University, Cambridge, MA, USA, jbaldwin@fas.harvard.edu*
[3]*Brandeis University, Waltham, MA, USA, ilyabron@brandeis.edu*
[4]*Department of Mathematics, Brandeis University, Waltham, MA, USA, tkpeters@brandeis.edu*

received 14 February 2003, revised 14th April 2003, accepted .

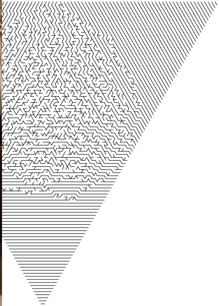


cle theorem for Groves

tersen[a], David Speyer[b]

s, Brandeis University, Waltham, MA, 02454, USA
versity of California Berkeley, Berkeley, CA, 94720, USA

Received 13 July 2004
able online 18 January 2005



Napkin Problems

Sweden, winter 2005

INSTITUT
MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Sweden, winter 2005

INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Anders Claesson

Sweden, winter 2005

INSTITUT
MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Anders Claesson

Anders: "Hej Kyle! Nice to meet you! Knuth is giving a talk in a few minutes."

Sweden, winter 2005

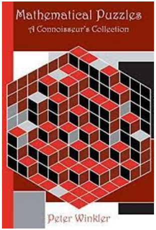
INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Anders Claesson

Anders: "Hej Kyle! Nice to meet you! Knuth is giving a talk in a few minutes."

Kyle: "I don't know, I'm pretty tired..."

Sweden, winter 2005



Knuth: "I learned about this great puzzle from Pete Winkler's new book. . . "

The Malicious Maitre d'

THE MALICIOUS MAITRE D'

At a mathematics conference banquet, 48 male mathematicians, none of them knowledgeable about table etiquette, find themselves assigned to a big circular table. On the table, between each pair of settings, is a coffee cup containing a cloth napkin. As each person is seated (by the maitre d'), he takes a napkin from his left or right; if both napkins are present, he chooses randomly (but the maitre d' doesn't get to see which one he chose).

In what order should the seats be filled to maximize the expected number of mathematicians who don't get napkins?

... This problem can be traced to a particular event. Princeton mathematician John H. Conway came to Bell Labs on March 30, 2001 to give a "General Research Colloquium." At lunchtime, [Winkler] found himself sitting between Conway and computer scientist Rob Pike (now of Google), and the napkins and coffee cups were as described in the puzzle. Conway asked how many diners would be without napkins if they were seated in *random* order, and Pike said: "Here's an easier question—what's the *worst* order?"

Napkins in a random setting

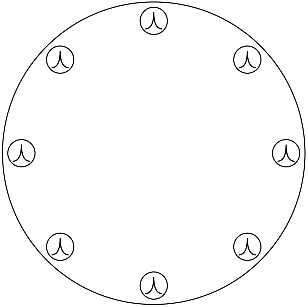
NAPKINS IN A RANDOM SETTING

Remember the conference banquet, where a bunch of mathematicians find themselves assigned to a big circular table? Again, on the table, between each pair of settings, is a coffee cup containing a cloth napkin. As each person sits down, he takes a napkin from his left or right; if both napkins are present, he chooses randomly.

This time there is no maitre d'; the seats are occupied in random order. If the number of mathematicians is large, what fraction of them (asymptotically) will end up without a napkin?

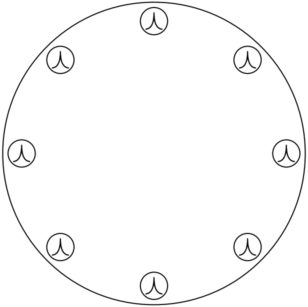
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):



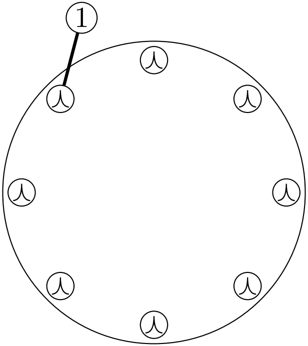
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R)$,



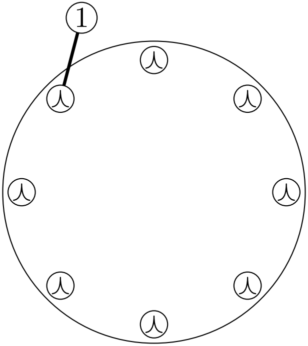
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R)$,



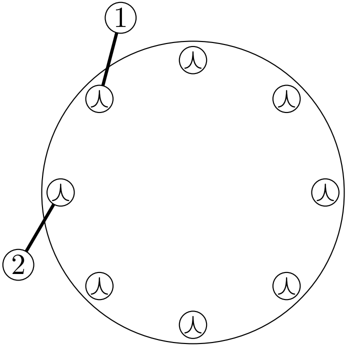
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L),$



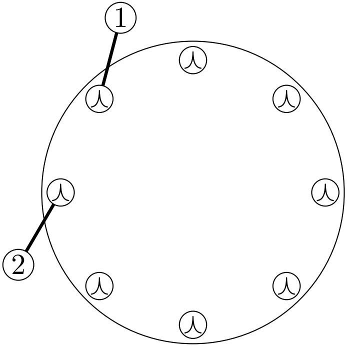
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L)$,



