Descents, Peaks, and P-partitions

A Dissertation

Presented to

The Faculty of the Graduate School of Arts and Sciences

Brandeis University

Department of Mathematics

Ira M. Gessel, Advisor

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

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May, 2006

This dissertation, directed and approved by T. Kyle Petersen's committee, has been accepted and approved by the Faculty of Brandeis University in partial fulfillment of the requirements for the degree of:

DOCTOR OF PHILOSOPHY

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Dedication

To my love.

Acknowledgments

This document being, primarily, a mathematical effort, I must first thank those who influenced this work mathematically. Thanks to my advisor, Ira Gessel. Many of the main theorems regarding descents were his ideas, and I'm sure if he had wanted to take the time, he could have produced proofs of all the theorems I present. I thank him for encouraging me to steal his ideas and for helping me to work out examples with the computer software Maple in order to build the proper conjectures for peaks. Thanks go to Nantel Bergeron for his encouragement and useful suggestions for future work. Though I only talked to him about it after the fact, I also want to acknowledge John Stembridge for his paper on enriched P-partitions. It provided an accessible and suggestive guide for my work with peak algebras.

This dissertation represents the culmination of my studies at Brandeis. My time here has been very rewarding. I would like to thank the department staff, faculty and graduate students for making Brandeis the intimate and welcoming place it is. Special thanks to Janet Ledda for her hard work, support, and conversation.

Lastly, I must thank my wife Rebecca. Though this document is a work of mathematics, and hence a creative endeavor, it also represents an effort of will. Without the example Rebecca provided me, and the encouragement she gave me, I would not have been able to finish this paper as quickly as I did (if at all!). She is my inspiration and the love of my life.

Abstract

Descents, Peaks, and P-partitions

A dissertation presented to the Faculty of the
Graduate School of Arts and Sciences of Brandeis University,
Waltham, Massachusetts

by T. Kyle Petersen

We use a variation on Richard Stanley's P-partitions to study "Eulerian" descent subalgebras of the group algebra of the symmetric group and of the hyperoctahedral group. In each case we give explicit structure polynomials for orthogonal idempotents (including q-analogues in many cases). Much of the study of descents carries over similarly to the study of peaks, where we replace the use of Stanley's P-partitions with John Stembridge's enriched P-partitions.

Preface

The structure of the group algebra of the symmetric group has been studied by many. Work on this group algebra has its roots in the early days of representation theory—an area where properties of the group algebra provide useful tools for understanding. One aspect of this investigation is the study of certain subalgebras of the group algebra, called *descent algebras*.

Louis Solomon is credited with defining the first type of descent algebras [**Sol76**]. For a symmetric group on n letters, Solomon's descent algebra is the subalgebra defined as the linear span of elements u_I, where u_I is the sum of all permutations having descent set I (the set of all i such that $\pi(i) > \pi(i+1)$). In fact, Solomon's notion of descent algebra extends to any finite Coxeter group.

A variation on Solomon's theme arises from taking the span of the elements E_i, where E_i is the sum of all permutations with $i-1$ descents. The number of summands in E_i is an Eulerian number, and so the E_i are called "Eulerian" elements, and the subalgebra they span is called the Eulerian descent algebra. Eulerian descent algebras comprise the initial focus of study in this paper.

Eulerian descent algebras exist in most Coxeter groups, and as was shown in some generality by Paola Cellini [**Cel95a, Cel95b, Cel98**], one can modify the definition of descent and still obtain a subalgebra spanned by sums of permutations with the same number of descents. We call these different sorts of descents *cyclic* descents.

The novelty of this manuscript lies primarily in its approach to the subject. Ira Gessel [**Ges84**] showed that a combinatorial tool called P-partitions, first defined by Richard Stanley [**Sta72, Sta97**], could be used to obtain nice formulas for the structure of the Eulerian descent algebra of the symmetric group. (In fact, he was looking at the internal product on quasisymmetric functions, the descent algebra result being a nice corollary.) Here we take Gessel's approach as a starting point and try to interpret as many descent algebra results as possible in the same way. A slightly modified notion of P-partitions becomes necessary, and several useful group algebra formulas arise.

A more recent development in the study of the group algebra of the symmetric group is the study of *peak algebras*. The basic idea for peak algebras is the same as that for descent algebras except that we group permutations according to peaks: positions i such that $\pi(i-1) < \pi(i) > \pi(i+1)$. John Stembridge [**Ste97**] laid the groundwork for the study of peak algebras, by introducing a tool he called *enriched* P-partitions. Kathryn Nyman [**Nym03**] built on his idea to show that in the group algebra of the symmetric group there is a subalgebra generated by the span of sums of permutations with the same peak set. Later, Marcelo Aguiar, Nantel Bergeron, and Nyman [**ABN04**] showed that another subalgebra could be obtained by grouping permutations according to the number of peaks: an "Eulerian" peak algebra (see also the work of Manfred Schocker [**Sch05**]). Moreover, they modified the definition of peak slightly and found another peak subalgebra. They showed that these peak algebras are homomorphic images of descent algebras of the hyperoctahedral group. We will not exhibit these relationships in this manuscript, though our formulas are certainly suggestive of them.

In the latter part of this work we study the Eulerian peak algebras of the symmetric group, using formulas for enriched P-partitions similar to those found in the case of descents. We conclude by providing a variation on enriched P-partitions for the hyperoctahedral group and examining the consequences, leading to the Eulerian peak algebra of the hyperoctahedral group. The author knows of no prior description of this subalgebra.

Chapter 1 provides an introduction to Stanley's P-partitions and some basic applications to studying descents, including the Eulerian descent algebra and the cyclic descent algebra for the symmetric group (type A Coxeter group). Chapter 2 carries out a similar investigation for Coxeter groups of type B, noting some interesting differences. Many of these results are included in [**Pet05**]. Chapter 3 introduces Stembridge's enriched P-partitions and gives results for the type A peak algebras. Chapter 4 introduces type B enriched P-partitions and the type B peak algebra. The results of chapters 3 and 4 can also be found in [**Pet**].

The remaining pages of this preface give a summary of the main results of this paper. Not all of the results are new, but the P-partition approach is new, and provides a way to see them as part of the same phenomenon.

<u>Definitions</u>

Type A

• A *descent* of a permutation $\pi \in \mathfrak{S}_n$ is any $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. The set of all descents is denoted $\operatorname{Des}(\pi)$, the number of descents is $\operatorname{des}(\pi) = |\operatorname{Des}(\pi)|$.

• A *cyclic descent* is any $i \in [n]$ such that $\pi(i) > \pi(i+1 \mod n)$. The set of all cyclic descents is denoted $\operatorname{cDes}(\pi)$, the number of cyclic descents is $\operatorname{cdes}(\pi) = |\operatorname{cDes}(\pi)|$.

• An *internal peak* is any $i \in \{2, 3, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. The set of all internal peaks is denoted $\operatorname{Pk}(\pi)$, the number of internal peaks is $\operatorname{pk}(\pi) = |\operatorname{Pk}(\pi)|$.

• A *left peak* is any $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. The set of all left peaks is denoted $\operatorname{Pk}^{(\ell)}(\pi)$, the number of left peaks is $\operatorname{pk}^{(\ell)}(\pi) = |\operatorname{Pk}^{(\ell)}(\pi)|$.

Type B

• A *descent* of a signed permutation $\pi \in \mathfrak{B}_n$ is any $i \in [0, n-1] := \{0\} \cup [n-1]$ such that $\pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. The set of all descents is denoted $\operatorname{Des}(\pi)$, the number of descents is $\operatorname{des}(\pi) = |\operatorname{Des}(\pi)|$.

• A *cyclic descent* (or *augmented descent*) is any $i \in [0, n]$ such that $\pi(i) > \pi(i+1 \mod (n+1))$. The set of all cyclic descents is denoted $\operatorname{aDes}(\pi)$, the number of cyclic descents is $\operatorname{ades}(\pi) = |\operatorname{aDes}(\pi)|$.

• A *peak* is any $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where $\pi(0) = 0$. The set of all peaks is denoted $\operatorname{Pk}(\pi)$, the number of peaks is $\operatorname{pk}(\pi) = |\operatorname{Pk}(\pi)|$.

Type A

Eulerian descent algebra

The Eulerian descent algebra is the span of the E_i, where E_i is the sum of all permutations with $i-1$ descents. It is a commutative, n-dimensional subalgebra of the group algebra.

Order polynomial:

$$\Omega_\pi(x) = \binom{x+n-1-\operatorname{des}(\pi)}{n}$$

Structure polynomial:

$$\begin{aligned} \phi(x) &= \sum_{\pi \in \mathfrak{S}_n} \Omega_\pi(x)\pi \\ &= \sum_{i=1}^{n} \Omega_i(x) E_i \\ &= \sum_{i=1}^{n} e_i x^i \end{aligned}$$

Multiplication rule:

$$\phi(x)\phi(y) = \phi(xy)$$

Therefore we have orthogonal idempotents

$$e_i e_j = \begin{cases} e_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\operatorname{Span}\{E_1, E_2, \ldots, E_n\} = \operatorname{Span}\{e_1, e_2, \ldots, e_n\}$$

Cyclic Eulerian descent algebra

The cyclic Eulerian descent algebra is the span of the $E_i^{(c)}$, where $E_i^{(c)}$ is the sum of all permutations with i cyclic descents. It is a commutative, $(n-1)$-dimensional subalgebra of the group algebra.

Structure polynomial:

$$\begin{aligned}\varphi(x) &= \frac{1}{n}\sum_{\pi\in\mathfrak{S}_n}\binom{x+n-1-\mathrm{cdes}(\pi)}{n-1}\pi \\ &= \frac{1}{n}\sum_{i=1}^{n-1}\binom{x+n-1-i}{n-1}E_i^{(c)} \\ &= \sum_{i=1}^{n-1} e_i^{(c)}x^i\end{aligned}$$

Multiplication rule:

$$\varphi(x)\varphi(y) = \varphi(xy)$$

Therefore we have orthogonal idempotents

$$e_i^{(c)}e_j^{(c)} = \begin{cases} e_i^{(c)} & \text{if } i = j \\ 0 & \text{otherwise}\end{cases}$$

$$\mathrm{Span}\{E_1^{(c)}, E_2^{(c)}, \ldots, E_{n-1}^{(c)}\} = \mathrm{Span}\{e_1^{(c)}, e_2^{(c)}, \ldots, e_{n-1}^{(c)}\}$$

Interior peak algebra

The interior peak algebra is the span of the E_i', where E_i' is the sum of all permutations with $i-1$ interior peaks. It is a commutative, $\lfloor (n+1)/2 \rfloor$-dimensional subalgebra of the group algebra.

Enriched order polynomial:

$$\Omega_\pi'(x)$$

with generating function

$$\sum_{k \geq 0} \Omega_\pi'(k) t^k = \frac{1}{2} \frac{(1+t)^{n+1}}{(1-t)^{n+1}} \cdot \left(\frac{4t}{(1+t)^2} \right)^{\mathrm{pk}(\pi)+1}$$

Structure polynomial:

$$\begin{aligned} \rho(x) &= \sum_{\pi \in \mathfrak{S}_n} \Omega_\pi'(x/2) \pi = \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \Omega_i'(x/2) E_i' \\ &= \begin{cases} \displaystyle\sum_{i=1}^{n/2} e_i' x^{2i} & \text{if } n \text{ is even} \\ \displaystyle\sum_{i=1}^{(n+1)/2} e_i' x^{2i-1} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Multiplication rule:

$$\rho(x)\rho(y) = \rho(xy)$$

Therefore we have orthogonal idempotents

$$e_i' e_j' = \begin{cases} e_i' & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Span}\{E_1', E_2', \ldots, E_{\lfloor (n+1)/2 \rfloor}'\} = \mathrm{Span}\{e_1', e_2', \ldots, e_{\lfloor (n+1)/2 \rfloor}'\}$$

Left peak algebra

The left peak algebra is the span of the $E_i^{(\ell)}$, where $E_i^{(\ell)}$ is the sum of all permutations with $i-1$ left peaks. It is a commutative, $(\lfloor n/2 \rfloor + 1)$-dimensional subalgebra of the group algebra.

Left enriched order polynomial:

$$\Omega_\pi^{(\ell)}(x)$$

with generating function

$$\sum_{k\geq 0} \Omega_\pi^{(\ell)}(k)t^k = \frac{(1+t)^n}{(1-t)^{n+1}} \cdot \left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}^{(\ell)}(\pi)}$$

Structure polynomial:

$$\begin{aligned} \rho^{(\ell)}(x) &= \sum_{\pi \in \mathfrak{S}_n} \Omega_\pi^{(\ell)}((x-1)/2)\pi = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} \Omega_i^{(\ell)}((x-1)/2) E_i^{(\ell)} \\ &= \begin{cases} \displaystyle\sum_{i=0}^{n/2} e_i^{(\ell)} x^{2i} & \text{if } n \text{ is even,} \\ \displaystyle\sum_{i=0}^{(n-1)/2} e_i^{(\ell)} x^{2i+1} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Multiplication rule:

$$\rho^{(\ell)}((x-1)/2)\rho^{(\ell)}((y-1)/2) = \rho^{(\ell)}((xy-1)/2)$$

Therefore we have orthogonal idempotents

$$e_i^{(\ell)} e_j^{(\ell)} = \begin{cases} e_i^{(\ell)} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Span}\{E_1^{(\ell)}, E_2^{(\ell)}, \ldots, E_{\lfloor \frac{n}{2} \rfloor + 1}^{(\ell)}\} = \mathrm{Span}\{e_0^{(\ell)}, e_1^{(\ell)}, \ldots, e_{\lfloor \frac{n}{2} \rfloor}^{(\ell)}\}$$

The double peak algebra

The double peak algebra is the multiplicative closure of the interior and left peak algebras. It is a commutative, n-dimensional subalgebra of the group algebra. The interior peak algebra is an ideal within the double peak algebra.

Multiplication rule:

$$\rho(y)\rho^{(\ell)}(x) = \rho^{(\ell)}(x)\rho(y) = \rho(xy)$$

Therefore we have multiplication of idempotents from before as well as

$$e_i^{(\ell)} e_j' = \begin{cases} e_i' & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} &\text{Span}\{E_1', E_2', \ldots, E'_{\lfloor (n+1)/2 \rfloor}, E_1^{(\ell)}, E_2^{(\ell)}, \ldots, E^{(\ell)}_{\lfloor \frac{n}{2} \rfloor + 1}\} \\ &= \text{Span}\{e_1', e_2', \ldots, e'_{\lfloor (n+1)/2 \rfloor}, e_0^{(\ell)}, e_1^{(\ell)}, \ldots, e^{(\ell)}_{\lfloor \frac{n}{2} \rfloor}\} \end{aligned}$$

with the relation

$$\sum_{i=1}^{\lfloor (n+1)/2 \rfloor} E_i' = \sum_{\pi \in \mathfrak{S}_n} \pi = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} E_i^{(\ell)}$$

Type B

Eulerian descent algebra

The Eulerian descent algebra of type B is the span of the E_i, where E_i is the sum of all permutations with $i-1$ descents. It is a commutative, $(n+1)$-dimensional subalgebra of the group algebra.

Order polynomial:

$$\Omega_\pi(x) = \binom{x+n-\operatorname{des}(\pi)}{n}$$

Structure polynomial:

$$\begin{aligned} \phi(x) &= \sum_{\pi\in\mathfrak{B}_n} \Omega_\pi((x-1)/2)\pi \\ &= \sum_{i=1}^{n+1} \Omega_i((x-1)/2)E_i \\ &= \sum_{i=0}^{n} e_i x^i \end{aligned}$$

Multiplication rule:

$$\phi(x)\phi(y) = \phi(xy)$$

Therefore we have orthogonal idempotents

$$e_i e_j = \begin{cases} e_i & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

$$\operatorname{Span}\{E_1, E_2, \ldots, E_{n+1}\} = \operatorname{Span}\{e_0, e_1, \ldots, e_n\}$$

Augmented descent algebra

The cyclic Eulerian descent algebra, or augmented descent algebra, is the span of the $E_i^{(a)}$, where $E_i^{(a)}$ is the sum of all permutations with i augmented descents. It is a commutative, n-dimensional subalgebra of the group algebra.

Order polynomial:

$$\Omega_\pi(x) = \binom{x + n - \text{ades}(\pi)}{n}$$

Structure polynomial:

$$\begin{aligned}\psi(x) &= \sum_{\pi \in \mathfrak{B}_n} \Omega_\pi(x/2)\pi \\ &= \sum_{i=1}^{n} \Omega_i(x/2) E_i^{(a)} \\ &= \sum_{i=1}^{n} e_i^{(a)} x^i\end{aligned}$$

Multiplication rule:

$$\psi(x)\psi(y) = \psi(xy)$$

Therefore we have orthogonal idempotents

$$e_i^{(a)} e_j^{(a)} = \begin{cases} e_i^{(a)} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Span}\{E_1^{(a)}, E_2^{(a)}, \ldots, E_n^{(a)}\} = \text{Span}\{e_1^{(a)}, e_2^{(a)}, \ldots, e_n^{(a)}\}$$

The double descent algebra is the sum of the type B Eulerian descent algebra and the augmented descent algebra. It is a commutative, $2n$-dimensional subalgebra of the group algebra. The augmented descent algebra is an ideal within the double descent algebra.

Multiplication rule:

$$\psi(y)\phi(x) = \phi(x)\psi(y) = \psi(xy)$$

Therefore we have multiplication of idempotents from before as well as

$$e_i e_j^{(a)} = \begin{cases} e_i^{(a)} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Span}\{E_1, E_2, \ldots, E_{n+1}, E_1^{(a)}, E_2^{(a)}, \ldots, E_n^{(a)}\}$$

$$= \mathrm{Span}\{e_0, e_1, \ldots, e_n, e_1^{(a)}, e_2^{(a)}, \ldots, e_n^{(a)}\}$$

with the relation

$$\sum_{i=1}^{n+1} E_i = \sum_{\pi \in \mathfrak{B}_n} \pi = \sum_{i=1}^{n} E_i^{(a)}$$

The Eulerian peak algebra

The Eulerian peak algebra of type B is the span of the $E_i^{\pm}$, where E_i^+ is the sum of all signed permutations π with i peaks and $\pi(1) > 0$, E_i^- is the sum of all signed permutations π with i peaks and $\pi(1) < 0$. It is a commutative, $(n+1)$-dimensional subalgebra of the group algebra.

Enriched order polynomial:

$$\Omega'_\pi(x)$$

with generating function

$$\begin{aligned}\sum_{k\geq 0}\Omega'_\pi(k)t^k &= \frac{(1+t)^n}{(1-t)^{n+1}}\cdot\left(\frac{2t}{1+t}\right)^{\varsigma(\pi)}\cdot\left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}(\pi)} \\ &= \left(\frac{1}{2}\right)^{\varsigma(\pi)}\cdot\frac{(1+t)^{n+\varsigma(\pi)}}{(1-t)^{n+1}}\cdot\left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}(\pi)+\varsigma(\pi)}\end{aligned}$$

where $\varsigma(\pi) = 0$ if $\pi(1) > 0$, $\varsigma(\pi) = 1$ if $\pi(1) < 0$.

Structure polynomial:

$$\begin{aligned}\rho(x) &= \sum_{\pi\in\mathfrak{B}_n}\Omega'_\pi((x-1)/4)\pi \\ &= \sum_{i=0}^{\lfloor n/2\rfloor}\left(\Omega'_{i+}((x-1)/4)E_i^+ + \Omega'_{i-}((x-1)/4)E_i^-\right) \\ &= \sum_{i=0}^{n} e'_i x^i,\end{aligned}$$

Multiplication rule:

$$\rho(x)\rho(y) = \rho(xy)$$

Therefore we have orthogonal idempotents

$$e'_i e'_j = \begin{cases} e'_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Span}\{E_0^{\pm}, E_1^{\pm}, \ldots, E_{\lfloor n/2 \rfloor}^{\pm}\} = \text{Span}\{e'_0, e'_1, \ldots, e'_n\}$$

Contents

List of Figures xxii

Chapter 1. P-partitions and descent algebras of type A 1

1.1. Ordinary P-partitions 2

1.2. Descents of permutations 5

1.3. The Eulerian descent algebra 7

1.4. The P-partition approach 8

1.5. The cyclic descent algebra 17

Chapter 2. Descent algebras of type B 21

2.1. Type B posets, P-partitions of type B 22

2.2. Augmented descents and augmented P-partitions 28

2.3. The augmented descent algebra 31

Chapter 3. Enriched P-partitions and peak algebras of type A 43

3.1. Peaks of permutations 44

3.2. Enriched P-partitions 45

3.3. The interior peak algebra 51

3.4. Left enriched P-partitions 55

3.5. The left peak algebra 60

Chapter 4. The peak algebra of type B 64

4.1. Type B peaks 65

4.2. Enriched P-partitions of type B 67
4.3. The peak algebra of type B 73
Bibliography 77

List of Figures

1.1 Linear extensions of a poset P. 4

1.2 Splitting solutions. 11

1.3 The "zig-zag" poset P_I for $I = \{2, 3\} \subset [5]$. 11

2.1 Two $\mathfrak{B}_3$ posets. 22

2.2 Linear extensions of a $\mathfrak{B}_2$ poset P. 24

2.3 The augmented lexicographic order. 33

3.1 The up-down order for $[l]' \times [k]'$. 53

3.2 The up-down order for $[l]^{(\ell)} \times [k]^{(\ell)}$. 62

4.1 One realization of the total order on $\mathbb{Z}'$. 68

4.2 The up-down order on $\pm[l]' \times \pm[k]'$ with points greater than or equal to $(0, 0)$. 75

CHAPTER 1

P-partitions and descent algebras of type A

In this chapter we will provide the basic definitions and primary examples that will motivate our study of descents. Section 1.1 defines Richard Stanley's P-partitions and outlines their most basic properties. Sections 1.2 and 1.3 give some background on our primary object of study: descents and descent algebras. Section 1.4 is devoted to showing the how P-partitions can be used to study descent algebras in the simplest case, followed by q-analogs.

Section 1.5 examines another "Eulerian" descent algebra for the symmetric group. This one differs from the ordinary one in its definition of a descent. We call this other type of descent a *cyclic* descent. Paola Cellini studied cyclic descents more generally in the papers [**Cel98**], [**Cel95a**], and [**Cel95b**]. She proved the existence of the cyclic descent algebra we study in this chapter and generalized her result to any Coxeter group that has an affine extension. While the existence of the cyclic descent algebra is now a foregone conclusion, using the P-partition approach is novel. In particular, the formulas we derive describe its structure in a new way.

1.1. Ordinary P-partitions

Let P denote a partially ordered set, or *poset*, defined by a set of elements, $E = \{e_1, e_2, \ldots\}$, and a partial order, $<_P$, among the elements. Until otherwise specified we will only consider *labeled* posets with a finite number of elements labeled by the integers $1, 2, \ldots, n$. We will then refer to an element of a poset by its label, so for practical purposes we can assume $E = \{1, 2, \ldots, n\}$, denoted $[n]$. We generally represent a partially ordered set by its Hasse diagram.[1] An example of a partially ordered set is given by $1 >_P 3 <_P 2$; its Hasse diagram is shown in Figure 1.1.

Definition 1.1.1. *Let $X = \{x_1, x_2, \ldots\}$ be a countable, totally ordered set. For a given poset P, a* P-partition *is an order-preserving map $f : [n] \to X$ such that:*

- $f(i) \leq f(j)$ *if* $i <_P j$
- $f(i) < f(j)$ *if* $i <_P j$ *and* $i > j$ *in* $\mathbb{Z}$

We should note that Stanley [**Sta97**] actually refers to this as a *reverse* P-partition. We choose this definition mainly for ease of notation later on. For our purposes we usually think of X as a subset of the positive integers. Let $\mathcal{A}(P)$ denote the set of all P-partitions. When X has a finite number of elements, the number of P-partitions is finite. In this case, if $|X| = k$, define the *order polynomial*, denoted $\Omega_P(k)$, to be the number of P-partitions $f : [n] \to X$. With the example of a poset from before, $1 >_P 3 <_P 2$, the set $\mathcal{A}(P)$ is all functions $f : \{1, 2, 3\} \to X$ such that $f(3) < f(1)$ and $f(3) < f(2)$.

These partitions of partially ordered sets are not the same as integer partitions, but there is a connection. Consider the q-variant of the order polynomial, or q-order

[1] See Stanley's book [**Sta97**] for the formal definition of Hasse diagram and other terms related to partially ordered sets.

polynomial,[2] where $X = \{0, 1, \ldots, k-1\}$:

$$\Omega_P(q;k) = \sum_{f \in \mathcal{A}(P)} \left(\prod_{i=1}^{n} q^{f(i)} \right).$$

If P is the chain $1 <_P 2 <_P \cdots <_P n$ then the q-order polynomial counts certain integer partitions. Specifically, the coefficient of q^r is the number of integer partitions of r with at most n parts of size at most $k-1$. This fact is of some interest, and there are similar results related to P-partitions, many of which we will not discuss here. See chapters 3 and 4 of [**Sta97**] for a broad treatment, including all the facts presented in this section. Our main interest will be in applying P-partitions to the study of permutations.

We will consider any permutation $\pi \in \mathfrak{S}_n$ to be a poset with the total order $\pi(s) <_\pi \pi(s+1)$, $s = 1, 2, \ldots, n-1$. For example, the permutation $\pi = (\pi(1), \pi(2), \pi(3), \pi(4)) = (3, 2, 1, 4)$ has $3 <_\pi 2 <_\pi 1 <_\pi 4$ as a poset. With this convention, the set of all π-partitions is easily characterized. The set $\mathcal{A}(\pi)$ is the set of all functions $f : [n] \to X$ such that

$$f(\pi(1)) \leq f(\pi(2)) \leq \cdots \leq f(\pi(n)),$$

and whenever $\pi(s) > \pi(s+1)$, then $f(\pi(s)) < f(\pi(s+1))$, $s = 1, 2, \ldots, n-1$. The set of all π-partitions where $\pi = (3, 2, 1, 4)$ is all maps f such that

$$f(3) < f(2) < f(1) \leq f(4).$$

For any poset P with n elements, let $\mathcal{L}(P)$ denote the Jordan-Hölder set, the set of all permutations of n which extend P to a total order. This set is sometimes

[2] Properly speaking, this q-analog of the order polynomial is not a polynomial in k. However, we will refer to it as the "q-order polynomial," even if it might be more appropriate to call it the "q-analog of the order polynomial."

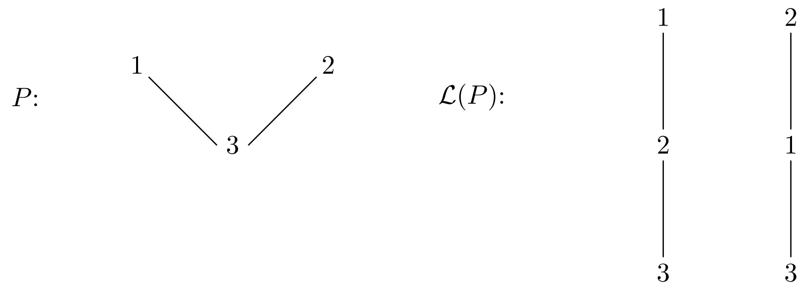


Figure 1.1. Linear extensions of a poset P.

called the set of "linear extensions" of P. For example, let P be the poset defined by $1 >_P 3 <_P 2$. In "linearizing" P we form a total order by retaining all the relations of P but introducing new relations so that every element is comparable to every other. In this case, 1 and 2 are not comparable, so we have exactly two ways of linearizing P: $3 < 2 < 1$ and $3 < 1 < 2$. These correspond to the permutations $(3, 2, 1)$ and $(3, 1, 2)$. Let us make the following observation.

Observation 1.1.1. *A permutation π is in $\mathcal{L}(P)$ if and only if $i <_P j$ implies $\pi^{-1}(i) < \pi^{-1}(j)$.*

In other words, if i is "below" j in the Hasse diagram of the poset P, it had better be below j in any linear extension of the poset. We also now prove what is sometimes called the fundamental theorem of P-partitions.

Theorem 1.1.1 (FTPP). *The set of all P-partitions of a poset P is the disjoint union of the set of π-partitions of all linear extensions π of P:*

$$\mathcal{A}(P) = \coprod_{\pi \in \mathcal{L}(P)} \mathcal{A}(\pi).$$

Proof. The proof follows from induction on the number of incomparable pairs of elements of P. If there are no incomparable pairs, then P has a total order and already represents a permutation. Suppose i and j are incomparable in P. Let P_{ij}

be the poset formed from P by introducing the relation $i < j$. Then it is clear that $\mathcal{A}(P) = \mathcal{A}(P_{ij}) \coprod \mathcal{A}(P_{ji})$. We continue to split these posets (each with strictly fewer incomparable pairs) until we have a collection of totally ordered chains corresponding to distinct linear extensions of P. $\square$

Corollary 1.1.1.

$$\Omega_P(k) = \sum_{\pi \in \mathcal{L}(P)} \Omega_\pi(k).$$

We have shown that the study of P-partitions boils down to the study of π-partitions. With this framework, we are ready to begin our main discussion.

1.2. Descents of permutations

A classical problem in enumerative combinatorics is to count permutations according to the number of descents: the study of Eulerian numbers. We can generalize this notion by considering which permutations have prescribed descents, and how these permutations interact in the group algebra.

For any permutation $\pi \in \mathfrak{S}_n$, we say π has a *descent* in position i if $\pi(i) > \pi(i+1)$. Define the set $\mathrm{Des}(\pi) = \{\, i \mid 1 \leq i \leq n-1, \pi(i) > \pi(i+1)\,\}$ and let $\mathrm{des}(\pi)$ denote the number of elements in $\mathrm{Des}(\pi)$. We call $\mathrm{Des}(\pi)$ the *descent set* of π, and $\mathrm{des}(\pi)$ the *descent number* of π. For example, the permutation $\pi = (1, 4, 3, 2)$ has descent set $\{2, 3\}$ and descent number 2. The number of permutations of n with descent number k is denoted by the Eulerian number $A_{n,k+1}$, and we recall that the Eulerian polynomial is defined as

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)+1} = \sum_{i=1}^{n} A_{n,i} t^i.$$

The Eulerian polynomials can be obtained by using P-partitions. Consider the generating function for the order polynomial (we take the formula from [**Sta97**] without proof):

$$\sum_{k\geq 0}\Omega_P(k)t^k = \frac{\sum_{\pi\in\mathcal{L}(P)} t^{\mathrm{des}(\pi)+1}}{(1-t)^{|P|+1}}$$

where $|P| = n$ is the number of elements in P. Let P be an antichain—that is, a poset with no relations—of n elements. Then $\Omega_P(k) = k^n$ since each of the n elements of P is free to be mapped to any of k places. Furthermore, $\mathcal{L}(P) = \mathfrak{S}_n$, so we get the following equation,

$$\sum_{k\geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

The Eulerian polynomials are interesting and well-studied objects, but we will not devote much more attention to them for now. We conclude this section with a nice formula for computing the order polynomial of a permutation.

Notice that for any permutation π and any positive integer k

$$\binom{k+n-1-\mathrm{des}(\pi)}{n} = \left(\!\!\binom{k-\mathrm{des}(\pi)}{n}\!\!\right),$$

where $\left(\!\binom{a}{b}\!\right)$ denotes the "multi-choose" function—the number of ways to choose b objects from a set of a objects with repetitions. Another interpretation of $\left(\!\binom{a}{b}\!\right)$ is the number of integer solutions to the set of inequalities

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_b \leq a.$$

With this in mind, $\binom{k+n-1-\mathrm{des}(\pi)}{n}$ is the same as the number of solutions to

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq k - \mathrm{des}(\pi).$$

Better still, we can say it is the number of solutions (though not in general the same *set* of solutions) to

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq k,$$

where $i_s < i_{s+1}$ if $s \in \mathrm{Des}(\pi)$. (For example, the number of solutions to $1 \leq i < j < 4$ is the same as the number of solutions to $1 \leq i \leq j - 1 \leq 2$ or the solutions to $1 \leq i \leq j' \leq 2$.) Now if we take $f(\pi(s)) = i_s$ it is clear that

$$\Omega_\pi(k) = \binom{k + n - 1 - \mathrm{des}(\pi)}{n}.$$

1.3. The Eulerian descent algebra

For each subset I of $\{1, 2, \ldots, n-1\}$, let

$$u_I := \sum_{\mathrm{Des}(\pi) = I} \pi,$$

the sum, in the group algebra of $\mathfrak{S}_n$, of all permutations with descent set I. Louis Solomon [**Sol76**] showed that the linear span of the u_I forms a subalgebra of the group algebra, called the *descent algebra*. The concept of descent generalizes naturally, and in fact Solomon defined a descent algebra for any finite Coxeter group.

For now consider the descent algebra of the symmetric group. This descent algebra has is presented in great detail in the work of Adriano Garsia and Christophe Reutenauer [**GR89**]. It has a commutative subalgebra, sometimes called the "Eulerian subalgebra," defined as follows. For $1 \leq i \leq n$, let

$$E_i := \sum_{\mathrm{des}(\pi) = i-1} \pi,$$

the sum of all permutations in $\mathfrak{S}_n$ with descent number $i-1$. Let

$$\phi(x) = \sum_{\pi \in \mathfrak{S}_n} \binom{x+n-1-\mathrm{des}(\pi)}{n} \pi = \sum_{i=1}^{n} \binom{x+n-i}{n} E_i.$$

Then the structure of the Eulerian subalgebra is described by the following:

THEOREM 1.3.1. *As polynomials in x and y with coefficients in the group algebra, we have*

$$\phi(x)\phi(y) = \phi(xy). \tag{1}$$

Define elements e_i in the group algebra (in fact they are in the span of the E_i) by $\phi(x) = \sum_{i=1}^{n} e_i x^i$. By (1) it is clear that the e_i are orthogonal idempotents: $e_i^2 = e_i$ and $e_i e_j = 0$ if $i \neq j$. This shows immediately that the Eulerian descent algebra is commutative of dimension n. Theorem 1.3.1 can be proved in several ways, but we will focus on one that employs Richard Stanley's theory of P-partitions. More specifically, the approach we take follows from work of Ira Gessel—the formula (1) is in fact an easy corollary of Theorem 11 from [**Ges84**]. In section 1.4 we will give a proof of Theorem 1.3.1 that derives from Gessel's work. Throughout the rest of this work we will mimic this method to prove similar formulas related to different notions of descents and peaks, both in the symmetric group algebra and in the hyperoctahedral group algebra.

1.4. The P-partition approach

Before presenting the P-partition proof of Theorem 1.3.1, let us point out that in order to prove that the formula holds as polynomials in x and y, it will suffice to prove that it holds for all pairs of positive integers. It is not hard to prove this fact, and we rely on it throughout this work.

Proof of Theorem 1.3.1. If we write out $\phi(xy) = \phi(x)\phi(y)$ using the definition, we have

$$\sum_{\pi \in \mathfrak{S}_n} \binom{xy + n - 1 - \operatorname{des}(\pi)}{n} \pi = \sum_{\sigma \in \mathfrak{S}_n} \binom{x + n - 1 - \operatorname{des}(\sigma)}{n} \sigma \sum_{\tau \in \mathfrak{S}_n} \binom{y + n - 1 - \operatorname{des}(\tau)}{n} \tau$$

$$= \sum_{\sigma, \tau \in \mathfrak{S}_n} \binom{x + n - 1 - \operatorname{des}(\sigma)}{n} \binom{y + n - 1 - \operatorname{des}(\tau)}{n} \sigma\tau$$

If we equate the coefficients of π we have

$$\binom{xy + n - 1 - \operatorname{des}(\pi)}{n} = \sum_{\sigma\tau = \pi} \binom{x + n - 1 - \operatorname{des}(\sigma)}{n} \binom{y + n - 1 - \operatorname{des}(\tau)}{n}. \tag{2}$$

Clearly, if formula (2) holds for all π, then formula (1) is true. Let us interpret the left hand side of this equation.

Let $x = k$, and $y = l$ be positive integers. Then the left hand side of equation (2) is just the order polynomial $\Omega_\pi(kl)$. To compute this order polynomial we need to count the number of π-partitions $f : [n] \to X$, where X is some totally ordered set with kl elements. But instead of using $[kl]$ as our image set, we will use a different totally ordered set of the same cardinality. Let us count the π-partitions $f : [n] \to [l] \times [k]$. This is equal to the number of solutions to

$$(1, 1) \leq (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_n, j_n) \leq (l, k) \tag{3}$$

where $(i_s, j_s) < (i_{s+1}, j_{s+1})$ if $s \in \operatorname{Des}(\pi)$. Here we take the *lexicographic ordering* on pairs of integers. Specifically, $(i, j) < (i', j')$ if $i < i'$ or else if $i = i'$ and $j < j'$.

To get the result we desire, we will sort the set of all solutions to (3) into distinct cases indexed by subsets $I \subset [n-1]$. The sorting depends on π and proceeds as follows. Let $F = ((i_1, j_1), \ldots, (i_n, j_n))$ be any solution to (3). For any $s = 1, 2, \ldots, n-1$, if $\pi(s) < \pi(s+1)$, then $(i_s, j_s) \leq (i_{s+1}, j_{s+1})$, which falls into one of two mutually exclusive cases:

$$i_s \leq i_{s+1} \quad \text{and} \quad j_s \leq j_{s+1}, \text{ or} \tag{4}$$

$$i_s < i_{s+1} \quad \text{and} \quad j_s > j_{s+1}. \tag{5}$$

If $\pi(s) > \pi(s+1)$, then $(i_s, j_s) < (i_{s+1}, j_{s+1})$, which means either:

$$i_s \leq i_{s+1} \quad \text{and} \quad j_s < j_{s+1}, \text{ or} \tag{6}$$

$$i_s < i_{s+1} \quad \text{and} \quad j_s \geq j_{s+1}, \tag{7}$$

also mutually exclusive. Define $I_F = \{s \in [n-1] \setminus \mathrm{Des}(\pi) \mid j_s > j_{s+1}\} \cup \{s \in \mathrm{Des}(\pi) \mid j_s \geq j_{s+1}\}$. Then I_F is the set of all s such that either (5) or (7) holds for F. Notice that in both cases, $i_s < i_{s+1}$. Now for any $I \subset [n-1]$, let S_I be the set of all solutions F to (3) satisfying $I_F = I$. We have split the solutions of (3) into 2^{n-1} distinct cases indexed by all the different subsets I of $[n-1]$.