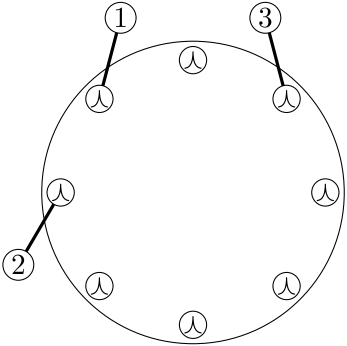
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L),$



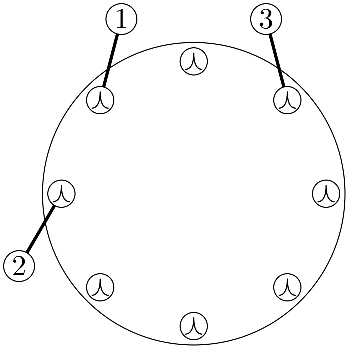
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L),$



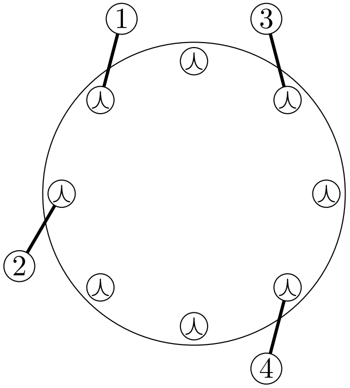
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R),$



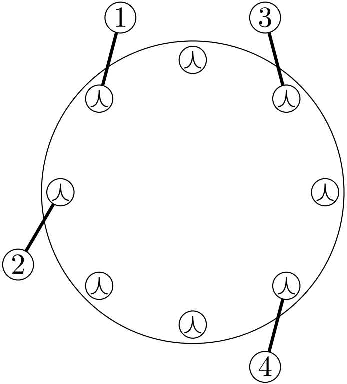
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R),$



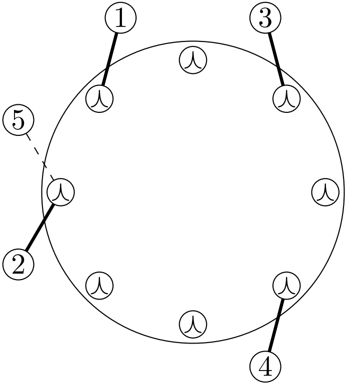
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R), (2, R),$



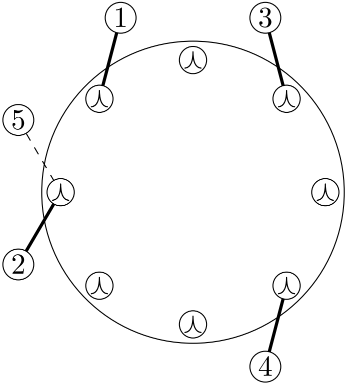
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R), (2, R),$



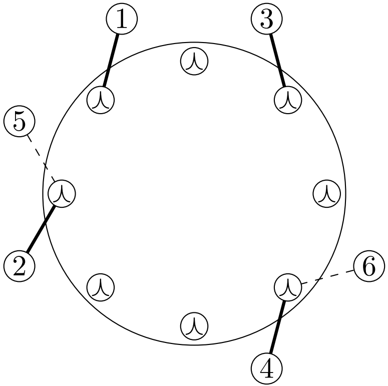
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R), (2, R), (6, L),$



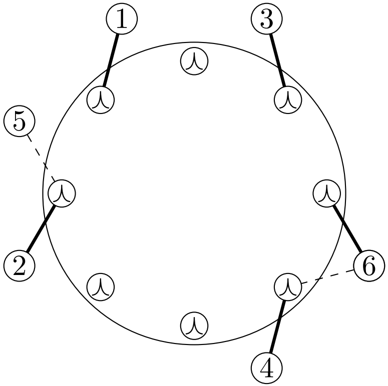
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R), (2, R), (6, L),$



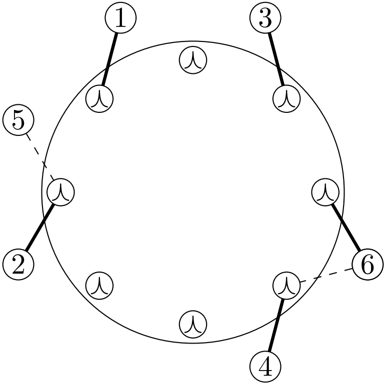
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R), (2, R), (6, L),$



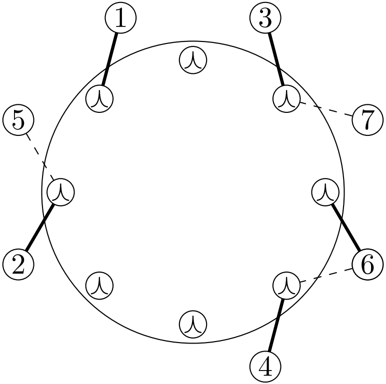
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R), (2, R), (6, L), (7, R),$



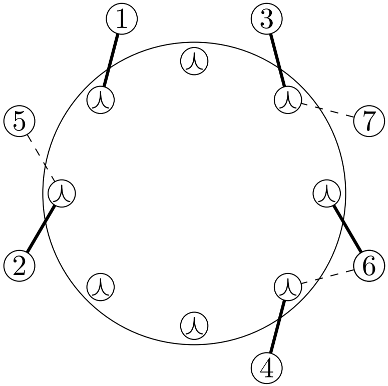
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R), (2, R), (6, L), (7, R),$