Say $\pi = (2, 1, 3)$. Then we want to count the number of solutions to

$$(1, 1) \leq (i_1, j_1) < (i_2, j_2) \leq (i_3, j_3) \leq (l, k),$$

which splits into four distinct cases, indexed by the subsets $I \subset \{1, 2\}$.

We now want to count all the solutions contained in each of these cases and add them up. For a fixed subset I we will use the theory of P-partitions to count the number of solutions for the set of inequalities first for the j_s's and then for the i_s's. Multiplying will give us the number of solutions in S_I; we do the same for the remaining subsets and sum to obtain the final result. For $I = \{\,1\,\}$ in the example above, we would count first the number of integer solutions to $j_1 \geq j_2 \leq j_3$, with $1 \leq j_s \leq k$, and then we multiply this number by the number of solutions to

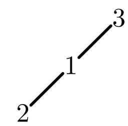
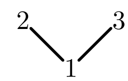
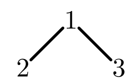
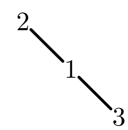
I	i_s	j_s	P_I
$\emptyset$	$i_1 \leq i_2 \leq i_3$	$j_1 < j_2 \leq j_3$	
$\{1\}$	$i_1 < i_2 \leq i_3$	$j_1 \geq j_2 \leq j_3$	
$\{2\}$	$i_1 \leq i_2 < i_3$	$j_1 < j_2 > j_3$	
$\{1,2\}$	$i_1 < i_2 < i_3$	$j_1 \geq j_2 > j_3$	

Figure 1.2. Splitting solutions.

$1 \leq i_1 < i_2 \leq i_3 \leq l$ to obtain the cardinality of $S_{\{1\}}$. We will now carry out the computation in general.

For any particular $I \subset [n-1]$, form the poset P_I of the elements $1, 2, \ldots, n$ by $\pi(s) <_{P_I} \pi(s+1)$ if $s \notin I$, $\pi(s) >_{P_I} \pi(s+1)$ if $s \in I$. We form a "zig-zag" poset of n elements labeled consecutively by $\pi(1), \pi(2), \ldots, \pi(n)$, with downward zigs corresponding to the elements of I. For example, if $I = \{2,3\}$ for $n = 5$, then P_I has $\pi(1) < \pi(2) > \pi(3) > \pi(4) < \pi(5)$.

P_I:

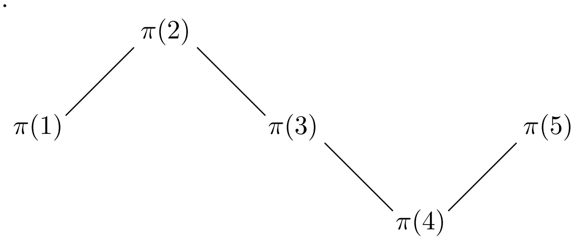


Figure 1.3. The "zig-zag" poset P_I for $I = \{2,3\} \subset [5]$.

For any solution in S_I, let $f : [n] \rightarrow [k]$ be defined by $f(\pi(s)) = j_s$ for $1 \leq s \leq n$. We will show that f is a P_I-partition. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in

$\mathbb{Z}$, then (4) tells us that $f(\pi(s)) = j_s \leq j_{s+1} = f(\pi(s+1))$. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (6) tells us that $f(\pi(s)) = j_s < j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (5) gives us that $f(\pi(s)) = j_s > j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (7) gives us that $f(\pi(s)) = j_s \geq j_{s+1} = f(\pi(s+1))$. In other words, we have verified that f is a P_I-partition. So for any particular solution in S_I, the j_s's can be thought of as a P_I-partition. Conversely, any P_I-partition f gives a solution in S_I since if $j_s = f(\pi(s))$, then $((i_1, j_1), \ldots, (i_n, j_n)) \in S_I$ if and only if $1 \leq i_1 \leq \cdots \leq i_n \leq l$ and $i_s < i_{s+1}$ for all $i \in I$. We can therefore turn our attention to counting P_I-partitions.

Let $\sigma \in \mathcal{L}(P_I)$. Then for any σ-partition f, we get a chain

$$1 \leq f(\sigma(1)) \leq f(\sigma(2)) \leq \cdots \leq f(\sigma(n)) \leq k$$

with $f(\sigma(s)) < f(\sigma(s+1))$ if $s \in \mathrm{Des}(\sigma)$. The number of solutions to this set of inequalities is

$$\Omega_\sigma(k) = \binom{k + n - 1 - \mathrm{des}(\sigma)}{n}.$$

Recall by Observation 1.1.1 that $\sigma^{-1}\pi(s) < \sigma^{-1}\pi(s+1)$ if $\pi(s) <_{P_I} \pi(s+1)$, i.e., if $s \notin I$. If $\pi(s) >_{P_I} \pi(s+1)$ then $\sigma^{-1}\pi(s) > \sigma^{-1}\pi(s+1)$ and $s \in I$. We get that $\mathrm{Des}(\sigma^{-1}\pi) = I$ if and only if $\sigma \in \mathcal{L}(P_I)$. Set $\tau = \sigma^{-1}\pi$. The number of solutions to

$$1 \leq i_1 \leq \cdots \leq i_n \leq l \quad \text{ and } i_s < i_{s+1} \text{ if } s \in \mathrm{Des}(\tau)$$

is given by

$$\Omega_\tau(l) = \binom{l + n - 1 - \mathrm{des}(\tau)}{n}.$$

Now for a given I, the number of solutions in S_I is

$$\sum_{\substack{\sigma \in \mathcal{L}(P_I) \\ \sigma\tau=\pi}} \binom{k+n-1-\mathrm{des}(\sigma)}{n}\binom{l+n-1-\mathrm{des}(\tau)}{n}.$$

Summing over all subsets $I \subset [n-1]$, we can write the number of all solutions to (3) as

$$\sum_{\sigma\tau=\pi} \binom{k+n-1-\mathrm{des}(\sigma)}{n}\binom{l+n-1-\mathrm{des}(\tau)}{n},$$

and so we have derived formula (2). □

Earlier we introduced the q-order polynomial $\Omega_P(q;k)$ as a refinement of the ordinary order polynomial that allowed us to be able to say something about the relationship between integer partitions and P-partitions. We can obtain similar refinements for formulas like (1). In later chapters we will present a q-analog of our formulas whenever possible.

Let $n_q! := (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$ and define the *q-binomial coefficent* $\binom{a}{b}_q$ in the natural way:

$$\binom{a}{b}_q := \frac{a_q!}{b_q!(a-b)_q!}$$

An equivalent way to interpret the q-binomial coefficient is as the coefficient of $x^b y^{a-b}$ in $(x+y)^a$ where x and y "q-commute" via the relation $yx = qxy$. These interpretations are good for some purposes, but we will use a third point of view. We will define the q-multi-choose function $\left(\!\binom{a}{b}\!\right)_q = \binom{a+b-1}{b}_q$ as the following:

$$\sum_{0 \le i_1 \le \cdots \le i_b \le a-1} \left(\prod_{s=1}^{n} q^{i_s}\right).$$

One might recognize this formula as the q-order polynomial $\Omega_P(q;a)$ where P is the chain $1 <_P 2 <_P \cdots <_P b$. Let us build on this notion.

For any permutation $\pi \in \mathfrak{S}_n$, the q-order polynomial may be expressed as

$$\Omega_\pi(q;k) = \sum_{\substack{0 \le i_1 \le \cdots \le i_n \le k-1 \\ s \in \mathrm{Des}(\pi) \Rightarrow i_s < i_{s+1}}} \left(\prod_{s=1}^{n} q^{i_s} \right).$$

When we computed the ordinary order polynomial we only cared about the *number* of solutions, rather than the *set* of solutions to the inequalities

$$0 \le i_1 \le \cdots \le i_n \le k-1,$$

where $i_s < i_{s+1}$ if $s \in \mathrm{Des}(\pi)$. Since we only cared how many solutions there were and not what the solutions were, we could count solutions to a system where all the inequalities were weak. We will still follow the same basic procedure, but as we manipulate our system of inequalities we need to keep track of how we modify the set of solutions. The q-order polynomial will be seen to be simply a power of q (depending on π) times a q-binomial coefficient.

Consider the set of solutions to

$$0 \le i_1 \le \cdots \le i_n \le k-1,$$

where $i_s < i_{s+1}$ if $s \in \mathrm{Des}(\pi)$. We can form a new system of inequalities that has the same number of solutions, but in which every inequality is weak:

$$0 \le i'_1 \le \cdots \le i'_n \le k-1-\mathrm{des}(\pi).$$

There is a bijection between these sets of solutions given by $i'_s = i_s - a(s)$ where $a(s)$ is the number of descents to the left of s. Therefore the q-order polynomial is given by

$$\begin{aligned}
\Omega_\pi(q;k) &= \sum_{\substack{0\le i_1\le\cdots\le i_n\le k-1\\ s\in\mathrm{Des}(\pi)\Rightarrow i_s<i_{s+1}}} \left(\prod_{s=1}^n q^{i_s}\right)\\
&= \sum_{0\le i'_1\le\cdots\le i'_n\le k-1-\mathrm{des}(\pi)} \left(\prod_{s=1}^n q^{i'_s+a(s)}\right)\\
&= q^{\sum_{s=1}^n a(s)}\cdot\left(\sum_{0\le i'_1\le\cdots\le i'_n\le k-1-\mathrm{des}(\pi)} \left(\prod_{s=1}^n q^{i'_s}\right)\right).
\end{aligned}$$

The sum of all $a(s)$ can be expressed as $\sum_{s\in\mathrm{Des}(\pi)}(n-s)$, which is sometimes referred to as the *comajor index*, denoted $\mathrm{comaj}(\pi)$.[3] The rest of the sum is now recognizable as a q-binomial coefficient. In summary, we have

(8)
$$\Omega_\pi(q;k) = q^{\mathrm{comaj}(\pi)}\binom{k+n-1-\mathrm{des}(\pi)}{n}_q$$

Now we will prove a formula for the group algebra expressed in terms of q-binomial coefficients. Define

$$\phi(q;x) = \sum_{\pi\in\mathfrak{S}_n} q^{\mathrm{comaj}(\pi)}\binom{x+n-1-\mathrm{des}(\pi)}{n}_q \pi$$

THEOREM 1.4.1. *As polynomials in x and y (and q) with coefficients in the group algebra we have*

$$\phi(q;x)\phi(q^x;y) = \phi(q;xy).$$

PROOF. The proof will follow nearly identical lines of reasoning as in the ordinary ($q=1$) case. See section 1.4 for more details. Here we sketch the proof with emphasis

[3]The *major* index of a permutation is $\sum_{s\in\mathrm{Des}(\pi)} s$. Indeed, had we adopted Stanley's original definition of a P-partition, we would have gotten $q^{\mathrm{maj}(\pi)}$ rather than comaj above.

on the major differences. Again, we will decompose the coefficient of π:

$$q^{\mathrm{comaj}(\pi)}\binom{kl+n-1-\mathrm{des}(\pi)}{n}_q$$

By (8), we have that the coefficient of π is the order polynomial $\Omega_\pi(q;kl)$ so we will examine the π-partitions $f:[n]\to\{0,1,\ldots,l-1\}\times\{0,1,\ldots,k-1\}$. Notice that we are still mapping into a set with kl elements. As before we impose the lexicographic ordering on this image set. To ensure that we keep the proper powers of q, we think of the order polynomial now as:

$$\Omega_\pi(q;kl)=\sum_{\substack{(0,0)\le(i_1,j_1)\le\cdots\le(i_n,j_n)\le(l-1,k-1)\\ s\in\mathrm{Des}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}}\left(\prod_{s=1}^{n}q^{ki_s+j_s}\right).$$

We give each point (i,j) the weight $ki+j$ so that the weight corresponds to the position of the point in the lexicographic ordering on $\{0,1,\ldots,l-1\}\times\{0,1,\ldots,k-1\}$. We now proceed exactly as in the proof of Theorem 1.3.1.

$$\begin{aligned}\Omega_\pi(q;kl) &= \sum_{\substack{(0,0)\le(i_1,j_1)\le\cdots\le(i_n,j_n)\le(l-1,k-1)\\ s\in\mathrm{Des}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}}\left(\prod_{s=1}^{n}q^{ki_s+j_s}\right)\\ &= \sum_{I\subset[n-1]}\left(\sum_{\substack{0\le i_1\le\cdots\le i_n\le l-1\\ s\in I\Rightarrow i_s<i_s+1}}q^{ki_s}\right)\left(\sum_{\sigma\in\mathcal{L}(P_I)}\Omega_\sigma(q;k)\right)\\ &= \sum_{\sigma\tau=\pi}\Omega_\sigma(q;k)\Omega_\tau(q^k;l),\end{aligned}$$

as desired.

□

1.5. The cyclic descent algebra

We now modify the notion of descent and explore some consequences. For a permutation $\pi \in \mathfrak{S}_n$ we define a *cyclic descent* at position i if $\pi(i) > \pi(i+1)$, or if $i = n$ and $\pi(n) > \pi(1)$. Define $\mathrm{cDes}(\pi)$ to be the set of cyclic descent positions of π, called the *cyclic descent set*. Let the *cyclic descent number*, $\mathrm{cdes}(\pi)$, be the number of cyclic descents. The number of cyclic descents is between 1 and $n-1$. One can observe that a permutation π has the same number of cyclic descents as $\pi\omega^i$ for $i = 0, 1, \ldots, n-1$, where ω is the n-cycle $(1\ 2\ \cdots\ n)$. Define the *cyclic Eulerian polynomial* to be

$$A_n^{(c)}(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{cdes}(\pi)}.$$

We can make the following

Proposition 1.5.1. *The cyclic Eulerian polynomial is expressible in terms of the ordinary Eulerian polynomial:*

$$A_n^{(c)}(t) = nA_{n-1}(t).$$

Proof. We will compare the coefficient of t^d on each side of the equation to show $A_{n,d}^{(c)} = nA_{n-1,d}$. Let $\pi \in \mathfrak{S}_{n-1}$ be any permutation of $[n-1]$ such that $\mathrm{des}(\pi)+1 = d$. Let $\widetilde{\pi} \in \mathfrak{S}_n$ be the permutation defined by $\widetilde{\pi}(i) = \pi(i)$ for $i = 1, 2, \ldots, n-1$ and $\widetilde{\pi}(n) = n$. Then we have $\mathrm{des}(\widetilde{\pi}) = \mathrm{des}(\pi)$ and $\mathrm{cdes}(\widetilde{\pi}) = d$. Let $\langle \widetilde{\pi} \rangle = \{\, \widetilde{\pi}\omega^i \mid i = 0, 1, \ldots, n-1 \,\}$, the set consisting of all n cyclic permutations of $\widetilde{\pi}$. Every permutation in the set has exactly d cyclic descents. There is a bijection between permutations of $\mathfrak{S}_{n-1}$ and such subsets of $\mathfrak{S}_n$ given by the map

$$\pi \mapsto \langle \widetilde{\pi} \rangle,$$

and so the proposition follows. $\square$

In this section we describe the structure of the cyclic descent algebra by way of a formula for the group algebra similar to equation (1). Let

$$E_i^{(c)} := \sum_{\mathrm{cdes}(\pi)=i} \pi,$$

the sum in the group algebra of all those permutations with i cyclic descents. Then we define

$$\varphi(x) = \frac{1}{n}\sum_{\pi\in\mathfrak{S}_n} \binom{x+n-1-\mathrm{cdes}(\pi)}{n-1}\pi = \frac{1}{n}\sum_{i=1}^{n-1}\binom{x+n-1-i}{n-1}E_i^{(c)}.$$

Theorem 1.5.1. *As polynomials in x and y with coefficients in the group algebra of the symmetric group we have*

$$\varphi(x)\varphi(y) = \varphi(xy).$$

Now if we define elements $e_i^{(c)}$ by $\varphi(x) = \sum_{i=1}^{n-1} e_i^{(c)} x^i$, we see that $\left(e_i^{(c)}\right)^2 = e_i^{(c)}$ and $e_i^{(c)}e_j^{(c)} = 0$ if $i \neq j$. Therefore the elements $e_i^{(c)}$ are orthogonal idempotents, showing that the cyclic descent algebra is commutative of dimension $n-1$. Similar to the bijection given in the proof of Proposition 1.5.1, the map

$$\pi \mapsto \sum_{\sigma\in\langle\widetilde{\pi}\rangle} \sigma$$

gives an isomorphism between the ordinary Eulerian descent algebra of $\mathfrak{S}_{n-1}$ and the cyclic descent algebra of $\mathfrak{S}_n$. We will prove Theorem 1.5.1 using formula (2).

Proof of Theorem 1.5.1. If we write out the definition for $\varphi(x)$ in the statement of Theorem 1.5.1, multiply both sides by n^2, and equate coefficients, we have for any $\pi \in \mathfrak{S}_n$,

$$n\binom{xy+n-1-\mathrm{cdes}(\pi)}{n-1} = \sum_{\sigma\tau=\pi}\binom{x+n-1-\mathrm{cdes}(\sigma)}{n-1}\binom{y+n-1-\mathrm{cdes}(\tau)}{n-1}.$$

For some i, we can write $\pi = \nu\omega^i$ where ω is the n-cycle $(\,1\;2\;\cdots\;n\,)$ and $\nu = (n, \nu(2), \ldots, \nu(n))$. Observe that $\mathrm{cdes}(\pi) = \mathrm{cdes}(\nu) = \mathrm{des}(\nu)$. Form the permutation $\widehat{\nu} \in \mathfrak{S}_{n-1}$ by $\widehat{\nu}(s) = \nu(s+1)$, $s = 1, 2, \ldots, n-1$. Then we can see that $\mathrm{cdes}(\pi) = \mathrm{des}(\widehat{\nu}) + 1$. We have

$$\binom{xy+n-1-\mathrm{cdes}(\pi)}{n-1} = \binom{xy+(n-1)-1-\mathrm{des}(\widehat{\nu})}{n-1}.$$

Now we can apply equation (2) to give us

$$\begin{aligned}(9)\quad &\binom{xy+(n-1)-1-\mathrm{des}(\widehat{\nu})}{n-1}\\ &\quad= \sum_{\sigma\tau=\widehat{\nu}}\binom{x+(n-1)-1-\mathrm{des}(\sigma)}{n-1}\cdot\binom{y+(n-1)-1-\mathrm{des}(\tau)}{n-1}.\end{aligned}$$

For each pair of permutations $\sigma, \tau \in \mathfrak{S}_{n-1}$ such that $\sigma\tau = \widehat{\nu}$, define the permutations $\widetilde{\sigma}, \widetilde{\tau} \in \mathfrak{S}_n$ as follows. For $s = 1, 2, \ldots, n-1$, let $\widetilde{\sigma}(s) = \sigma(s)$ and $\widetilde{\tau}(s+1) = \tau(s)$. Let $\widetilde{\sigma}(n) = n$ and $\widetilde{\tau}(1) = n$. Then by construction we have $\widetilde{\sigma}\widetilde{\tau} = \nu$ and a quick observation tells us that $\mathrm{cdes}(\widetilde{\sigma}) = \mathrm{des}(\sigma) + 1$ and $\mathrm{cdes}(\widetilde{\tau}) = \mathrm{des}(\tau) + 1$. On the other hand, from any pair of permutations $\widetilde{\sigma}, \widetilde{\tau} \in \mathfrak{S}_n$ such that $\widetilde{\sigma}\widetilde{\tau} = \nu$, $\widetilde{\sigma}(n) = n$, we can construct a pair of permutations $\sigma, \tau \in \mathfrak{S}_{n-1}$ such that $\sigma\tau = \widehat{\nu}$ by reversing the process. Observe now that if $\widetilde{\sigma}(n) = n$ and $\widetilde{\sigma}\widetilde{\tau} = \nu$, then $\widetilde{\tau}(1) = n$. Therefore we have that (9) is equal to

$$\begin{aligned}
&\sum_{\substack{\widetilde{\sigma}\widetilde{\tau}=\nu \\ \widetilde{\sigma}(n)=n}} \binom{x+n-1-\mathrm{cdes}(\widetilde{\sigma})}{n-1}\binom{y+n-1-\mathrm{cdes}(\widetilde{\tau})}{n-1} \\
=&\sum_{\substack{\widetilde{\sigma}(\widetilde{\tau}\omega^i)=\pi \\ \widetilde{\sigma}(n)=n}} \binom{x+n-1-\mathrm{cdes}(\widetilde{\sigma})}{n-1}\binom{y+n-1-\mathrm{cdes}(\widetilde{\tau}\omega^i)}{n-1} \\
=&\sum_{\substack{(\widetilde{\sigma}\omega^{n-j})(\omega^j\widetilde{\tau}\omega^i)=\pi \\ \widetilde{\sigma}(n)=n}} \binom{x+n-1-\mathrm{cdes}(\widetilde{\sigma}\omega^{n-j})}{n-1}\binom{y+n-1-\mathrm{cdes}(\omega^j\widetilde{\tau}\omega^i)}{n-1} \\
=&\sum_{\substack{\sigma\tau=\pi \\ \sigma(j)=n}} \binom{x+n-1-\mathrm{cdes}(\sigma)}{n-1}\binom{y+n-1-\mathrm{cdes}(\tau)}{n-1},
\end{aligned}$$

where the last two formulas hold for any $j \in [n]$. Notice that the number of cyclic descents of $\tau = \omega^j\widetilde{\tau}\omega^i$ is still the same as the number of cyclic descents of $\widetilde{\tau}$. We take the sum over all $j = 1, \ldots, n$, yielding

$$n\binom{xy+n-1-\mathrm{cdes}(\pi)}{n-1} = \sum_{\sigma\tau=\pi} \binom{x+n-1-\mathrm{cdes}(\sigma)}{n-1}\binom{y+n-1-\mathrm{cdes}(\tau)}{n-1}$$

as desired. □

CHAPTER 2

Descent algebras of type B

In this chapter we move from the symmetric group to the hyperoctahedral group, the group of signed permutations. In Section 2.1 we will present some of the definitions and results for the hyperoctahedral group that mirror the results for the symmetric group presented in Chapter 1. Many of these results are due to Chak-On Chow [**Cho01**].

In what remains of the chapter we introduce type B cyclic descents, or augmented descents. While the basic idea for cyclic descents is the same in the hyperoctahedral group as in the symmetric group, the algebraic structure related to type B cyclic descents seems to be richer. For example, as will be seen in Chapter 3, they are related to peak algebras of the symmetric group. We will prove a group algebra formula that gives the structure for the type B cyclic descent algebra, as well as a formula that combines both ordinary and cyclic descents of type B.

As with type A, the existence of the type B cyclic descent algebra is proven by Cellini, [**Cel95a**], [**Cel95b**]. The algebraic structure implied by our Theorem 2.3.2 is given in her paper [**Cel98**], as well as the paper [**ABN04**] of Marcelo Aguiar, Nantel Bergeron, and Kathryn Nyman. Interesting variations of Theorem 2.3.1 can be found in work of Jason Fulman [**Ful01**]. His techniques employ card shuffling and seem very interesting. Also noteworthy is work on the descent algebra of the hyperoctahedral group carried out in detail by Francois Bergeron and Nantel Bergeron, [**Ber92**], [**BB92a**], [**BB92b**]. In particular, [**BB92b**] points to some possible applications of the formulas derived in our main theorems.

2.1. Type B posets, P-partitions of type B

Let $\pm[n]$ denote the set $\{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$. Let $\mathfrak{B}_n$ denote the hyperoctahedral group, the group of all bijections $\pi : \pm[n] \to \pm[n]$ with the property that $\pi(-s) = -\pi(s)$, for $s = 0, 1, \ldots, n$ (note that $\pi(0) = 0$ as a consequence). Since the elements of the hyperoctahedral group are uniquely determined by where they map $1, 2, \ldots, n$, we can think of them as signed permutations. For a signed permutation $\pi \in \mathfrak{B}_n$ we will write $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$.

The definition of signed permutations necessitates the definition of a new type of partially ordered set. What is desired is a poset whose linear extensions are given by signed permutations. The following definitions are taken from Chak-On Chow's Ph.D. thesis [**Cho01**], though they derive from earlier work by Victor Reiner [**Rei93**]. In [**Rei92**], Reiner extends the concept of poset and P-partition to any finite Coxeter group.

Definition 2.1.1. *A* $\mathfrak{B}_n$ poset *is a poset P whose elements are* $0, \pm 1, \pm 2, \ldots, \pm n$ *such that if* $i <_P j$ *then* $-j <_P -i$.

Note that if we are given a poset with $n+1$ elements labeled by $0, a_1, \ldots, a_n$ where $a_i = i$ or $-i$, then we can extend it to a $\mathfrak{B}_n$ poset of $2n+1$ elements.

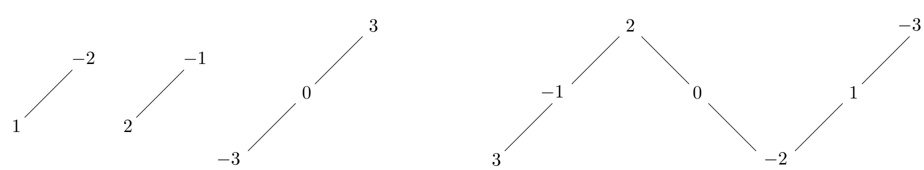


Figure 2.1. Two $\mathfrak{B}_3$ posets.

Let $X = \{x_0, x_1, x_2, \ldots\}$ be a countable, totally ordered set with total order

$$x_0 < x_1 < x_2 < \cdots .$$

Then define $\pm X$ to be the set $\{\ldots, -x_1, x_0, x_1, \ldots\}$ with total order

$$\cdots < -x_2 < -x_1 < x_0 < x_1 < x_2 < \cdots .$$

Definition 2.1.2. *For any $\mathfrak{B}_n$ poset P, a P-*partition of type B *is an order preserving map $f : \pm[n] \to \pm X$ such that:*

- *$f(i) \leq f(j)$ if $i <_P j$*
- *$f(i) < f(j)$ if $i <_P j$ and $i > j$ in $\mathbb{Z}$*
- *$f(-i) = -f(i)$*

Note that type B P-partitions differ from ordinary P-partitions only in the addition of the property $f(-i) = -f(i)$. Let $\mathcal{A}(P)$ denote the set of all type B P-partitions. We usually think of X as a subset of the nonnegative integers, and when X has finite cardinality $k+1$, then the *type B order polynomial*, denoted $\Omega_P(k)$, is the number of P-partitions $f : \pm[n] \to \pm X$. We use the same notation as in the ordinary case, but the context will make clear which definition we are using.

As before, we can think of any signed permutation $\pi \in \mathfrak{B}_n$ as a $\mathfrak{B}_n$ poset with the total order $\pi(s) <_\pi \pi(s+1)$, $0 \leq s \leq n-1$. For example, the signed permutation $(-2, 1)$ has $-1 <_\pi 2 <_\pi 0 <_\pi -2 <_\pi 1$ as a poset. Note that $\mathcal{A}(\pi)$ is the set of all functions $f : \pm[n] \to \pm X$ such that for $0 \leq s \leq n$, $f(-s) = -f(s)$ and

$$x_0 = f(\pi(0)) \leq f(\pi(1)) \leq f(\pi(2)) \leq \cdots \leq f(\pi(n)),$$

where if $\pi(s) > \pi(s+1)$, then $f(\pi(s)) < f(\pi(s+1))$, $s = 0, 1, \ldots, n-1$. The type B π-partitions where $\pi = (-2, 1)$ are all maps f such that $x_0 < f(-2) \leq f(1)$.

For a $\mathfrak{B}_n$ poset P, let $\mathcal{L}(P)$ denote the set of all signed permutations of n extending P to a total order. For example let P be the $\mathfrak{B}_2$ poset defined by $0 > 1 < -2$ (and hence $2 < -1 > 0$ as well). Then linearizing gives $2 < 1 < 0 < -1 < -2$,

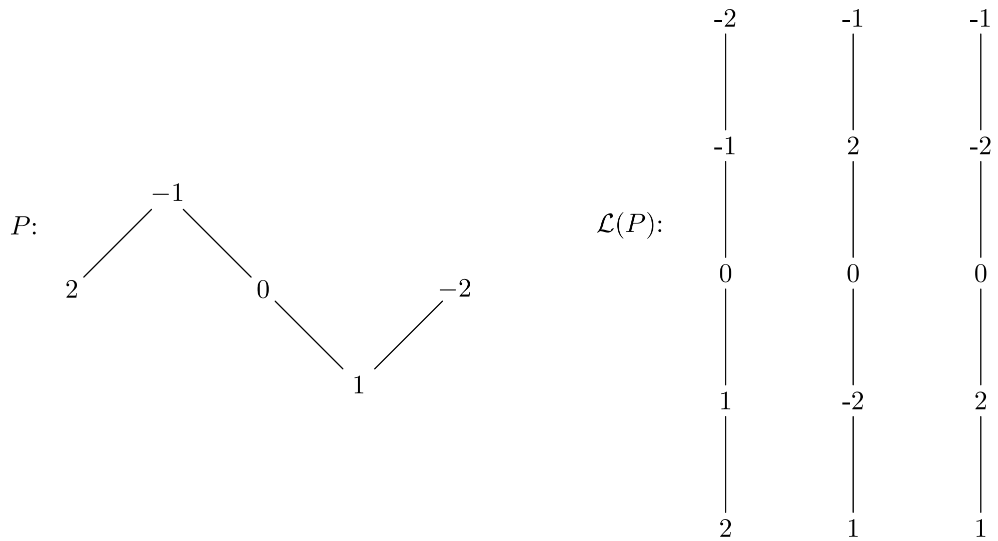


Figure 2.2. Linear extensions of a $\mathfrak{B}_2$ poset P.

$1 < -2 < 0 < 2 < -1$, or $1 < 2 < 0 < -2 < -1$, corresponding to signed permutations $(-1,-2)$, $(2,1)$, and $(-2,-1)$. Proofs of some of the basic facts of type B P-partitions are identical to the proofs of analogous statements for ordinary P-partitions and may be omitted.

Observation 2.1.1. *A signed permutation π is in $\mathcal{L}(P)$ if and only if $i <_P j$ implies $\pi^{-1}(i) < \pi^{-1}(j)$.*

We have a fundamental theorem for P-partitions of type B.

Theorem 2.1.1 (FTPPB). *The set of all type B P-partitions of a $\mathfrak{B}_n$ poset P is the disjoint union of the set of π-partitions of all linear extensions π of P:*

$$\mathcal{A}(P) = \coprod_{\pi \in \mathcal{L}(P)} \mathcal{A}(\pi).$$

Corollary 2.1.1.

$$\Omega_P(k) = \sum_{\pi \in \mathcal{L}(P)} \Omega_\pi(k).$$

In moving from the symmetric group to the hyperoctahedral group, we vary the definition of descent slightly. Define the *descent set* $\mathrm{Des}(\pi)$ of a signed permutation $\pi \in \mathfrak{B}_n$ to be the set of all $i \in \{0, 1, 2, \ldots, n-1\}$ such that $\pi(i) > \pi(i+1)$, where we always take $\pi(0) = 0$. The *descent number* of π is again denoted $\mathrm{des}(\pi)$ and is equal to the cardinality of $\mathrm{Des}(\pi)$. As a simple example, the signed permutation $(-2, 3, 1)$ has descent set $\{0, 2\}$ and descent number 2. For any permutation $\pi \in \mathfrak{B}_n$, it is easy to compute the order polynomial $\Omega_\pi(k)$. Any π-partition $f : \pm[n] \to \pm[k]$ is determined by where we map $\pi(1), \pi(2), \ldots, \pi(n)$. To count them we can look at the number of integer solutions to the set of inequalities

$$0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq k,$$

where $\mathrm{des}(\pi)$ of the inequalities are strict. This is the same as the number of solutions to

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq k + 1 - \mathrm{des}(\pi),$$

which we know to be $\left(\!\binom{k+1-\mathrm{des}(\pi)}{n}\!\right)$. We have

$$\Omega_\pi(k) = \binom{k + n - \mathrm{des}(\pi)}{n}.$$

There is an Eulerian descent algebra of the hyperoctahedral group. For $1 \leq i \leq n+1$ let E_i be the sum of all permutations in $\mathfrak{B}_n$ with $i - 1$ descents. Define

$$\phi(x) = \sum_{\pi \in \mathfrak{B}_n} \binom{(x-1)/2 + n - \mathrm{des}(\pi)}{n} \pi = \sum_{i=1}^{n+1} \binom{(x-1)/2 + n + 1 - i}{n} E_i.$$

We can prove the following using an argument nearly identical to that for Theorem 1.3.1. See [**Cho01**], where the following formula can be found as Proposition 2.4.2, a specialization of one of his theorems for type B quasisymmetric functions.

THEOREM 2.1.2. *As polynomials in x and y with coefficients in the group algebra of the hyperoctahedral group,*

$$\phi(x)\phi(y) = \phi(xy).$$

We therefore have orthogonal idempotents e_i defined by $\phi(x) = \sum_{i=0}^{n} e_i x^i$, telling us that the Eulerian descent algebra of the hyperoctahedral group is commutative of dimension $n+1$.

PROOF. The main difference between this proof and the proof in the case of the symmetric group (Theorem 1.3.1) is that we want to count π-partitions $f : \pm[n] \to \pm[l] \times \pm[k]$. We notice that because of the property $f(-s) = -f(s)$ of type B P-partitions this is just like counting all $f : [n] \to \{0, 1, \ldots, l\} \times \{-k, \ldots, -1, 0, 1, \ldots, k\}$ where for $s = 1, 2, \ldots, n$, $f(\pi(s)) = (i_s, j_s)$ with $(0, 0) \leq (i_s, j_s) \leq (l, k)$ in the lexicographic order. The image set of f then has $2kl + k + l + 1$ elements, and so for each π we can count all these maps with $\Omega_\pi(2kl + k + l) = \binom{2kl+k+l-\mathrm{des}(\pi)}{n}$. We use similar arguments to those of Theorem 1.3.1 for splitting the lexicographic solutions to

$$(0, 0) \leq (i_1, j_1) \leq \cdots \leq (i_n, j_n) \leq (l, k),$$

where $(i_s, j_s) < (i_{s+1}, j_{s+1})$ if $s \in \mathrm{Des}(\pi)$. Once we have properly grouped the set of solutions it is not much more work to obtain the crucial formula:

$$\Omega_\pi(2kl + k + l) = \sum_{\sigma\tau=\pi} \Omega_\sigma(k)\Omega_\tau(l).$$

$\square$

There is also a q-analog of Theorem 2.1.2. We can define the q-order polynomial for a signed permutation π as

$$\Omega_\pi(q;k) = \sum_{\substack{0\le i_1\le\cdots\le i_n\le k\\ s\in\mathrm{Des}(\pi)\Rightarrow i_s<i_{s+1}}} \left(\prod_{s=1}^{n} q^{i_s}\right) = q^{\mathrm{comaj}(\pi)}\binom{k+n-\mathrm{des}(\pi)}{n}_q.$$

Let

$$\phi(q;x) = \sum_{\pi\in\mathfrak{B}_n} q^{\mathrm{comaj}(\pi)}\binom{(x-1)/2+n-\mathrm{des}(\pi)}{n}_q \pi.$$

Theorem 2.1.3. *The following relation holds as polynomials in x and y (and q) with coefficients in the group algebra of the hyperoctahedral group:*

$$\phi(q;x)\phi(q^x;y) = \phi(q;xy).$$

Proof. We will omit most of the details, but the crucial step is to keep the proper exponent on q. We convert each point (i,j) to the weight $(2k+1)i+j$ so that the weight corresponds to the position of the point in the lexicographic order on the set $\{0,1,\ldots,l\}\times\{-k,\ldots,-1,0,1,\ldots,k\}$. The proof is outlined in two steps below. For any π and any pair of positive integers k,l,

$$\begin{aligned}\Omega_\pi(q;2kl+k+l) &= \sum_{\substack{(0,0)\le(i_1,j_1)\le\cdots\le(i_n,j_n)\le(l,k)\\ s\in\mathrm{Des}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}} \left(\prod_{s=1}^{n} q^{(2k+1)i_s+j_s}\right)\\ &= \sum_{\sigma\tau=\pi}\Omega_\sigma(q;k)\Omega_\tau(q^{2k+1};l).\end{aligned}$$

□

2.2. Augmented descents and augmented P-partitions

For a permutation $\pi \in \mathfrak{B}_n$, position i is an *augmented descent* (or *type B cyclic descent*[1]) if $\pi(i) > \pi(i+1)$ or if $i = n$ and $\pi(n) > 0 = \pi(0)$. If we consider that signed permutations always begin with 0, then augmented descents are the natural choice for a type B version of cyclic descents.[2] The set of all augmented descent positions is denoted $\mathrm{aDes}(\pi)$, the *augmented descent set.* It is the ordinary descent set of π along with n if $\pi(n) > 0$. The *augmented descent number*, $\mathrm{ades}(\pi)$, is the number of augmented descents. With these definitions, $(-2, 3, 1)$ has augmented descent set $\{0, 2, 3\}$ and augmented descent number 3. Note that while $\mathrm{aDes}(\pi) \subset \{0, 1, \ldots, n\}$, $\mathrm{aDes}(\pi) \neq \emptyset$, and $\mathrm{aDes}(\pi) \neq \{0, 1, \ldots, n\}$. Denote the number of signed permutations with k augmented descents by $A^{(a)}_{n,k}$ and define the *augmented Eulerian polynomial* as

$$A^{(a)}_n(t) = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{ades}(\pi)} = \sum_{i=1}^{n} A^{(a)}_{n,i} t^i.$$

After we introduce a new type of P-partition, we will prove the following observation.

Proposition 2.2.1. *The number of signed permutations with $i+1$ augmented descents is 2^n times the number of unsigned permutations with i descents, $0 \leq i \leq n-1$:*

$$A^{(a)}_n(t) = 2^n A_n(t).$$

We now give the definition of an augmented P-partition and basic tools related to their study. Let $X = \{x_0, x_1, \ldots, x_\infty\}$ be a countable, totally ordered set with a

[1]The term cyclic descent seems appropriate for this definition, but Gessel has also used the term augmented. We will also adopt this term to avoid confusion with type A cyclic descents.