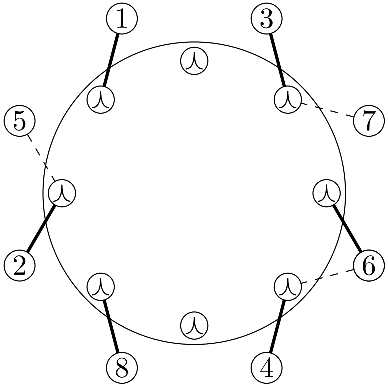
Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R), (2, R), (6, L), (7, R), (4, L)$



Napkins in a random setting

Diners in the queue (assigned seat, napkin preference):
$(1, R), (3, L), (8, L), (5, R), (2, R), (6, L), (7, R), (4, L)$



Winkler's solution

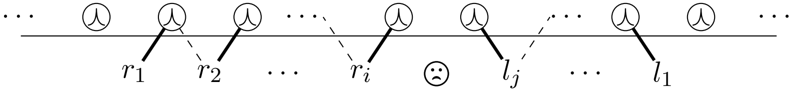
Winkler's solution

$p =$(expected proportion napkinless)$=$(prob. YOU are napkinless)

Winkler's solution

p =(expected proportion napkinless)=(prob. YOU are napkinless)

A minimal block:

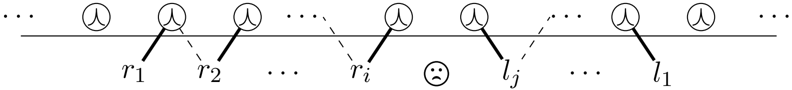


with $k = i + j + 1$

Winkler's solution

p =(expected proportion napkinless)=(prob. YOU are napkinless)

A minimal block:



with $k = i + j + 1$

Then

$$p_k = \frac{2 \cdot \#\text{proper nonempty subsets}}{\#\text{seating pref.}} = \frac{2(2^{k-1} - 2)}{2^k k!}$$

Winkler's solution

Sum over all $k \geq 3$ (and let $n \to \infty$):

Winkler's solution

Sum over all $k \geq 3$ (and let $n \to \infty$):

$$p = \sum_{k \geq 3} p_k$$

Winkler's solution

Sum over all $k \geq 3$ (and let $n \to \infty$):

$$p = \sum_{k \geq 3} p_k$$
$$= \sum_{k \geq 3} \frac{2^k - 2^2}{2^k k!}$$

Winkler's solution

Sum over all $k \geq 3$ (and let $n \to \infty$):

$$p = \sum_{k \geq 3} p_k$$
$$= \sum_{k \geq 3} \frac{2^k - 2^2}{2^k k!}$$
$$= \sum_{k \geq 3} \frac{1}{k!} - 4 \sum_{k \geq 3} \frac{1}{2^k k!}$$

Winkler's solution

Sum over all $k \geq 3$ (and let $n \to \infty$):

$$\begin{aligned} p &= \sum_{k\geq 3} p_k \\ &= \sum_{k\geq 3} \frac{2^k - 2^2}{2^k k!} \\ &= \sum_{k\geq 3} \frac{1}{k!} - 4\sum_{k\geq 3} \frac{1}{2^k k!} \\ &= e - 4\sqrt{e} + 4 \end{aligned}$$

Winkler's solution

Sum over all $k \geq 3$ (and let $n \to \infty$):

$$\begin{aligned} p &= \sum_{k\geq 3} p_k \\ &= \sum_{k\geq 3} \frac{2^k - 2^2}{2^k k!} \\ &= \sum_{k\geq 3} \frac{1}{k!} - 4\sum_{k\geq 3} \frac{1}{2^k k!} \\ &= e - 4\sqrt{e} + 4 \\ &= (2-\sqrt{e})^2 \approx 0.1234 \end{aligned}$$

Napkins in a random setting

Theorem (Winkler's solution to Conway's napkin problem)

The expected number of napkinless diners at a table for n diners is approximately

$$(2 - \sqrt{e})^2 \cdot n,$$

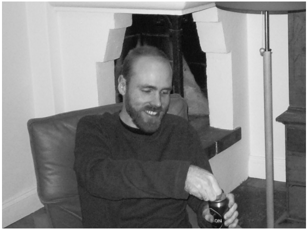
or about 12.34% of the table.

Napkins in a random setting

Knuth: "Someone should do this with generating functions."

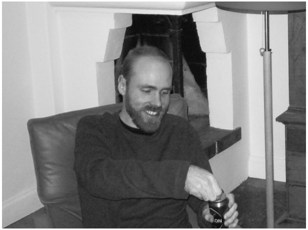
- model seatings with signed permutations $\pi \in C_n$
- $\nu(\pi)$ denotes the number of napkinless diners
- let $C_n(x) = \sum_{\pi \in C_n} x^{\nu(\pi)}$
- want
  $C(x,z) = \sum_{n \geq 0} C_n(x) \frac{z^n}{2^n n!} = \sum_{n,k \geq 0} p(n,k) x^k z^n$

Napkins in a random setting



Kyle and Anders: "Let's do it!"

Napkins in a random setting



Kyle and Anders: "Let's do it!" (and we did!)