[2]Most generally, Cellini [**Cel95a**] uses the term "descent in zero" to represent this concept for any Weyl group.

maximal element x_∞. The total ordering on X is given by

$$x_0 < x_1 < x_2 < \cdots < x_\infty.$$

Define $\pm X$ to be $\{-x_\infty, \ldots, -x_1, x_0, x_1, \ldots, x_\infty\}$ with the total order

$$-x_\infty < \cdots < -x_1 < x_0 < x_1 < \cdots < x_\infty.$$

Definition 2.2.1. *For any* $\mathfrak{B}_n$ *poset* P*, an* augmented P-partition *is a function* $f : \pm[n] \to \pm X$ *such that:*

- $f(i) \leq f(j)$ *if* $i <_P j$
- $f(i) < f(j)$ *if* $i <_P j$ *and* $i > j$ *in* $\mathbb{Z}$
- $f(-i) = -f(i)$
- *if* $0 < i$ *in* $\mathbb{Z}$*, then* $f(i) < x_\infty$*.*

Note that augmented P-partitions differ from P-partitions of type B only in the addition of maximal and minimal elements of the image set $\pm X$ and in the last criterion. Let $\mathcal{A}^{(a)}(P)$ denote the set of all augmented P-partitions. When X has finite cardinality $k+1$ (and so $\pm X$ has cardinality $2k+1$), then the *augmented order polynomial*, denoted $\Omega_P^{(a)}(k)$, is the number of augmented P-partitions.

For any signed permutation $\pi \in \mathfrak{B}_n$, note that $\mathcal{A}^{(a)}(\pi)$ is the set of all functions $f : \pm[n] \to \pm X$ such that for $0 \leq s \leq n$, $f(-s) = -f(s)$ and

$$x_0 = f(\pi(0)) \leq f(\pi(1)) \leq f(\pi(2)) \leq \cdots \leq f(\pi(n)) \leq x_\infty.$$

Whenever $\pi(s) > \pi(s+1)$, then $f(\pi(s)) < f(\pi(s+1))$, $s = 0, 1, \ldots, n-1$. In addition, we have $f(\pi(n)) < x_\infty$ whenever $\pi(n) > 0$. The set of all augmented π-partitions where $\pi = (-2, 1)$ is all maps f such that $x_0 < f(-2) \leq f(1) < x_\infty$.

The proof of the fundamental theorem of augmented P-partitions is similar that of ordinary or type B P-partitions.

THEOREM 2.2.1 (FTAPP). *The set of all augmented P-partitions of a $\mathfrak{B}_n$ poset P is the disjoint union of the set of π-partitions of all linear extensions π of P:*

$$\mathcal{A}^{(a)}(P) = \coprod_{\pi \in \mathcal{L}(P)} \mathcal{A}^{(a)}(\pi).$$

COROLLARY 2.2.1.

$$\Omega_P^{(a)}(k) = \sum_{\pi \in \mathcal{L}(P)} \Omega_\pi^{(a)}(k).$$

It is fairly easy to compute the augmented order polynomial for a signed permutation. The number of augmented π-permutations $f : \pm[n] \to \pm[k]$ is equal to the number of integer solutions to the set of inequalities

$$0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq k,$$

where $\mathrm{ades}(\pi)$ of the inequalities are strict. This is the same as the number of solutions to

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq k + 1 - \mathrm{ades}(\pi),$$

which we know to be $\left(\!\binom{k+1-\mathrm{ades}(\pi)}{n}\!\right)$. In other words,

$$\Omega_\pi^{(a)}(k) = \binom{k + n - \mathrm{ades}(\pi)}{n}.$$

We conclude this section with the proof of Proposition 2.2.1.

Proof of Proposition 2.2.1. Recall from Section 1.2 that we have the following formula for the ordinary Eulerian polynomials:

$$\sum_{k\geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Now let P be the poset given by an antichain of $2n+1$ elements labeled $0, \pm 1$, $\pm 2, \ldots, \pm n$. The number of augmented P-partitions $f : \pm[n] \to \pm[k]$ is determined by the choices for $f(1), f(2), \ldots, f(n)$, which can take any of the $2k$ different values in the set $\{-k, -k+1, \ldots, k-1\}$. Therefore $\Omega_P^{(a)}(k) = (2k)^n$. For $\mathfrak{B}_n$ posets P, it is not difficult to show that we have the identity

$$\sum_{k\geq 0} \Omega_P^{(a)}(k) t^k = \frac{\sum_{\pi \in \mathcal{L}(P)} t^{\mathrm{ades}(\pi)}}{(1-t)^{n+1}},$$

similarly to the ordinary case. For our antichain we have $\mathcal{L}(P) = \mathfrak{B}_n$, and therefore

$$\frac{A_n^{(a)}(t)}{(1-t)^{n+1}} = \sum_{k\geq 0} (2k)^n t^k = 2^n \sum_{k\geq 0} k^n t^k = \frac{2^n A_n(t)}{(1-t)^{n+1}},$$

so the theorem is proved. □

2.3. The augmented descent algebra

The theorems that we prove in this section establish the existence of the augmented descent algebra. We will also show that the augmented descent algebra and the Eulerian descent algebra are related in a nice way, and actually can be taken together to form another subalgebra of the group algebra. We will state both theorems before the proof of either.

Let

$$E_i^{(a)} := \sum_{\mathrm{ades}(\pi)=i} \pi,$$

the sum in the group algebra of all permutations with i augmented descents. Define

$$\psi(x) = \sum_{\pi \in \mathfrak{B}_n} \binom{x/2 + n - \operatorname{ades}(\pi)}{n} \pi = \sum_{i=1}^{n} \binom{x/2 + n - i}{n} E_i^{(a)}.$$

THEOREM 2.3.1. *As polynomials in x and y with coefficients in the group algebra of the hyperoctahedral group we have*

$$\psi(x)\psi(y) = \psi(xy).$$

We get orthogonal idempotents $e_i^{(a)}$ defined by $\psi(x) = \sum_{i=1}^{n} e_i^{(a)} x^i$.

THEOREM 2.3.2. *As polynomials in x and y with coefficients in the group algebra of the hyperoctahedral group we have*

$$\phi(y)\psi(x) = \psi(x)\phi(y) = \psi(xy).$$

Theorem 2.3.2 implies that $e_i^{(a)} e_i = e_i e_i^{(a)} = e_i^{(a)}$ and that $e_i^{(a)} e_j = 0$ if $i \neq j$. We can take the span of the e_i and the $e_i^{(a)}$ to form a subalgebra of the group algebra of dimension $2n$ in which the augmented Eulerian descent algebra is an ideal. This relationship shows up again in the case of peak algebras of type A. See Chapter 3 as well as the paper of Aguiar, Bergeron, and Nyman [**ABN04**] for more.

The dimension of this subalgebra is $2n$ and not $2n+1$ since the only dependency relation between the sets $\{E_i\}$ and $\{E_i^{(a)}\}$ is $\sum_{i=1}^{n+1} E_i = \sum_{\pi \in \mathfrak{S}_n} \pi = \sum_{i=1}^{n} E_i^{(a)}$. Alternatively, for $i = 1, 2, \ldots, n$, let F_i^- be the sum of all permutations with i augmented descents and $\pi(n) < 0$, let F_i^+ be the sum of all permutations with i augmented descents and $\pi(n) > 0$. Then

$$E_1 = F_1^+,$$

$$E_{n+1} = F_n^-,$$

$$E_i = F_{i-1}^- + F_i^+ \text{ for } 1 < i < n+1,$$
$$\text{and } E_i^{(a)} = F_i^- + F_i^+ \text{ for } 1 \leq i \leq n.$$

Then we see that the F_i^+, F_i^-, which are obviously linearly independent, span the E_i, $E_i^{(a)}$.

The proofs of Theorems 2.3.1 and 2.3.2 will follow the same basic structure as the proof of Theorem 1.3.1, but with some important changes in detail. In both cases we will rely on a slightly different total ordering on the integer points (i, j), where i and j are bounded both above and below. Let us now define the *augmented lexicographic order*.

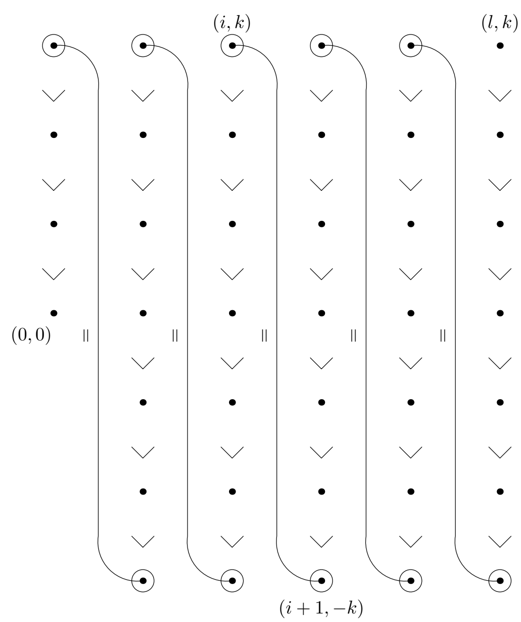


FIGURE 2.3. The augmented lexicographic order.

Consider all points (i, j) with $0 \leq i \leq l$, $-k \leq j \leq k$. We have $(i, j) < (i', j')$ if $i < i'$ or else if $i = i'$ and $j < j'$ as before, except in the important special case that follows. We now say $(i, j) = (i', j')$ in one of two situations. Either

$$i = i' \quad \text{and} \quad j = j'$$

or

$$i+1=i' \quad \text{and} \quad j=k=-j'.$$

If we have $0 \le i \le l$, $-2 \le j \le 2$, then in the augmented lexicographic order, the first few points $(0,0) \le (i,j) \le (l,2)$ are:

$$\begin{aligned}(0,0) < (0,1) < (0,2) = (1,-2) < (1,-1) < (1,0) < (1,1) < (1,2)& \\ = (2,-2) < (2,-1) < (2,0) \cdots&\end{aligned}$$

To be more precise, what we have done is to form equivalence classes of points and to introduce a total order on these equivalence classes. If $j \neq \pm k$, then the class represented by (i,j) is just the point itself. Otherwise, the classes consist of the two points (i,k) and $(i+1,-k)$. When we write $(i,j)=(i',j')$, what we mean is that the two points are in the same equivalence class. In the proofs that follow, it will be important to remember the original points as well as the equivalence classes to which they belong. This special ordering will be very apparent in deriving the q-analogs of Theorem 2.3.1 and Theorem 2.3.2. We will now prove the theorems.

Proof of Theorem 2.3.1. As before, we equate coefficients and prove that a simpler formula,

$$\binom{2kl+n-\mathrm{ades}(\pi)}{n} = \sum_{\sigma\tau=\pi} \binom{k+n-\mathrm{ades}(\sigma)}{n}\binom{l+n-\mathrm{ades}(\tau)}{n}, \tag{10}$$

holds for any $\pi \in \mathfrak{B}_n$.

We recognize the left-hand side of equation (10) as $\Omega_\pi^{(a)}(2kl)$, so we want to count augmented P-partitions $f : \pm[n] \to \pm X$, where X is a totally ordered set of order

$2kl+1$. We interpret this as the number of solutions, in the augmented lexicographic ordering, to

$$(0,0) \leq (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_n, j_n) \leq (l, 0), \tag{11}$$

where we have

- $0 \leq i_s \leq l$,
- $-k < j_s \leq k$ if $\pi(s) < 0$,
- $-k \leq j_s < k$ if $\pi(s) > 0$, and
- $(i_s, j_s) < (i_{s+1}, j_{s+1})$ if $s \in \mathrm{aDes}(\pi)$.

Let us clarify. There are $2kl + l + 1$ points (i, j) with $0 \leq i \leq l$ and $-k \leq j \leq k$, not including the points $(0, j)$ with $j < 0$, or the points (l, j) with $j > 0$. Under the augmented lexicographic ordering, l of these points are identified: points of the form $(i, k) = (i+1, -k)$, for $i = 0, 1, \ldots, l-1$. Any particular (i_s, j_s) can only occupy one of (i, k) or $(i+1, -k)$, but not both. So there are truly $2kl + 1$ distinct classes in which the n points can fall. This confirms our interpretation of the order polynomial.

Now as before, we will split the solutions to the inequalities into distinct cases. Let $\pi(0) = \pi(n+1) = 0$, $i_0 = j_0 = 0$, $i_{n+1} = l$, and $j_{n+1} = 0$. Let $F = ((i_1, j_1), \ldots, (i_n, j_n))$ be any solution to (11). If $\pi(s) < \pi(s+1)$, then $(i_s, j_s) \leq (i_{s+1}, j_{s+1})$, which falls into one of two mutually exclusive cases:

$$i_s \leq i_{s+1} \quad \text{and} \quad j_s \leq j_{s+1}, \text{ or} \tag{12}$$

$$i_s < i_{s+1} \quad \text{and} \quad j_s > j_{s+1}. \tag{13}$$

If $\pi(s) > \pi(s+1)$, then $(i_s, j_s) < (i_{s+1}, j_{s+1})$, which we split as:

$$i_s \leq i_{s+1} \quad \text{and} \quad j_s < j_{s+1}, \text{ or} \tag{14}$$

$$i_s < i_{s+1} \quad \text{and} \quad j_s \geq j_{s+1}, \tag{15}$$

also mutually exclusive. Define $I_F = \{s \in \{0, 1, \ldots, n\} \setminus \mathrm{aDes}(\pi) \mid j_s > j_{s+1}\} \cup \{s \in \mathrm{aDes}(\pi) \mid j_s \geq j_{s+1}\}$. Then I_F is the set of all s such that either (13) or (15) holds for F. Now for any $I \subset \{0, 1, \ldots, n\}$, let S_I be the set of all solutions F to (11) satisfying $I_F = I$. We have split the solutions of (11) into 2^{n+1} distinct cases indexed by all the different subsets I of $\{0, 1, \ldots, n\}$.

However, $S_\emptyset$ is empty, since

$$0 \leq i_1 \leq \cdots \leq i_n \leq l$$

yields

$$0 \leq j_1 \leq \cdots \leq j_n \leq 0 \quad \text{ with } j_s < j_{s+1} \text{ if } s \in \mathrm{aDes}(\pi).$$

As discussed before, the augmented descent set of a signed permutation is never empty, so we would get $0 < 0$, a contradiction. At the other extreme, the set $S_{\{0,1,\ldots,n\}}$ has no solutions either. Here we get

$$0 < i_1 < \cdots < i_n < l$$

and consequently

$$0 \geq j_1 \geq \cdots \geq j_n \geq 0 \quad \text{ with } j_s > j_{s+1} \text{ if } s \notin \mathrm{aDes}(\pi).$$

But $\mathrm{aDes}(\pi)$ cannot equal $\{0, 1, \ldots, n\}$, so we get the contradiction $0 > 0$.

Now let I be any nonempty, proper subset of $\{0, 1, \ldots, n\}$. Form the poset P_I by $\pi(s) >_{P_I} \pi(s+1)$ if $s \in I$, $\pi(s) <_{P_I} \pi(s+1)$ otherwise. The poset P_I looks like a zig-zag, labeled consecutively by $0 = \pi(0), \pi(1), \pi(2), \ldots, \pi(n), 0 = \pi(n+1)$ with downward zigs corresponding to the elements of I. Because I is neither empty nor full, we never have $0 <_{P_I} 0$, so P_I is a well-defined, nontrivial type B poset.

For a given $F \in S_I$, let $f : \pm[n] \to \pm[k]$ be defined by $f(\pi(s)) = j_s$ and $f(-s) = -f(s)$ for $s = 0, 1, \ldots, n$. We will show that f is an augmented P_I partition. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (12) tells us that $f(\pi(s)) = j_s \leq j_{s+1} = f(\pi(s+1))$. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (14) tells us that $f(\pi(s)) = j_s < j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (13) gives us that $f(\pi(s)) = j_s > j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (15) gives us that $f(\pi(s)) = j_s \geq j_{s+1} = f(\pi(s+1))$. Since we required that $-k < j_s \leq k$ if $\pi(s) < 0$ and $-k \leq j_s < k$ if $\pi(s) > 0$, we have that for any particular solution in S_I, the j_s's can be thought of as an augmented P_I-partition. Conversely, any augmented P_I-partition f gives a solution in S_I since if $j_s = f(\pi(s))$, then $((i_1, j_1), \ldots, (i_n, j_n)) \in S_I$ if and only if $0 \leq i_1 \leq \cdots \leq i_n \leq l$ and $i_s < i_{s+1}$ for all $i \in I$. We can therefore turn our attention to counting augmented P_I-partitions.

Let $\sigma \in \mathcal{L}(P_I)$. Then we get for any σ-partition f,

$$0 \leq f(\sigma(1)) \leq f(\sigma(2)) \leq \cdots \leq f(\sigma(n)) \leq k,$$

and $f(\sigma(s)) < f(\sigma(s+1))$ whenever $s \in \mathrm{aDes}(\sigma)$, where we take $f(\sigma(n+1)) = k$. The number of solutions to this set of inequalities is

$$\Omega_\sigma^{(a)}(k) = \binom{k + n - \mathrm{ades}(\sigma)}{n}.$$

Recall by Observation 2.1.1 that $\sigma^{-1}\pi(s) < \sigma^{-1}\pi(s+1)$ if $\pi(s) <_{P_I} \pi(s+1)$, i.e., if $s \notin I$. If $\pi(s) >_{P_I} \pi(s+1)$ then $\sigma^{-1}\pi(s) > \sigma^{-1}\pi(s+1)$ and $s \in I$. We get that $\mathrm{aDes}(\sigma^{-1}\pi) = I$ if and only if $\sigma \in \mathcal{L}(P_I)$. Set $\tau = \sigma^{-1}\pi$. The number of solutions to

$$0 \le i_1 \le \cdots \le i_n \le l \quad \text{ and } i_s < i_{s+1} \text{ if } s \in \mathrm{aDes}(\tau)$$

is given by

$$\Omega_\tau(l) = \binom{l+n-\mathrm{ades}(\tau)}{n}.$$

Now for a given I, the number of solutions to (11) is

$$\sum_{\substack{\sigma\in\mathcal{L}(P_I)\\ \sigma\tau=\pi}} \binom{k+n-\mathrm{ades}(\sigma)}{n}\binom{l+n-\mathrm{ades}(\tau)}{n}.$$

Summing over all subsets $I \subset \{0,1,\ldots,n\}$, we can write the number of all solutions to (11) as

$$\sum_{\sigma\tau=\pi} \binom{k+n-\mathrm{ades}(\sigma)}{n}\binom{l+n-\mathrm{ades}(\tau)}{n},$$

and so the theorem is proved.

□

The proof of Theorem 2.3.2 is very similar, so we will omit unimportant details in the proof below.

Proof of Theorem 2.3.2. We equate coefficients and prove that

$$\binom{2kl+k+n-\mathrm{ades}(\pi)}{n} = \sum_{\sigma\tau=\pi} \binom{k+n-\mathrm{ades}(\sigma)}{n}\binom{l+n-\mathrm{des}(\tau)}{n}, \tag{16}$$

holds for any $\pi \in \mathfrak{B}_n$.

We recognize the left-hand side of equation (16) as $\Omega^{(a)}_\pi(2kl+k)$, so we want to count augmented P-partitions $f : \pm[n] \to \pm X$, where X is a totally ordered set of order $2kl + k + 1$. We interpret this as the number of solutions, in the augmented lexicographic ordering, to

$$(0,0) \leq (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_n, j_n) \leq (l, k), \tag{17}$$

where we have

- $0 \leq i_s \leq l$,
- $-k < j_s \leq k$ if $\pi(s) < 0$,
- $-k \leq j_s < k$ if $\pi(s) > 0$, and
- $(i_s, j_s) < (i_{s+1}, j_{s+1})$ if $s \in \mathrm{aDes}(\pi)$.

With these restrictions, we split the solutions to (17) by our prior rules. Let $F = ((i_1, j_1), \ldots, (i_n, j_n))$ be any particular solution. If $\pi(s) < \pi(s+1)$, then $(i_s, j_s) \leq (i_{s+1}, j_{s+1})$, which falls into one of two mutually exclusive cases:

$$i_s \leq i_{s+1} \quad \text{and} \quad j_s \leq j_{s+1}, \text{ or}$$

$$i_s < i_{s+1} \quad \text{and} \quad j_s > j_{s+1}.$$

If $\pi(s) > \pi(s+1)$, then $(i_s, j_s) < (i_{s+1}, j_{s+1})$, giving:

$$i_s \leq i_{s+1} \quad \text{and} \quad j_s < j_{s+1}, \text{ or}$$

$$i_s < i_{s+1} \quad \text{and} \quad j_s \geq j_{s+1},$$

also mutually exclusive. With (i_n, j_n), there is only one case, depending on π. If $\pi(n) > 0$, then $(i_n, j_n) < (l, k)$ and $i_n \leq l$ and $-k \leq j_n < k$. Similarly, if $\pi(n) < 0$, then $(i_n, j_n) \leq (l, k)$ and we have $i_n \leq l$ and $-k < j_n \leq k$. Define I_F and S_I as before. We get 2^n mutually exclusive sets S_I indexed by subsets $I \subset \{0, 1, \ldots, n-1\}$ (these subsets will correspond to ordinary descent sets).

Now for any $I \subset \{0, 1, \ldots, n-1\}$, define the $\mathfrak{B}_n$ poset P_I to be the poset given by $\pi(s) >_{P_I} \pi(s+1)$ if $s \in I$, and $\pi(s) <_{P_I} \pi(s+1)$ if $s \notin I$, for $s = 0, 1, \ldots, n-1$. We form a zig-zag poset labeled consecutively by $\pi(0) = 0, \pi(1), \pi(2), \ldots, \pi(n)$.

For a given solution $F \in S_I$, let $f : \pm[n] \to \pm[k]$ be defined by $f(\pi(s)) = j_s$ for $0 \le s \le n$, with $f(-s) = -f(s)$. It is not too difficult to check that f is an augmented P_I-partition, and that any augmented P_I-partition corresponds to a solution in S_I. Let $\sigma \in \mathcal{L}(P_I)$. Then for any σ-partition f we get

$$f(\sigma(0)) = 0 \le f(\sigma(1)) \le \cdots \le f(\sigma(n)) \le k,$$

with $f(\sigma(s)) < f(\sigma(s+1))$ whenever $s \in \mathrm{aDes}(\sigma)$. The number of solutions to this set of inequalities is

$$\Omega_\sigma^{(a)}(k) = \binom{k + n - \mathrm{ades}(\sigma)}{n}.$$

We see that for $s = 0, 1, \ldots, n-1$, $\sigma^{-1}\pi(s) < \sigma^{-1}\pi(s+1)$ if $\pi(s) <_{P_I} \pi(s+1)$, i.e., if $s \notin I$. Also, if $\pi(s) >_{P_I} \pi(s+1)$ then $\sigma^{-1}\pi(s) > \sigma^{-1}\pi(s+1)$ and $s \in I$. This time we get that $\mathrm{Des}(\sigma^{-1}\pi) = I$, an ordinary descent set, if and only if $\sigma \in \mathcal{L}_{P_I}$. Set $\tau = \sigma^{-1}\pi$. The number of solutions to

$$0 \le i_1 \le \cdots \le i_n \le l \quad \text{and } i_s < i_{s+1} \text{ if } s \in \mathrm{Des}(\tau)$$

is given by

$$\Omega_\tau(l) = \binom{l + n - \mathrm{des}(\tau)}{n}.$$

We take the sum over all subsets I to show the number of solutions to (16) is

$$\sum_{\sigma\tau=\pi} \binom{k + n - \mathrm{ades}(\sigma)}{n}\binom{l + n - \mathrm{des}(\tau)}{n},$$

and the theorem is proved. □

There is an augmented version of the q-order polynomial. We can write it quite nicely for a signed permutation π. We have

$$\begin{aligned}
\Omega_{\pi}^{(a)}(q;k) &= \sum_{\substack{0\le i_1\le\cdots\le i_n\le k\\ s\in\mathrm{aDes}(\pi)\Rightarrow i_s<i_{s+1}}} \left(\prod_{s=1}^{n} q^{i_s}\right)\\
&= \sum_{0\le i_1\le\cdots\le i_n\le k-\mathrm{ades}(\pi)} \left(\prod_{s=1}^{n} q^{i_s+a(s)}\right)\\
&= q^{\mathrm{acomaj}(\pi)}\binom{k+n-\mathrm{ades}(\pi)}{n}_q.
\end{aligned}$$

Here again $a(s)$ is the number of descents of π to the left of s and the *augmented comajor index*, $\mathrm{acomaj}(\pi)$, is the sum over all s of the numbers $a(s)$. There are also q-analogs of Theorems 2.3.1 and 2.3.2, which we will now state. The proofs of the q-analogs are very similar to the proofs of the theorems themselves, so we only sketch them. Define

$$\psi(q;x) = \sum_{\pi\in\mathfrak{B}_n} q^{\mathrm{acomaj}(\pi)}\binom{x/2+n-\mathrm{ades}(\pi)}{n}_q \pi.$$

THEOREM 2.3.3. *As polynomials in x and y (and q) with coefficients in the group algebra of the hyperoctahedral group we have*

$$\psi(q;x)\psi(q^x;y) = \psi(q;xy).$$

PROOF. The crucial step is that we want to give the integer pairs (i,j) the proper weight in the augmented lexicographic ordering. If we take $2ki+j$ as the weight of the point (i,j) then we get that the points (i,k) and $(i+1,-k)$ have the same weight. As desired, the weight corresponds to the position of (i,j) in the augmented lexicographic ordering. Everything else follows as in the proof of Theorem 2.3.1. For any π and any pair of positive integers k, l,

$$\begin{aligned}\Omega_\pi^{(a)}(q;2kl) &= \sum_{\substack{(0,0)\leq(i_1,j_1)\leq\cdots\leq(i_n,j_n)\leq(l+1,0)\\ s\in\mathrm{aDes}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}} \left(\prod_{s=1}^{n} q^{2ki_s+j_s}\right)\\ &= \sum_{\sigma\tau=\pi} \Omega_\sigma^{(a)}(q;k)\Omega_\tau^{(a)}(q^{2k};l).\end{aligned}$$

□

Theorem 2.3.4. *As polynomials in x and y (and q) with coefficients in the group algebra of the hyperoctahedral group we have*

$$\psi(q;x)\phi(q^x;y) = \psi(q;xy).$$

Proof. Because we exploit the augmented lexicographic order in the proof of Theorem 2.3.2 (the $q = 1$ case), we will use the same weighting scheme as in the proof of Theorem 2.3.3 for the points (i, j). We have:

$$\begin{aligned}\Omega_\pi^{(a)}(q;2kl+k) &= \sum_{\substack{(0,0)\leq(i_1,j_1)\leq\cdots\leq(i_n,j_n)\leq(l+1,k+1)\\ s\in\mathrm{Des}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}} \left(\prod_{s=1}^{n} q^{2ki_s+j_s}\right)\\ &= \sum_{\sigma\tau=\pi} \Omega_\sigma^{(a)}(q;k)\Omega_\tau(q^{2k};l).\end{aligned}$$

□

CHAPTER 3

Enriched P-partitions and peak algebras of type A

In this chapter we begin the investigation of different commutative subalgebras of the group algebra of the symmetric group, called (Eulerian) *peak algebras*. We will introduce two definitions of peaks, "interior" and "left," each giving rise to a different subalgebra. Taking the closure of both the interior and left peak algebras gives another subalgebra in which the interior peak algebra is an ideal. This situation closely resembles the relationship between the Eulerian and the augmented descent algebras of the hyperoctahedral group algebra. See the work of Aguiar, Bergeron, and Nyman [**ABN04**] for more on this relationship.

To study peaks, we begin by following the work of John Stembridge [**Ste97**]. We first survey Stembridge's *enriched* P-partitions, which will be useful for studying interior peaks, and a variation of Stembridge's maps called *left* enriched P-partitions for the study of left peaks.

3.1. Peaks of permutations

A *peak* of a permutation $\pi \in \mathfrak{S}_n$ is a position i such that $\pi(i-1) < \pi(i) > \pi(i+1)$. The only difference between *interior peaks* and *left peaks* is the values of i that we allow. The notion of peak that Stembridge [**Ste97**] defines is that of an interior peak. An *interior peak* is any $i \in \{2, 3, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. We define the *interior peak set*, $\mathrm{Pk}(\pi) \subset \{2, 3, \ldots, n-1\}$, to be the set of all such i. The number of interior peaks is denoted $\mathrm{pk}(\pi)$. For example, the permutation $\pi = (2, 1, 4, 3, 5)$ has $\mathrm{Pk}(\pi) = \{3\}$ and $\mathrm{pk}(\pi) = 1$. Notice that we always have $0 \leq \mathrm{pk}(\pi) \leq \lfloor \frac{n-1}{2} \rfloor$.

Aguiar, Bergeron, and Nyman [**ABN04**] study another type of peak, which we call a *left peak*. A left peak of a permutation is any position $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. This definition of peak varies from the prior one only in allowing a peak in the first position if $\pi(1) > \pi(2)$. We denote the *left peak set* by $\mathrm{Pk}^{(\ell)}(\pi) \subset [n-1]$, and the number of left peaks by $\mathrm{pk}^{(\ell)}(\pi)$. With $\pi = (2, 1, 4, 3, 5)$ as above, $\mathrm{Pk}^{(\ell)}(\pi) = \{1, 3\}$ and $\mathrm{pk}^{(\ell)}(\pi) = 2$. The number of left peaks always falls in the range $0 \leq \mathrm{pk}^{(\ell)}(\pi) \leq \lfloor n/2 \rfloor$.

Just as there are Eulerian numbers, counting the number of permutations with the same descent number, we also have peak numbers, counting the number of permutations with the same number of peaks. We will not devote much time to this topic, but state only those properties that are easy observations given the theory of enriched P-partitions developed in this chapter. We denote the number of permutations of n with k left peaks by $P^{(\ell)}_{n,k}$. We define the interior peak polynomial as

$$W_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{pk}(\pi)+1} = \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} P_{n,i} t^i.$$

Similarly, we define the left peak polynomial as

$$W_n^{(\ell)}(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{pk}^{(\ell)}(\pi)} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} P_{n,i}^{(\ell)} t^i.$$

Later in the chapter we will have the tools to prove the following observations relating peak polynomials to Eulerian polynomials. The first observation appears in Remark 4.8 of [**Ste97**]. In both cases, the second equality follows from Proposition 2.2.1.

Observation 3.1.1. *We have the following relation between the interior peak polynomial, the Eulerian polynomial, and the augmented Eulerian polynomial:*

$$W_n\left(\frac{4t}{(1+t)^2}\right) = \frac{2^{n+1}}{(1+t)^{n+1}} A_n(t) = \frac{2}{(1+t)^{n+1}} A_n^{(a)}(t).$$

Observation 3.1.2. *We have the following relation between the left peak polynomial, the Eulerian polynomial, and the augmented Eulerian polynomial:*

$$\begin{aligned} W_n^{(\ell)}\left(\frac{4t}{(1+t)^2}\right) &= \frac{1}{(1+t)^n}\sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}2^iA_i(t) \\ &= \frac{1}{(1+t)^n}\sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}A_i^{(a)}(t). \end{aligned}$$

3.2. Enriched P-partitions

We now introduce much of Stembridge's basic theory of enriched P-partitions. For a more detailed treatment see [**Ste97**]. We only provide proofs where our method is new, or where the old proof is enlightening. As in the first chapter, we will assume that all of our posets P are finite and labeled with the positive integers $1, 2, \ldots, n$. Throughout this section, by "peaks" we mean interior peaks.

To begin, Stembridge defines $\mathbb{P}'$ to be the set of nonzero integers with the following total order:

$$-1 < 1 < -2 < 2 < -3 < 3 < \cdots$$

In general, we can define X' for any totally ordered set $X = \{x_1, x_2, \ldots\}$ to be the set $\{-x_1, x_1, -x_2, x_2, \ldots\}$ with total order

$$-x_1 < x_1 < -x_2 < x_2 < \cdots$$

(which we can think of as two interwoven copies of X). In particular, for any positive integer k, $[k]'$ is the set

$$-1 < 1 < -2 < 2 < \cdots < -k < k.$$

For any $x \in X$, we say $x > 0$, or x is *positive*. On the other hand, we say $-x < 0$ and $-x$ is *negative*. The absolute value forgets any minus signs: $|\pm x| = x$ for any $x \in X$.

Definition 3.2.1. *An* enriched P-partition *is a map* $f : P \to X'$ *such that for all* $i <_P j$ *in* P*,*

- $f(i) \leq f(j)$
- $f(i) = f(j) > 0$ *only if* $i < j$ *in* $\mathbb{Z}$
- $f(i) = f(j) < 0$ *only if* $i > j$ *in* $\mathbb{Z}$

We let $\mathcal{E}(P)$ denote the set of all enriched P-partitions. When X has a finite number of elements, k, then the number of enriched P-partitions is finite. In this case, define the *enriched order polynomial*, denoted $\Omega'_P(k)$, to be the number of enriched P-partitions $f : P \to X'$.

Just as with ordinary P-partitions, we have what Stembridge calls the *fundamental lemma of enriched P-partitions* (or what Gessel would call the fundamental theorem).

Lemma 3.2.1 (FLEPP). *For any poset P, the set of all enriched P-partitions is the disjoint union of all enriched π-partitions for linear extensions π of P. Or,*

$$\mathcal{E}(P) = \coprod_{\pi \in \mathcal{L}(P)} \mathcal{E}(\pi)$$

The proof of the lemma is identical to the proof of the analogous statement for ordinary P-partitions, and the following corollary is immediate.

Corollary 3.2.1.

$$\Omega'_P(k) = \sum_{\pi \in \mathcal{L}(P)} \Omega'_\pi(k).$$

Therefore when studying enriched P-partitions it is enough (as before) to consider the case where P is a permutation. It is easy to describe the set of all enriched π-partitions in terms of descent sets. For any $\pi \in \mathfrak{S}_n$ we have

$$\begin{aligned} \mathcal{E}(\pi) = \{\, f : [n] \to X' \quad | \quad & f(\pi(1)) \leq f(\pi(2)) \leq \cdots \leq f(\pi(n)), \\ & f(\pi(i)) = f(\pi(i+1)) > 0 \Rightarrow i \notin \mathrm{Des}(\pi), \\ & f(\pi(i)) = f(\pi(i+1)) < 0 \Rightarrow i \in \mathrm{Des}(\pi) \,\} \end{aligned}$$

To try to simplify notation, and perhaps make this characterization more closely resemble the case of ordinary P-partitions, let $i \leq^+ j$ mean that $i < j$ in X' or $i = j > 0$. Similarly define $i \leq^- j$ to mean that $i < j$ in X' or $i = j < 0$. The set of all enriched π-partitions $f : [n] \to X'$ is all solutions to

$$f(\pi(1)) \leq^{\pm} f(\pi(2)) \leq^{\pm} \cdots \leq^{\pm} f(\pi(n)) \tag{18}$$

where $f(\pi(s)) \leq^- f(\pi(s+1))$ if $s \in \mathrm{Des}(\pi)$ and $f(\pi(s)) \leq^+ f(\pi(s+1))$ otherwise.

Counting the number of solutions to a set of inequalities like (18) is not so simple as counting integers with ordinary inequalities as was the case with ordinary

P-partitions—we are not going to derive a nice binomial coefficient for the order polynomial. However, Stembridge provides us some characterizations of use.

Let $c_l(P)$ denote the number of enriched P-partitions f such that $\{\,|f(i)| : i = 1, 2, \dots, n\,\} = [l]$ as sets. Then we have the following formula for the enriched order polynomial:

$$\Omega'_P(k) = \sum_{l=1}^{n} \binom{k}{l} c_l(P).$$

This formula quickly shows that the enriched order polynomial has degree n. Though it may not be obvious in this formulation, Stembridge observes ([**Ste97**], Proposition 4.2) that enriched order polynomials satisfy a reciprocity relation:

$$\Omega'_P(-x) = (-1)^n \Omega'_P(x).$$

In fact, we can combine these facts to be precise:

Observation 3.2.1. *For n even, $\Omega'_P(x)$ is a polynomial of degree $n/2$ in x^2. For n odd, $x\Omega'_P(x)$ is a polynomial of degree $(n+1)/2$ in x^2.*

Before we get too far ahead of the story, we have yet to say why enriched order polynomials are useful for studying peaks of permutations. Clearly enriched π-partitions depend on the descent set of π. In fact they depend only on the number of peaks, as seen in Stembridge's formulation of the generating function for the order polynomial ([**Ste97**], Theorem 4.1). Here we give only the generating function for enriched order polynomials of permutations, and remark that by the fundamental Lemma 3.2.1, we can obtain the order polynomial generating function for any poset by summing the generating functions for its linear extensions.

THEOREM 3.2.1. *We have the following generating function for enriched π-partitions:*

$$\sum_{k\geq 0} \Omega'_\pi(k)t^k = \frac{1}{2}\frac{(1+t)^{n+1}}{(1-t)^{n+1}} \cdot \left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}(\pi)+1}$$

Notice that this formula implies that $\Omega'_\pi(x)$ has no constant term. We will sketch Stembridge's proof since it will be useful for dealing with both the left peaks case and the type B case.

PROOF. Fix any permutation $\pi \in \mathfrak{S}_n$. As seen in Chapter 1, we have the following formula for the generating function of ordinary order polynomials:

$$\sum_{k\geq 0} \Omega_\pi(k)t^k = \frac{t^{\mathrm{des}(\pi)+1}}{(1-t)^{n+1}}$$

For any set of integers D, let $D+1$ denote the set $\{d+1 \mid d \in D\}$. From Stembridge's Proposition 3.5 [**Ste97**], we see that an enriched order polynomial can be written as a sum of ordinary order polynomials:

$$\Omega'_\pi(k) = 2^{\mathrm{pk}(\pi)+1} \cdot \sum_{\substack{D\subset[n-1] \text{ and} \\ \mathrm{Pk}(\pi)\subset D\triangle(D+1)}} \Omega_D(k),$$

where $\Omega_D(k)$ denotes the ordinary order polynomial of any permutation with descent set D, and $\triangle$ denotes the symmetric difference of sets: $A \triangle B = (A\cup B)\backslash(A\cap B)$.