Conway's Napkin Problem

Anders Claesson and T. Kyle Petersen

1. INTRODUCTION. The problem studied in this article first appeared in the book *Mathematical Puzzles: A Connoisseur's Collection*, by Peter Winkler [5] and was in-

Napkins in a random setting

Theorem (Claesson-P., 2007)

The generating function $C(x,z)$ is:

$$C(x,z) = \frac{z(1-(1-x)(1-e^{z/2})^2)}{x(1-z)+(1-x)(2-e^{z/2})^2},$$

and thus the expectation for $\nu(\pi)$ is:

$$E_n(\nu(\pi)) = n\left(4 - 4\exp_n(1/2) + \exp_n(1)\right),$$

where $\exp_n$ is the truncated exponential function.

(Idea: straighten the table and use recursive structure to find generating functions)

Napkins in a random setting

Comments:

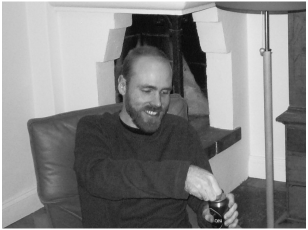
- if p is prob. diner takes left napkin,

$$E_n(\nu(\pi)) = \frac{n}{pq}(1 - p\exp_n(q) - q\exp_n(p) + pq\exp_n(1))$$

  (reproduces Sudbury)
- can also track "frustrated" diners; when $p = 1/2$, there are about 17% frustrated diners, so about 70% of diners are happy!
- follow-up paper by N. Eriksen in 2008 considered other variations (French diners, sloppy diners, greedy diners)

Napkin Problems

15 years go by. . .



15 years go by. . .

Chicago, winter 2022

"We want to do a reading course in generating functions."

Reed Acton

Blake Shirman

Daniel Toal

Chicago, winter 2022

"We want to do a reading course in generating functions."

Reed Acton

Blake Shirman

Daniel Toal

"Okay, read the first half of Wilf, then reproduce the results of my paper with Anders"

A mathematical microagression

From Claesson-P. (2007):

1. INTRODUCTION. The problem studied in this article first appeared in the book *Mathematical Puzzles: A Connoisseur's Collection*, by Peter Winkler [**5**] and was inspired by a true story. Rather than recounting the problem and the story ourselves, we prefer to quote directly from *Mathematical Puzzles* [**5**, p. 22]:

> THE MALICIOUS MAITRE D'
> At a mathematics conference banquet, 48 male mathematicians, none of them knowledgeable about table etiquette, find themselves assigned to a big circular table. On the table, between each pair of settings, is a coffee cup containing a cloth napkin. As each person is seated (by the maitre d'), he takes a napkin from his left or right; if both napkins are present, he chooses randomly (but the maitre d' doesn't get to see which one he chose).
>
> In what order should the seats be filled to maximize the expected number of mathematicians who don't get napkins?
>
> . . . This problem can be traced to a particular event. Princeton mathematician John H. Conway came to Bell Labs on March 30, 2001 to give a "General Research Colloquium." At lunchtime, [Winkler] found himself sitting between Conway and computer scientist Rob Pike (now of Google), and the napkins and coffee cups were as described in the puzzle. Conway asked how many diners would be without napkins if they were seated in *random* order, and Pike said: "Here's an easier question—what's the *worst* order?"

The problem of the malicious maitre d' is not horribly difficult; if you're having trouble finding a solution, you can see Winkler's book for a nice explanation. In this paper, it is Conway's problem that we focus on. Again, from the book [**5**, p. 122]:

A mathematical microagression

From Claesson-P. (2007):

lunchtime, [Winkler] found himself sitting between Conway and computer scientist Rob Pike (now of Google), and the napkins and coffee cups were as described in the puzzle. Conway asked how many diners would be without napkins if they were seated in *random* order, and Pike said: "Here's an easier question—what's the *worst* order?"

The problem of the malicious maitre d' is not horribly difficult; if you're having trouble finding a solution, you can see Winkler's book for a nice explanation. In this

A mathematical microagression

From Claesson-P. (2007):

lunchtime, [Winkler] found himself sitting between Conway and computer scientist Rob Pike (now of Google), and the napkins and coffee cups were as described in the puzzle. Conway asked how many diners would be without napkins if they were seated in *random* order, and Pike said: "Here's an easier question—what's the *worst* order?"

The problem of the malicious maitre d' is not horribly difficult; if you're having trouble finding a solution, you can see Winkler's book for a nice explanation. In this

Students: "We don't get it."

A mathematical microagression

From Claesson-P. (2007):

> lunchtime, [Winkler] found himself sitting between Conway and computer scientist Rob Pike (now of Google), and the napkins and coffee cups were as described in the puzzle. Conway asked how many diners would be without napkins if they were seated in *random* order, and Pike said: "Here's an easier question—what's the *worst* order?"
>
> The problem of the malicious maitre d' is not horribly difficult; if you're having trouble finding a solution, you can see Winkler's book for a nice explanation. In this

Students: "We don't get it."
. . . what they "didn't get" was a new puzzle Winkler introduced in his solution. . .

The adaptive maitre d'

From Winkler's solution:

If the maitre d' sees which napkin is grabbed each time he seats a diner (computer theorists would call him an "adaptive adversary"), it is not hard to see that his best strategy is as follows. If the first diner takes (say) his right napkin, the next is seated two spaces to his right so that the diner in between may be trapped. If the second diner also takes his right napkin, the maitre d' tries again by skipping another chair to the right. If the second diner takes his left napkin (leaving the space between him and the first diner napkinless), the third diner is seated directly to the second diner's right. Further diners are seated according to the same rule until the circle is closed, then the remaining diners (some of whom are doomed to be napkinless) are seated. This results in 1/6 of the diners without napkins, on average.

The adaptive maitre d'

From Winkler's solution:

If the maitre d' sees which napkin is grabbed each time he seats a diner (computer theorists would call him an "adaptive adversary"), it is not hard to see that his best strategy is as follows. If the first diner takes (say) his right napkin, the next is seated two spaces to his right so that the diner in between may be trapped. If the second diner also takes his right napkin, the maitre d' tries again by skipping another chair to the right. If the second diner takes his left napkin (leaving the space between him and the first diner napkinless), the third diner is seated directly to the second diner's right. Further diners are seated according to the same rule until the circle is closed, then the remaining diners (some of whom are doomed to be napkinless) are seated. This results in 1/6 of the diners without napkins, on average.

Students, week 6: "We don't get it."

The adaptive maitre d'

From Winkler's solution:

> If the maitre d' sees which napkin is grabbed each time he seats a diner (computer theorists would call him an "adaptive adversary"), it is not hard to see that his best strategy is as follows. If the first diner takes (say) his right napkin, the next is seated two spaces to his right so that the diner in between may be trapped. If the second diner also takes his right napkin, the maitre d' tries again by skipping another chair to the right. If the second diner takes his left napkin (leaving the space between him and the first diner napkinless), the third diner is seated directly to the second diner's right. Further diners are seated according to the same rule until the circle is closed, then the remaining diners (some of whom are doomed to be napkinless) are seated. This results in 1/6 of the diners without napkins, on average.

Students, week 6: "We don't get it."
Students, week 7: "We don't get it."

The adaptive maitre d'

From Winkler's solution:

If the maitre d' sees which napkin is grabbed each time he seats a diner (computer theorists would call him an "adaptive adversary"), it is not hard to see that his best strategy is as follows. If the first diner takes (say) his right napkin, the next is seated two spaces to his right so that the diner in between may be trapped. If the second diner also takes his right napkin, the maitre d' tries again by skipping another chair to the right. If the second diner takes his left napkin (leaving the space between him and the first diner napkinless), the third diner is seated directly to the second diner's right. Further diners are seated according to the same rule until the circle is closed, then the remaining diners (some of whom are doomed to be napkinless) are seated. This results in 1/6 of the diners without napkins, on average.

Students, week 6: "We don't get it."
Students, week 7: "We don't get it."
Students, week 8: "We can do better."

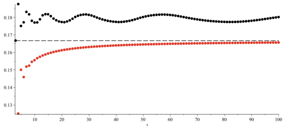
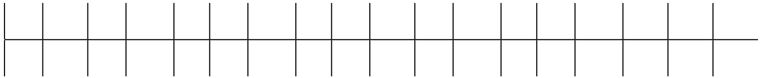


Figure 1. The expected proportion of napkinless diners, when seated using trap setting (in red) and napkin shunning (in black). The dashed line is at height 1/6.

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$



(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R																	
1																	

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R															
1		2															

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R		L													
1		2		3													

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R		L	R												
1		2		3	4												

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R		L	R		L										
1		2		3	4		5										

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R		L	R		L	R									
1		2		3	4		5	6									

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R		L	R		L	R		R							
1		2		3	4		5	6		7							

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R		L	R		L	R		R		L					
1		2		3	4		5	6		7		8					

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R		L	R		L	R		R		L	R				
1		2		3	4		5	6		7		8	9				

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R		L	R		L	R		R		L	R		L	
1		2		3	4		5	6		7		8	9		10	

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R		L	R		L	R		R		L	R		L	L	
1		2		3	4		5	6		7		8	9		10	11	

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	R		L	R		L	R		R		L	R		L	L	
1	12	2		3	4		5	6		7		8	9		10	11	

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	R	R	L	R		L	R		R		L	R		L	L	
1	12	2	**13**	3	4		5	6		7		8	9		10	11	

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	R	R	L	R	L	L	R		R		L	R		L	L
1	12	2	**13**	3	4	**14**	5	6		7		8	9		10	11

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	R	R	L	R	L	L	R	L	R		L	R		L	L		
1	12	2	**13**	3	4	**14**	5	6	15	7		8	9		10	11		

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	R	R	L	R	L	L	R	L	R	R	L	R		L	L
1	12	2	**13**	3	4	**14**	5	6	15	7	**16**	8	9		10	11

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	R	R	L	R	L	L	R	L	R	R	L	R	R	L	L
1	12	2	**13**	3	4	**14**	5	6	15	7	**16**	8	9	**17**	10	11

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	R	R	L	R	L	L	R	L	R	R	L	R	R	L	L	L
1	12	2	**13**	3	4	**14**	5	6	15	7	**16**	8	9	**17**	10	11	18

(table wraps around)

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	R	R	L	R	L	L	R	L	R	R	L	R	R	L	L	L
1	12	2	**13**	3	4	**14**	5	6	15	7	**16**	8	9	**17**	10	11	18

(table wraps around)
Notice at most $n/3$ traps can be set, and about half of these will succeed . . .

Trap setting

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	R	R	L	R	L	L	R	L	R	R	L	R	R	L	L	L
1	12	2	**13**	3	4	**14**	5	6	15	7	**16**	8	9	**17**	10	11	18

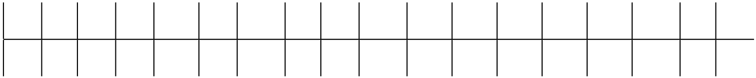
(table wraps around)

Notice at most $n/3$ traps can be set, and about half of these will succeed ... So expect $n/6$ napkinless diners!