Putting these two facts together, we get:

$$\begin{aligned}
\sum_{k\geq 0}\Omega'_\pi(k)t^k &= \sum_{k\geq 0} 2^{\mathrm{pk}(\pi)+1}\cdot \sum_{\substack{D\subset[n-1] \text{ and} \\ \mathrm{Pk}(\pi)\subset D\triangle(D+1)}} \Omega_D(k)t^k \\
&= 2^{\mathrm{pk}(\pi)+1}\cdot \sum_{\substack{D\subset[n-1] \text{ and} \\ \mathrm{Pk}(\pi)\subset D\triangle(D+1)}} \sum_{k\geq 0}\Omega_D(k)t^k \\
&= \frac{2^{\mathrm{pk}(\pi)+1}}{(1-t)^{n+1}}\cdot t \sum_{\substack{D\subset[n-1] \text{ and} \\ \mathrm{Pk}(\pi)\subset D\triangle(D+1)}} t^{|D|}
\end{aligned}$$

It is not hard to write down the generating function for the sets D by size. We have, for any $j\in \mathrm{Pk}(\pi)$, exactly one of j or $j-1$ will be in D. There are $n-2\,\mathrm{pk}(\pi)-1$ remaining elements of $[n-1]$, and they can be included in D or not:

$$\begin{aligned}
\sum_{\substack{D\subset[n-1] \text{ and} \\ \mathrm{Pk}(\pi)\subset D\triangle(D+1)}} t^{|D|} &= \underbrace{(t+t)(t+t)\cdots(t+t)}_{\mathrm{pk}(\pi)}\underbrace{(1+t)(1+t)\cdots(1+t)}_{n-2\,\mathrm{pk}(\pi)-1} \\
&= (2t)^{\mathrm{pk}(\pi)}(1+t)^{n-2\,\mathrm{pk}(\pi)-1}
\end{aligned}$$

Putting everything together, we get

$$\sum_{k\geq 0}\Omega'_\pi(k)t^k = \frac{1}{2}\frac{(1+t)^{n+1}}{(1-t)^{n+1}}\cdot\left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}(\pi)+1}$$

as desired. □

So while we may not have the order polynomial given by a simple binomial coefficient as in the earlier cases, we do know that we have polynomials that depend only on the number of peaks, and that have as many terms as there are realizable peak numbers. Recall that this is very similar to the case of descents, where we knew that our ordinary order polynomials depended on the number of descents, and that the number of terms in these polynomials corresponded to the number of realizable descent numbers. We are ready to discuss the application of enriched order polynomials to the interior peak algebra. We conclude the section with proof of Observation 3.1.1.

Proof of Observation 3.1.1. Recall from Section 1.2 that we have the following formula for the ordinary Eulerian polynomials:

$$\sum_{k\geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Now let P be an antichain of n elements labeled $1, 2, \ldots, n$. The number of enriched P-partitions $f : [n] \to [k]'$ is $(2k)^n$ since there are $2k$ elements in $[k]'$ and there are no relations among the elements of the antichain. Therefore $\Omega'_P(k) = (2k)^n$, and since we have $\mathcal{L}(P) = \mathfrak{S}_n$, Theorem 3.2.1 gives

$$\frac{1}{2}\frac{(1+t)^{n+1}}{(1-t)^{n+1}} W_n\left(\frac{4t}{(1+t)^2}\right) = \sum_{k\geq 0}(2k)^n t^k = 2^n \sum_{k\geq 0} k^n t^k = \frac{2^n A_n(t)}{(1-t)^{n+1}}.$$

Rearranging terms gives the desired result:

$$W_n\left(\frac{4t}{(1+t)^2}\right) = \frac{2^{n+1}}{(1+t)^{n+1}} A_n(t).$$

□

3.3. The interior peak algebra

In this section we will prove the existence of the interior peak algebra by describing a set of orthogonal idempotents as coefficients of certain "structure" polynomials. Let

$$\rho(x) = \sum_{\pi\in\mathfrak{S}_n} \Omega'_\pi(x/2)\pi = \sum_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} \Omega'_i(x/2)E'_i,$$

where E'_i is the sum of all permutations with $i-1$ peaks and $\Omega'_i(x)$ is the enriched order polynomial for any permutation with $i-1$ peaks.

Theorem 3.3.1. *As polynomials in x and y with coefficients in the group algebra of the symmetric group, we have*

$$\rho(x)\rho(y) = \rho(xy). \tag{19}$$

As in the case of descents, this formula gives us orthogonal idempotents for a subalgebra of the group algebra. If we let e'_i be the coefficient of x^{2i} for n even (the coefficient of x^{2i-1} for n odd), in $\rho(x) = \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} e'_i x^{2i}$, then $e'_i e'_j = 0$ if $i \neq j$ and $(e'_i)^2 = e'_i$. So we get that the interior peak algebra of the symmetric group is commutative of dimension $\lfloor (n+1)/2 \rfloor$.

Proof. We will try to imitate the proofs from earlier chapters, making adjustments only when necessary. By equating the coefficient of π on both sides of equation (19) we know that we need only prove the following claim: For any permutation $\pi \in \mathfrak{S}_{\mathfrak{n}}$ and positive integers k, l we have

$$\Omega'_\pi(2kl) = \sum_{\sigma\tau=\pi} \Omega'_\sigma(k)\Omega'_\tau(l).$$

We will interpret the left-hand side of the equation in such a way that we can split it apart to form the right hand side. Rather than considering $\Omega'(\pi; 2kl)$ to count maps $f : \pi \to [2kl]'$, we will understand it to count maps $f : \pi \to [l]' \times [k]'$, where we take the *up-down* order on $[l]' \times [k]'$. The up-down order is defined as follows (see Figure 3.1): $(i,j) < (i',j')$ if and only if

(1) $i < i'$, or

(2) $i = i' > 0$ and $j < j'$, or

(3) $i = i' < 0$ and $j > j'$.

So if the horizontal coordinate is negative, we read the columns from the top down, if the horizontal coordinate is positive, we read from the bottom up. Then $\Omega'(\pi; 2kl)$ is the number of solutions to

$$(-1, k) \leq (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_n, j_n) \leq (l, k) \tag{20}$$

where $(i_s, j_s) \leq^- (i_{s+1}, j_{s+1})$ if $s \in \mathrm{Des}(\pi)$ and $(i_s, j_s) \leq^+ (i_{s+1}, j_{s+1})$ otherwise. For example, if $\pi = (1, 3, 2)$, we will count the number of points

$$(-1, k) \leq (i_1, j_1) \leq^+ (i_2, j_2) \leq^- (i_3, j_3) \leq (l, k).$$

Here we write $(i, j) \leq^+ (i', j')$ in one of three cases: if $i < i'$, or if $i = i' > 0$ and $j \leq^+ j'$, or if $i = i' < 0$ and $j \geq^- j'$. Similarly, $(i, j) \leq^- (i', j')$ if $i < i'$, or if $i = i' > 0$ and $j \leq^- j'$, or if $i = i' < 0$ and $j \geq^+ j'$.

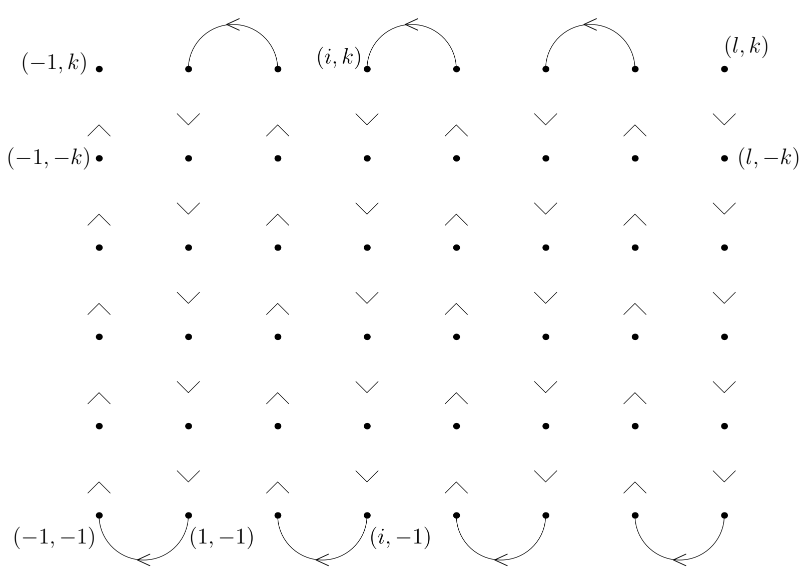


FIGURE 3.1. The up-down order for $[l]' \times [k]'$.

To get the result we desire, we will sort the set of all solutions to (20) into distinct cases indexed by subsets $I \subset [n-1]$. The sorting depends on π and proceeds as follows. Let $F = ((i_1, j_1), \ldots, (i_n, j_n))$ be any solution to (20). For any $s = 1, 2, \ldots, n-1$, if $\pi(s) < \pi(s+1)$, then $(i_s, j_s) \leq^+ (i_{s+1}, j_{s+1})$, which falls into one of two mutually exclusive cases:

$$i_s \leq^+ i_{s+1} \text{ and } j_s \leq^+ j_{s+1}, \text{ or} \tag{21}$$

$$i_s \leq^- i_{s+1} \text{ and } j_s \geq^- j_{s+1}. \tag{22}$$

If $\pi(s) > \pi(s+1)$, then $(i_s, j_s) \leq^- (i_{s+1}, j_{s+1})$, which we split as:

$$i_s \leq^+ i_{s+1} \text{ and } j_s \leq^- j_{s+1}, \text{ or} \tag{23}$$

$$i_s \leq^- i_{s+1} \text{ and } j_s \geq^+ j_{s+1}, \tag{24}$$

also mutually exclusive. Define I_F to be the set of all s such that either (22) or (24) holds for F. Notice that in both cases, $i_s \leq^- i_{s+1}$. Now for any $I \subset [n-1]$, let S_I be the set of all solutions F to (20) satisfying $I_F = I$. We have split the solutions of (20) into 2^{n-1} distinct cases indexed by all the different subsets I of $[n-1]$.

For any particular $I \subset [n-1]$, form the poset P_I of the elements $1, 2, \ldots, n$ by $\pi(s) <_{P_I} \pi(s+1)$ if $s \notin I$, $\pi(s) >_{P_I} \pi(s+1)$ if $s \in I$. We form a zig-zag poset (see Figure 1.4) of n elements labeled consecutively by $\pi(1), \pi(2), \ldots, \pi(n)$ with downward zigs corresponding to the elements of I.

For any solution F in S_I, let $f : [n] \to [k]'$ be defined by $f(\pi(s)) = j_s$. We will show that f is an enriched P_I-partition. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (21) tells us that $f(\pi(s)) = j_s \leq^+ j_{s+1} = f(\pi(s+1))$. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (23) tells us that $f(\pi(s)) = j_s \leq^- j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (22) gives us that $f(\pi(s)) = j_s \geq^-$

$j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (24) gives us that $f(\pi(s)) = j_s \geq^+ j_{s+1} = f(\pi(s+1))$. In other words, we have verified that f is a P_I-partition. So for any particular solution in S_I, the j_s's can be thought of as an enriched P_I-partition.

Conversely, any enriched P_I-partition f gives a solution in S_I since if $j_s = f(\pi(s))$, then

$$((i_1, j_1), \ldots, (i_n, j_n)) \in S_I$$

if and only if $1 \leq i_1 \leq \cdots \leq i_n \leq l$ and $i_s \leq^- i_{s+1}$ for all $s \in I$, $i_s \leq^+ i_{s+1}$ for $s \notin I$. We can therefore turn our attention to counting enriched P_I-partitions.

The remainder of the argument is identical to the latter half of the proof of Theorem 1.3.1. □

Note that the up-down order used in the proof is not immediately amenable to a q-analog for Theorem 3.3.1, though there may exist such a formula.

3.4. Left enriched P-partitions

In this section we modify the definition of enriched P-partitions in order to study the left peak algebra. Throughout this section, by "peaks" we mean left peaks unless otherwise noted.

Define $\mathbb{P}^{(\ell)}$ to be the integers with the following total order:

$$0 < -1 < 1 < -2 < 2 < -3 < 3 < \cdots$$

In general, we can define $X^{(\ell)}$ for any totally ordered set $X = \{x_0, x_1, x_2, \ldots\}$ to be the set $\{x_0, -x_1, x_1, -x_2, x_2, \ldots\}$ with total order

$$x_0 < -x_1 < x_1 < -x_2 < x_2 < \cdots$$

In particular, for any positive integer k, $[k]^{(\ell)}$ is the set

$$0 < -1 < 1 < -2 < 2 < \cdots < -k < k.$$

For any $x_i \in X$, we say $x_i \geq 0$, or x_i is *nonnegative*. On the other hand, if $i \neq 0$ we say $-x_i < 0$ and $-x_i$ is *negative*. The absolute value loses any minus signs: $|\pm x| = x$ for any $x \in X$.

DEFINITION 3.4.1. *A* left enriched P-partition *is a map* $f : P \to X^{(\ell)}$ *such that for all* $i <_P j$ *in* P,

- $f(i) \leq f(j)$
- $f(i) = f(j) \geq 0$ *only if* $i < j$ *in* $\mathbb{Z}$
- $f(i) = f(j) < 0$ *only if* $i > j$ *in* $\mathbb{Z}$

We let $\mathcal{E}^{(\ell)}(P)$ denote the set of all left enriched P-partitions. When X has a finite number of elements, k, then the number of left enriched P-partitions is finite. In this case, define the *left enriched order polynomial*, denoted $\Omega_P^{(\ell)}(k)$, to be the number of left enriched P-partitions $f : P \to X^{(\ell)}$.

We have the fundamental lemma and its corollary.

LEMMA 3.4.1 (FLLEPP). *For any poset P, the set of all left enriched P-partitions is the disjoint union of all left enriched π-partitions for linear extensions π of P. In other words,*

$$\mathcal{E}^{(\ell)}(P) = \coprod_{\pi \in \mathcal{L}(P)} \mathcal{E}^{(\ell)}(\pi)$$

COROLLARY 3.4.1.

$$\Omega_P^{(\ell)}(k) = \sum_{\pi \in \mathcal{L}(P)} \Omega_\pi^{(\ell)}(k).$$

The set of all left enriched π-partitions can be described in terms of descent sets. For any $\pi \in \mathfrak{S}_n$ we have

$$\begin{aligned}\mathcal{E}^{(\ell)}(\pi) = \{\, f : [n] \to X^{(\ell)} \quad | \quad & f(\pi(1)) \leq f(\pi(2)) \leq \cdots \leq f(\pi(n)), \\ & f(\pi(i)) = f(\pi(i+1)) \geq 0 \Rightarrow i \notin \mathrm{Des}(\pi), \\ & f(\pi(i)) = f(\pi(i+1)) < 0 \Rightarrow i \in \mathrm{Des}(\pi) \,\}\end{aligned}$$

Using different notation, we can write the set of all left enriched π-partitions $f : [n] \to X^{(\ell)}$ as all solutions to

$$f(\pi(1)) \leq^{\pm} f(\pi(2)) \leq^{\pm} \cdots \leq^{\pm} f(\pi(n)) \tag{25}$$

where $f(\pi(s)) \leq^{-} f(\pi(s+1))$ if $s \in \mathrm{Des}(\pi)$ and $f(\pi(s)) \leq^{+} f(\pi(s+1))$ otherwise.

Let $c_m^{(\ell)}(P)$ denote the number of left enriched P-partitions f such that $\{\, |f(i)| : i = 1, 2, \ldots, n \,\} = [m]$ as sets. Let $c_{m,0}^{(\ell)}(P)$ denote the number of left enriched P-partitions f such that $\{\, |f(i)| : i = 1, 2, \ldots, n \,\} = \{0\} \cup [m]$. Then we have the following formula for the left enriched order polynomial:

$$\Omega_P^{(\ell)}(k) = \sum_{m=1}^{n} \binom{k}{m} c_l^{(\ell)}(P) + \sum_{m=0}^{n-1} \binom{k}{m} c_{m,0}^{(\ell)}(P).$$

This formula shows that the left enriched order polynomial has degree n. The left enriched order polynomials also satisfy a reciprocity relation, though not quite the same as the interior case.

Observation 3.4.1. *We have*

$$\Omega_P^{(\ell)}(-x) = (-1)^n \Omega_P^{(\ell)}(x-1),$$

or by substituting $x \leftarrow x + 1/2$*,*

$$\Omega_P^{(\ell)}(-x-1/2) = (-1)^n \Omega_P^{(\ell)}(x-1/2).$$

The proof of this observation is omitted, though we will say it is straightforward given the generating function in Theorem 3.4.1 below. Since these order polynomials are even or odd, we have the following.

OBSERVATION 3.4.2. *For n even, $\Omega_P^{(\ell)}(x-1/2)$ is a polynomial of degree $n/2$ in x^2. For n odd, $x\Omega_P^{(\ell)}(x-1/2)$ is a polynomial of degree $(n+1)/2$ in x^2.*

It remains to show that left enriched order polynomials are somehow related to peaks. From the definition it is immediate that they depend on descents, but we will derive the generating function for these polynomials to show they depend only on the number of peaks. As before, we write down the case where the poset is a permutation.

THEOREM 3.4.1. *We have the following generating function for left enriched order polynomials:*

$$\sum_{k\geq 0} \Omega_\pi^{(\ell)}(k) t^k = \frac{(1+t)^n}{(1-t)^{n+1}} \cdot \left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}^{(\ell)}(\pi)}$$

Notice that this formula implies that left enriched order polynomials depend only on the number of left peaks.

PROOF. Fix any permutation $\pi \in \mathfrak{S}_n$. The key fact, proved in [**Pet**], is the following:

$$\Omega_\pi^{(\ell)}(k) = 2^{\mathrm{pk}^{(\ell)}(\pi)} \cdot \sum_{\substack{D\subset\{0\}\cup[n-1] \text{ and} \\ \mathrm{Pk}^{(\ell)}(\pi)\subset D\triangle(D+1)}} \Omega_{(B;D)}(k),$$

where $\Omega_{(B;D)}(k)$ denotes the type B order polynomial of any signed permutation with descent set D. It may seem strange to express a type A polynomial related to peaks in terms of type B polynomials related to descents, but as may be more clear in the next chapter, left peaks are basically a special case of type B peaks, which are quite naturally related to type B descents. The paper of Aguiar, Bergeron, and Nyman [**ABN04**] points out some connections between type B descents and type A peaks more formally than we will here.

The generating function for type B order polynomials is (see Reiner [**Rei93**] for example)

$$\sum_{k\geq 0} \Omega_{(B;\pi)}(k)t^k = \frac{t^{\mathrm{des}(\pi)}}{(1-t)^{n+1}}$$

Similarly to the interior enriched order polynomial case, we put these two facts together to get:

$$\begin{aligned}\sum_{k\geq 0} \Omega^{(\ell)}_{\pi}(k)t^k &= \frac{2^{\mathrm{pk}^{(\ell)}(\pi)}}{(1-t)^{n+1}} \cdot \sum_{\substack{D\subset\{0\}\cup[n-1] \text{ and} \\ \mathrm{Pk}^{(\ell)}(\pi)\subset D\triangle(D+1)}} t^{|D|} \\ &= \frac{2^{\mathrm{pk}^{(\ell)}(\pi)}}{(1-t)^{n+1}} \cdot (2t)^{\mathrm{pk}^{(\ell)}(\pi)}(1+t)^{n-2\,\mathrm{pk}^{(\ell)}(\pi)}\end{aligned}$$

By rearranging terms, we get

$$\sum_{k\geq 0} \Omega^{(\ell)}_{\pi}(k)t^k = \frac{(1+t)^n}{(1-t)^{n+1}} \cdot \left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}^{(\ell)}(\pi)}$$

as desired. □

Now that we have our left enriched order polynomials, $\Omega^{(\ell)}_{\pi}(x-1/2)$, that depend only on left peak numbers, and with the property that they have as many terms as realizable left peak numbers, we can use them to find orthogonal idempotents for the left peak subalgebra. We finish this section with proof of Observation 3.1.2.