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$



Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R																	
1																	

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R																	R
1																	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R																L	R
1																3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R								R								L	R
1								4								3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R							L	R								L	R
1							5	4								3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R			R				L	R								L	R
1			6				5	4								3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R		R	R				L	R								L	R
1		7	6				5	4								3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R				L	R								L	R
1	8	7	6				5	4								3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R		R		L	R								L	R
1	8	7	6		9		5	4								3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R	L	R		L	R								L	R
1	8	7	6	10	9		5	4								3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R	L	R	L	L	R								L	R
1	8	7	6	10	9	**11**	5	4								3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R	L	R	L	L	R				R				L	R
1	8	7	6	10	9	**11**	5	4				12				3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R	L	R	L	L	R			R	R				L	R
1	8	7	6	10	9	**11**	5	4			13	12				3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R	L	R	L	L	R		L	R	R				L	R
1	8	7	6	10	9	**11**	5	4		14	13	12				3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R	L	R	L	L	R	L	L	R	R				L	R
1	8	7	6	10	9	**11**	5	4	**15**	14	13	12				3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R	L	R	L	L	R	L	L	R	R		R		L	R
1	8	7	6	10	9	**11**	5	4	**15**	14	13	12		16		3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R	L	R	L	L	R	L	L	R	R	R	R		L	R
1	8	7	6	10	9	**11**	5	4	**15**	14	13	12	17	16		3	2

Napkin shunning

diner preferences order $\sigma \in \{\pm 1\}^n$:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	R	L	R	L	L	R	L	L	R	R	R	R	L	L	R
1	8	7	6	10	9	**11**	5	4	**15**	14	13	12	17	16	**18**	3	2

Trap setting vs. Napkin shunning

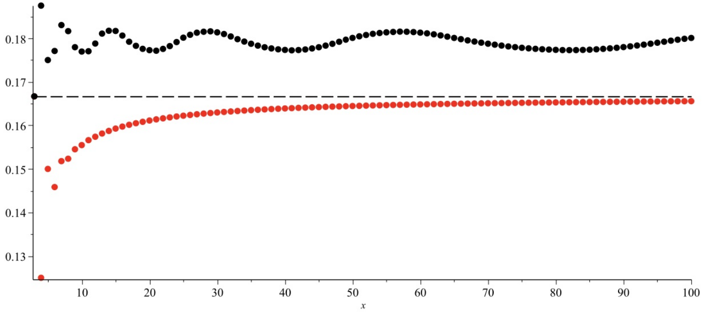


Figure 1. The expected proportion of napkinless diners, when seated using trap setting (in red) and napkin shunning (in black). The dashed line is at height $1/6$.

Trap setting vs. Napkin shunning

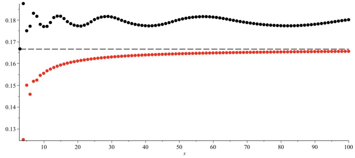


Figure 1. The expected proportion of napkinless diners, when seated using trap setting (in red) and napkin shunning (in black). The dashed line is at height 1/6.

Me: "Okay, I'm convinced! Let's figure this out."

Trap setting vs. Napkin shunning

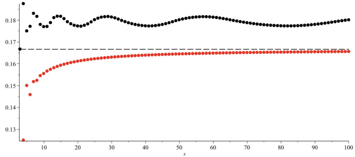


Figure 1. The expected proportion of napkinless diners, when seated using trap setting (in red) and napkin shunning (in black). The dashed line is at height 1/6.

Me: "Okay, I'm convinced! Let's figure this out." . . . and we did!

A More Malicious Maitre d'

Reed Acton, T. Kyle Petersen, Blake Shirman, and Daniel Toal

A More Malicious Maitre d'

Theorem (Acton-P.-Shirman-Toal, 2023)

The expected proportion of napkinless diners on a table with $n \geq 3$ seats:

1. *is given by $\frac{(3n-2)-16(-1/2)^n}{18n} \leq 1/6$ when using the trap setting strategy, and*
2. *is at least $1/6 = 8/48$ and at most $3/16 = 9/48$ when using the napkin shunning strategy. Moreover, for all $n \geq 5$, this proportion is between 0.1769 and 0.1831.*

(Idea: straighten the table and use recursive structure to find generating functions)

A More Malicious Maitre d' (trap setting detail)

Proposition (Acton-P.-Shirman-Toal, 2023)

Letting $W_n(t) = \sum_{\sigma \in \{\pm 1\}^n} t^{\nu_W(\sigma)}$,

$$W(t,z) = \sum_{n \geq 0} W_n(t) z^n = \frac{2 - 2z^2}{1 - z - 2z^2 - 2(t-1)z^3},$$

and with $E_n^W = W_n'(1)/2^n$ *(expectation),*

$$E^W(z) = \sum_{n \geq 0} E_n^W z^n = \frac{z^3(2-z)}{2(1-z)^2(2+z)}.$$

A More Malicious Maitre d' (napkin shunning detail)

Proposition (Acton-P.-Shirman-Toal, 2023)

Letting $S_n(t) = \sum_{\sigma \in \{\pm 1\}^n} t^{\nu_S(\sigma)}$*, and with* $E_n^S = S_n'(1)/2^n$*,* $E_1^S = E_2^S = 0$*,* $E_3^S = 1/2$*,* $E_4^S = 3/4$*, and for* $n \geq 5$*,*

$$E_n^S = \frac{1}{2}\left(E_{n-1}^S + E_{\lfloor \frac{n+1}{2} \rfloor}^S + E_{\lceil \frac{n+1}{2} \rceil}^S\right).$$

The generating function $E^S(z) = \sum_{n \geq 0} E_n^S z^n$ *has the form*

$$E^S(z) = \sum_{n \geq 0} \frac{z^{3 \cdot 2^n}(2 + z^{2^n})}{2(2 - z^{2^n})} \prod_{k=0}^{n-1} \frac{(1 + z^{2^k})^2}{z^{2^{k+1}}(2 - z^{2^k})}.$$

A More Malicious Maitre d' (open questions)

Question

We know

$$8/48 \leq E_n^S/n \leq 9/48$$

for all $n \geq 3$*, and we know*

$$0.17695 \approx \frac{3171}{17920} \leq E_n^S/n \leq \frac{3280}{17920} \approx 0.18304$$

for all $n \geq 5$.

A More Malicious Maitre d' (open questions)

Question

We know

$$8/48 \leq E_n^S/n \leq 9/48$$

for all $n \geq 3$*, and we know*

$$0.17695 \approx \frac{3171}{17920} \leq E_n^S/n \leq \frac{3280}{17920} \approx 0.18304$$

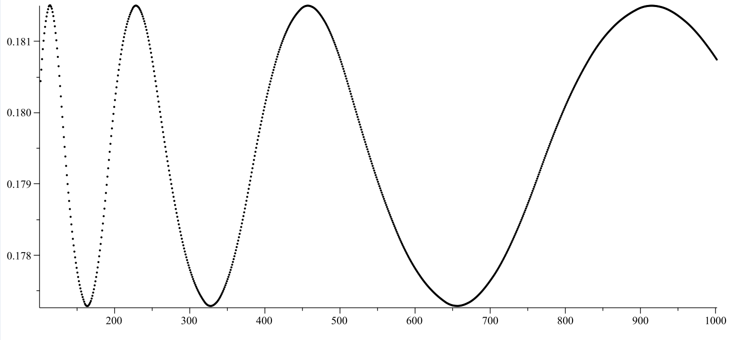
for all $n \geq 5$*.*

Does $\lim_{n\to\infty} E_n^S/n$ *exist?*

A More Malicious Maitre d' (open questions)

Question

$0.1772841670 \leq E_n^S/n \leq 0.1814994700$ *for* $n \geq 100$*:*



Does $\lim_{n\to\infty} E_n^S/n$ *exist?*

A More Malicious Maitre d' (open questions)

Optimality:

A More Malicious Maitre d' (open questions)

Optimality:
Trap setting is not optimal because we proved that napkin shunning was better

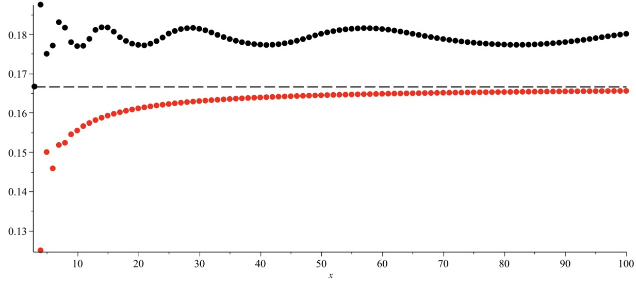
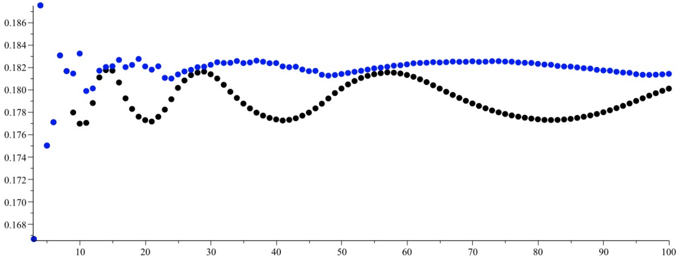


Figure 1. The expected proportion of napkinless diners, when seated using trap setting (in red) and napkin shunning (in black). The dashed line is at height $1/6$.

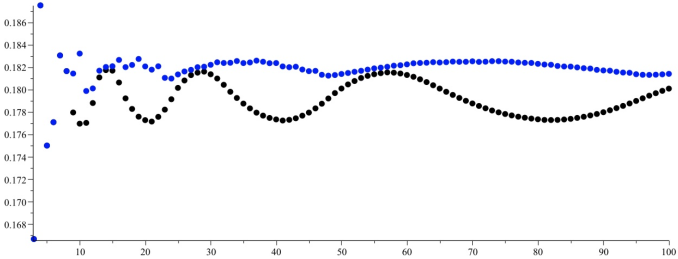
A More Malicious Maitre d' (open questions)

However, napkin shunning is NOT optimal. By adjusting the shunning algorithm very slightly (empty seats mod 3 matter), we find algorithm $\tilde{S}$ such that $E_n^S \leq E_n^{\tilde{S}}$.



A More Malicious Maitre d' (open questions)

However, napkin shunning is NOT optimal. By adjusting the shunning algorithm very slightly (empty seats mod 3 matter), we find algorithm $\tilde{S}$ such that $E_n^S \leq E_n^{\tilde{S}}$.



Question

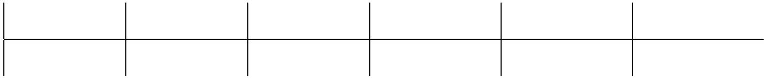
Is there an optimal strategy for the adaptive maitre d'?

Napkin Problems

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$



The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R					
1					

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R				
1	2				

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R $\quad$ L	R				
1 $\quad$ 3	2				

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R L	R	R				
1 3	2	4				

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R L	R L	R			
1 3	2 5	4			

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R L	R L	R	R		
1 3	2 5	4	6		

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R L	R L	R	R	R	
1 3	2 5	4	6	7	

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	L	R	L	R		R	
1	3	2	5	4	8	6		7	

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	L	R	L	R	R	R
1	3	2	5	4	8	6	7	9

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	L	R	L	R	L	R		R
1	3	2	5	4	8	6	10	7		9

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	L	R	L	R	L	R	L	R	L	R
1	3	2	5	4	8	6	10	7	11	9

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	L	R	L	R	L	R	L	R	L	R
1	**12**	3	2	5	4	8	6	10	7	11	9

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	L	R	R	L	R		L	R		L	R		L	R		
1	**12**	3	2	**13**	5	4		8	6		10	7		11	9		

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	L	R	R	L	R	L	R	L	R	L	R	L
1	**12**	3	2	**13**	5	4	8	6	10	7	11	9	14

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	L	R	R	L	R	L	L	R	L	R	L	R	L
1	**12**	3	2	**13**	5	4	**15**	8	6	10	7	11	9	14

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	L	R	R	L	R	L	L	R	R	L	R	L	R	L
1	**12**	3	2	**13**	5	4	**15**	8	6	**16**	10	7	11	9	14

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	L	R	R	L	R	L	L	R	R	L	R	R	L	R	L
1	**12**	3	2	**13**	5	4	**15**	8	6	**16**	10	7	**17**	11	9	14

The most malicious maitre d'

Suppose the Maitre d' ascertains the diner's napkin preference before they reach the table. This is the **Clairvoyant Maitre d'**, and they can realize the full potential of trap setting:

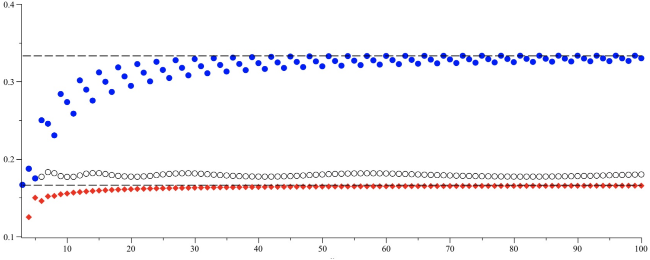
$$\sigma = (1, 1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1)$$

R	R	L	R	R	L	R	L	L	R	R	L	R	R	L	R	L	L
1	**12**	3	2	**13**	5	4	**15**	8	6	**16**	10	7	**17**	11	9	**18**	14

The most malicious maitre d'

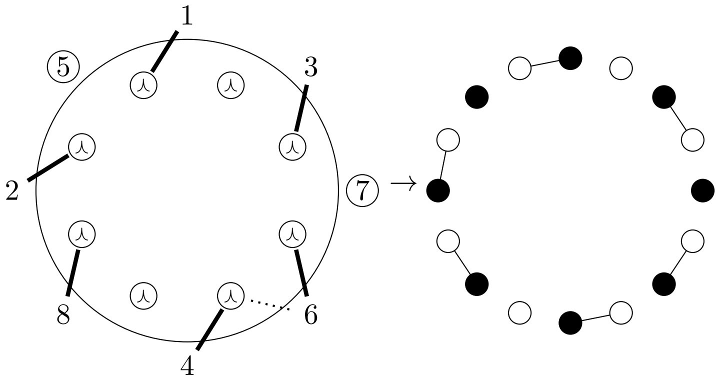
Theorem (Acton-P.-Shirman-Tenner, 2024)

For any sequence of napkin preferences σ, the clairvoyant maitre d' maximizes the number of napkinless diners for σ (despite only learning one preference at a time), and the limiting proportion of napkinless diners is $1/3$.

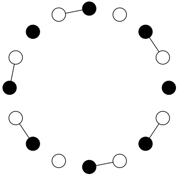


Napkin Problems

Maximal matchings



Maximal matchings



Seating the diners models the formation of a **maximal matching** of a cycle graph; the end result of "random sequential adsorption" in which napkinless diners correspond to "free radicals"

Došlić and Zubac (2016) find about 17.7% free radicals (when every matching is equally likely), while earlier work by Flory (1939) found $e^{-2} \approx 0.135335$

Thanks Jim, and Happy Birthday!

Thanks Jim, and Happy Birthday!

References

- Winkler, *Mathematical Puzzles: a Connoisseur's Collection*, AK Peters/CRC Press (2004).
- A. Claesson and T. K. Petersen, "Conway's Napkin Problem," American Mathematical Monthly, (2007).
- N. Erikson, "The Freshman's approach to Conway's Napkin Problem," American Mathematical Monthly, (2008).
- R. Acton, T. K. Petersen, B. Shirman, and D. Toal, "A more malicious maitre d'," American Mathematical Monthly, (2023).
- R. Acton, T. K. Petersen, B. Shirman, and B. E. Tenner, "The clairvoyant maitre d'," arXiv.org/abs/2401.11680