Proof of Observation 3.1.2. If we let P be an antichain of n elements, the number of left enriched P-partitions $f : [n] \to [k]^{(\ell)}$ is $(2k+1)^n$ since there are $2k+1$

elements in $[k]^{(\ell)}$ and there are no relations among the elements of the antichain. Therefore $\Omega_P^{(\ell)}(k) = (2k+1)^n$, and since we have $\mathcal{L}(P) = \mathfrak{S}_n$, Theorem 3.4.1 gives

$$\begin{aligned}\frac{(1+t)^n}{(1-t)^{n+1}} W_n^{(\ell)}\left(\frac{4t}{(1+t)^2}\right) = \sum_{k\geq 0}(2k+1)^n t^k &= \sum_{i=0}^{n}\binom{n}{i} 2^i \sum_{k\geq 0} k^i t^k \\ &= \sum_{i=0}^{n}\binom{n}{i}\frac{2^i A_i(t)}{(1-t)^{i+1}}.\end{aligned}$$

Rearranging terms gives the desired result:

$$\begin{aligned} W_n^{(\ell)}\left(\frac{4t}{(1+t)^2}\right) &= \frac{1}{(1+t)^n}\sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}2^i A_i(t) \\ &= \frac{1}{(1+t)^n}\sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}A_i^{(a)}(t). \end{aligned}$$

□

3.5. The left peak algebra

In this section we use the theory of left enriched P-partitions to prove the existence of the left peak algebra. Let

$$\rho^{(\ell)}(x) = \sum_{\pi\in\mathfrak{S}_n} \Omega_\pi^{(\ell)}((x-1)/2)\pi = \sum_{i=1}^{\lfloor \frac{n}{2}\rfloor+1} \Omega_i^{(\ell)}((x-1)/2)E_i^{(\ell)},$$

where $E_i^{(\ell)}$ is the sum of all permutations with $i-1$ left peaks and $\Omega_i^{(\ell)}(x)$ is the left enriched order polynomial for any permutation with $i-1$ left peaks.

Theorem 3.5.1. *As polynomials in x and y with coefficients in the group algebra of the symmetric group, we have*

$$\rho^{(\ell)}(x)\rho^{(\ell)}(y) = \rho^{(\ell)}(xy). \tag{26}$$

This formula gives us orthogonal idempotents for another commutative subalgebra of the group algebra. If we let $e_i^{(\ell)}$ be the coefficient of x^{2i} for n even (the coefficient of x^{2i+1} for n odd), in

$$\rho^{(\ell)}(x) = \begin{cases} \sum_{i=0}^{n/2} e_i^{(\ell)} x^{2i} & \text{if } n \text{ is even,} \\ \sum_{i=0}^{(n-1)/2} e_i^{(\ell)} x^{2i+1} & \text{if } n \text{ is odd,} \end{cases}$$

then $e_i^{(\ell)} e_j^{(\ell)} = 0$ if $i \neq j$ and $(e_i^{(\ell)})^2 = e_i^{(\ell)}$. So we get that the left peak algebra of the symmetric group is commutative of dimension $\lfloor n/2 \rfloor + 1$ (there is no constant term for n odd by Observation 3.4.1. The fact that there is a constant term for n even follows from a partial fraction decomposition of the generating function for $\Omega_\pi^{(\ell)}(x)$).

Proof. By equating the coefficient of π on both sides of equation (26) we know that we need only prove

$$\Omega^{(\ell)}(\pi; 2kl + k + l) = \sum_{\sigma\tau=\pi} \Omega^{(\ell)}(\sigma; k)\, \Omega^{(\ell)}(\tau; l).$$

We will think of the left-hand side of the equation as counting maps $f : \pi \to [l]^{(\ell)} \times [k]^{(\ell)}$, where, as in the proof of Theorem 3.3.1, we take the up-down order on $[l]^{(\ell)} \times [k]^{(\ell)}$.

Then $\Omega^{(\ell)}(\pi; 2kl + k + l)$ is the number of solutions to

$$(0, 0) \leq (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_n, j_n) \leq (l, k)$$

where $(i_s, j_s) \leq^- (i_{s+1}, j_{s+1})$ if $s \in \mathrm{Des}(\pi)$ and $(i_s, j_s) \leq^+ (i_{s+1}, j_{s+1})$ otherwise. Recall that in the up-down order we write $(i, j) \leq^+ (i', j')$ in one of three cases: if $i < i'$, or if $i = i' \geq 0$ and $j \leq^+ j'$, or if $i = i' < 0$ and $j \geq^- j'$. Similarly, $(i, j) \leq^- (i', j')$ if $i < i'$, or if $i = i' \geq 0$ and $j \leq^- j'$, or if $i = i' < 0$ and $j \geq^+ j'$. See Figure 3.2.

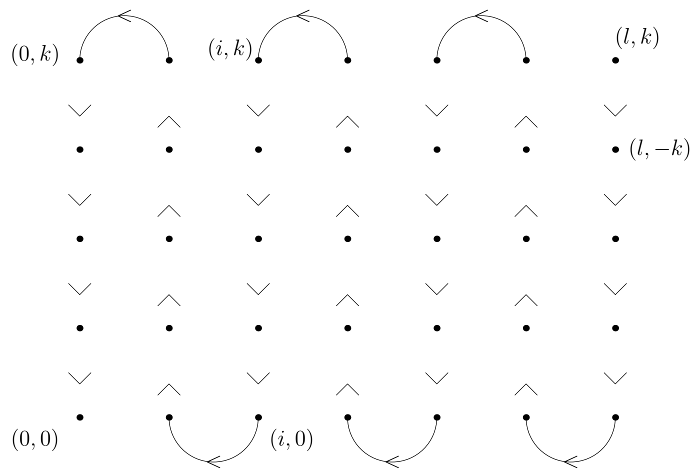


FIGURE 3.2. The up-down order for $[l]^{(\ell)} \times [k]^{(\ell)}$.

The rest of the proof is identical to that of Theorem 3.3.1. □

We also have the following way to combine interior and left peaks.

THEOREM 3.5.2. *As polynomials in x and y with coefficients in the group algebra of the symmetric group,*

$$\rho(y)\rho^{(\ell)}(x) = \rho^{(\ell)}(x)\rho(y) = \rho(xy).$$

PROOF. This proof varies from the previous proof only slightly. For any $\pi \in \mathfrak{S}_n$ we show that:

$$\Omega'(\pi; 2kl + l) = \sum_{\sigma\tau=\pi} \Omega^{(\ell)}(\sigma; k)\Omega'(\tau; l), \tag{27}$$

$$= \sum_{\sigma\tau=\pi} \Omega'(\sigma; l)\, \Omega^{(\ell)}(\tau; k). \tag{28}$$

For equation (27), the key is to think of the left-hand side of the equation as counting maps $f : \pi \to [l]' \times [k]^{(\ell)}$, with the up-down order on $[l]' \times [k]^{(\ell)}$.

For (28), we count enriched π-partitions $f : \pi \to [k]^{(\ell)} \times [l]'$ with the up-down order, and the theorem follows. □

The consequence of Theorem 3.5.2 is the multiplication for the two sets of idempotents found in this chapter. We have $e_i^{(\ell)} e_i' = e_i'$ and $e_i^{(\ell)} e_j' = 0$ if $i \neq j$. So if we take both sets of idempotents, they span a subalgebra of the group algebra of dimension n (rather than $n+1$ since these subalgebras have the relation $\sum_{i=1}^{\lfloor n/2 \rfloor + 1} E_i^{(\ell)} = \sum_{\pi \in \mathfrak{S}_n} \pi = \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} E_i'$, see [**ABN04**]). It is also clear from these relations that the interior peak algebra is an ideal, just as in the case of augmented descents. The reader is referred to [**ABN04**] for more about the relationships between descents and peaks.

CHAPTER 4

The peak algebra of type B

We now move to the (Eulerian) peak algebra of type B. Recent work of Schocker [**Sch05**] suggests that there should be a "peak-like" subalgebra of any finite Coxeter group formed by something like the sums of permutations with the same peak set. His result claims to be analogous to Solomon's result for descent algebras (formed by sums of permutations with the same peak set). As in the case of descents, we will not examine this problem at the level of the *set* of peaks, but rather the *number* of peaks. However, the linear span of sums of signed permutations with the same number of peaks does not give a subalgebra of the group algebra. The linear span of sums of signed permutations with the same number of peaks and the same sign on $\pi(1)$ does.

4.1. Type B peaks

We say a signed permutation π has a peak in position $i = 1, 2, \ldots, n-1$ if $\pi(i-1) < \pi(i) > \pi(i+1)$, where, as in our earlier dealings with signed permutations, we require that $\pi(0) = 0$. As before, we will denote the set of peaks by $\mathrm{Pk}(\pi)$, and the number of peaks by $\mathrm{pk}(\pi)$. For example, the permutation $\pi = (-2, 4, -5, 3, 1)$ has $\mathrm{Pk}(\pi) = \{2, 4\}$ and $\mathrm{pk}(\pi) = 2$. Note that the number of peaks of a signed permutation is between zero and $\lfloor n/2 \rfloor$.

A natural guess at the structure of an Eulerian peak algebra of type B might be the span of sums of permutations with the same number of peaks. However, this definition simply does not work. The following definition does work. Define the elements E_i^+, E_i^- in the group algebra of the hyperoctahedral group by:

$$E_i^+ = \sum_{\substack{\mathrm{pk}(\pi)=i \\ \pi(1)>0}} \pi$$

$$E_i^- = \sum_{\substack{\mathrm{pk}(\pi)=i \\ \pi(1)<0}} \pi$$

We will show that the linear span of these elements forms a subalgebra of the group algebra. These elements split the collection of permutations with the same number of peaks into two groups: those that begin with a positive number and those that begin with a negative number. This splitting of cases is similar to splitting left peaks apart from interior peaks, and once we introduce type B enriched order polynomials we will see that the generating functions for type B and type A enriched order polynomials are closely related. It is not hard to check that E_i^+ and E_i^- are nonzero for all $0 \le i < \lfloor n/2 \rfloor$. If n is odd, $E^+_{\frac{n-1}{2}}$ and $E^-_{\frac{n-1}{2}}$ are both nonzero, but if n is even, $E^+_{n/2}$ is nonzero while $E^-_{n/2} = 0$. In other words, the set $\{E_i^{\pm}\}$ has cardinality $n+1$ for any n.

We can define type B peak numbers and type B peak polynomials. We will denote the number of signed permutations of n with k peaks and $\pi(1) > 0$ by $P^+_{n,k}$. We denote the number of signed permutations of n with k peaks by $P^-_{n,k+1}$. We define the type B peak polynomials by

$$\begin{aligned} W_n^+(t) &= \sum_{\substack{\pi\in\mathfrak{B}_n\\ \pi(1)>0}} t^{\mathrm{pk}(\pi)} = \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor} P^+_{n,i}t^i \\ W_n^-(t) &= \sum_{\substack{\pi\in\mathfrak{B}_n\\ \pi(1)<0}} t^{\mathrm{pk}(\pi)+1} = \sum_{i=1}^{\lfloor \frac{n}{2}\rfloor+1} P^-_{n,i}t^i. \end{aligned}$$

Later in the chapter we will have the tools to prove the following observations relating type B peak polynomials to type A peak polynomials.

Observation 4.1.1. *We have the following relation between type B peak polynomials and the interior peak polynomial:*

$$\begin{aligned} W_n^+\left(\frac{4t}{(1+t)^2}\right) &+ \frac{1+t}{2}\cdot W_n^-\left(\frac{4t}{(1+t)^2}\right) \\ &= \sum_{i=0}^{n}\binom{n}{i}\frac{(1-t)^{n-i}}{(1+t)^{n-i-1}}2^{i-1}W_i\left(\frac{4t}{(1+t)^2}\right). \end{aligned}$$

Observation 4.1.2. *We have the following relation between type B peak polynomials and the left peak polynomial:*

$$\begin{aligned} W_n^+\left(\frac{4t}{(1+t)^2}\right) &+ \frac{1+t}{2}\cdot W_n^-\left(\frac{4t}{(1+t)^2}\right) \\ &= (-1)^n\sum_{i=0}^{n}\binom{n}{i}\frac{(1-t)^{n-i}}{(1+t)^{n-i}}(-2)^iW_i^{(\ell)}\left(\frac{4t}{(1+t)^2}\right). \end{aligned}$$

We omit the formal proofs of these observations, since they follow the approach taken in proving Observations 3.1.1 and 3.1.2. We will only mention that once we have the generating function for enriched order polynomials of type B, all that is needed is to notice is that we can expand $(4k+1)^n$ in the following two ways:

$$\sum_{i=0}^{n} \binom{n}{i} 2^i \cdot (2k)^i \quad \text{and} \quad (-1)^n \sum_{i=0}^{n} \binom{n}{i} (-2)^i (2k+1)^i$$

4.2. Enriched P-partitions of type B

We will slightly modify the notation for the set X' introduced in the previous chapter. Let $X = \{x_1, x_2, \ldots\}$ be any totally ordered set. Then we define the totally ordered set X' to be the set $\{x_1^{-1}, x_1, x_2^{-1}, x_2, \ldots\}$ with total order

$$x_1^{-1} < x_1 < x_2^{-1} < x_2 < \cdots$$

We introduce this new notation because now we would like to define the set $\mathbb{Z}' = \{\ldots, -2, -2^{-1}, -1, -1^{-1}, 0, 1^{-1}, 1, 2^{-1}, 2, \ldots\}$, with the total order

$$\cdots - 2 < -2^{-1} < -1 < -1^{-1} < 0 < 1^{-1} < 1 < 2^{-1} < 2 < \cdots$$

In general, if we recall the definition of $\pm X$ from Chapter 3, we have the total order on $\pm X'$ given by

$$\cdots - x_2 < -x_2^{-1} < -x_1 < -x_1^{-1} < x_0 < x_1^{-1} < x_1 < x_2^{-1} < x_2 < \cdots$$

In practice, however, we will usually refer only to $\mathbb{Z}'$ rather than the slightly more abstract $\pm X'$. We also have the special case for any positive integer k, $\pm[k]'$ has total order

$$-k < -k^{-1} < \cdots < -1 < -1^{-1} < 0 < 1^{-1} < 1 < \cdots < k^{-1} < k.$$

For any $x \in \pm X'$, let $\varepsilon(x)$ be the exponent on x, and let $|x|$ be a map from $\pm X' \to X$ that forgets signs and exponents. For example, if $x = -x_i^{-1}$, then $\varepsilon(x) = -1 < 0$ and $|x| = x_i$, while if $x = x_i$, then $\varepsilon(x) = 1 > 0$ and $|x| = x_i$. For $i = 0$, we require $\varepsilon(x_0) = 1 > 0$, $|x_0| = x_0$, and $-x_0 = x_0$.

Another way to think of $\mathbb{Z}'$ is as a total ordering of the integer points on the axes in $\mathbb{Z} \times \mathbb{Z}$:

$$\cdots (0,-2) < (-1,0) < (0,-1) < (0,0) < (0,1) < (1,0) < (0,2) \cdots$$

In particular, we have $(k,l) < (k',l')$ in $\mathbb{Z}'$ if $k+l < k'+l'$ (in $\mathbb{Z}$), if $k = l' < 0$ (in $\mathbb{Z}$), or if $l = k' > 0$ (also in $\mathbb{Z}$). We have $\varepsilon((k,0)) = 1$, $\varepsilon((0,k)) = -1$, and $|(k,l)| = |k+l|$. To negate a point we simply reflect across the perpendicular axis. Note that we could also use this model to understand $\mathbb{P}'$ from the previous chapter as all those points (i,j) with $i + j > 0$.

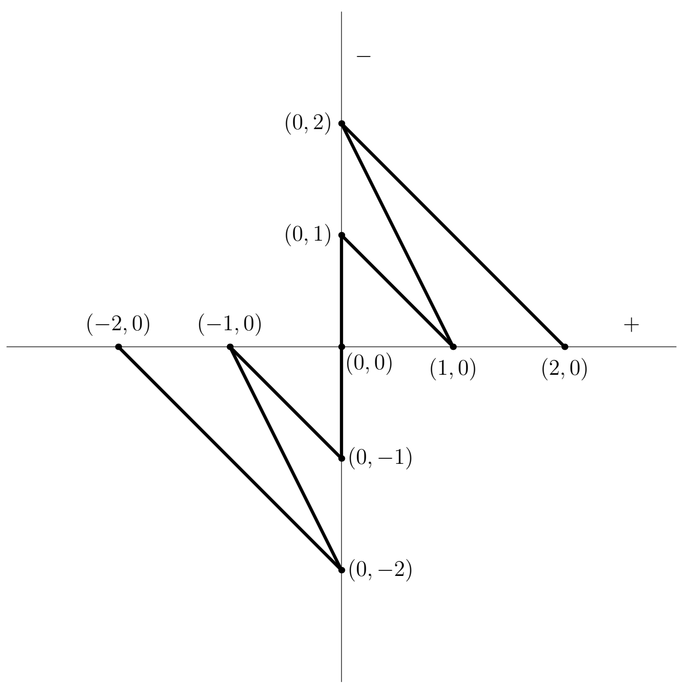


FIGURE 4.1. One realization of the total order on $\mathbb{Z}'$.

Now we will introduce our tool for studying the type B peak algebra.

DEFINITION 4.2.1. *For any* $\mathfrak{B}_n$ *poset* P*, an* enriched P-partition of type B *is a map* $f : \pm[n] \to \pm X'$ *such that for every* $i <_P j$*,*

- $f(i) \leq f(j)$
- $f(i) = f(j)$ *and* $\varepsilon(f(i)) > 0$ *only if* $i < j$ *in* $\mathbb{Z}$
- $f(i) = f(j)$ *and* $\varepsilon(f(i)) < 0$ *only if* $i > j$ *in* $\mathbb{Z}$
- $f(-i) = -f(i)$

As in the case of ordinary type B P-partitions, this definition varies from type A enriched P-partitions only in the last condition. Let $\mathcal{E}(P)$ denote the set of all type B enriched P-partitions. If we take X to have finite cardinality k, then define the *enriched order polynomial of type B*, denoted $\Omega'_P(k)$, to be the number of enriched P-partitions $f : P \to \pm X'$.

THEOREM 4.2.1. *The set of all type B enriched* P*-partitions is the disjoint union of all type B enriched* π*-partitions where* π *ranges over all linear extensions of* P*.*

$$\mathcal{E}(P) = \coprod_{\pi \in \mathcal{L}(P)} \mathcal{E}(\pi).$$

COROLLARY 4.2.1.

$$\Omega'_P(k) = \sum_{\pi \in \mathcal{L}(P)} \Omega'_\pi(k).$$

Notice that we can easily characterize the type B enriched π-partitions in terms of descent sets, keeping in mind that if we know where to map $\pi(i)$, then we know where to map $\pi(-i) = -\pi(i)$ by the reflexive property: $f(\pi(i)) = -f(-\pi(i))$. For any signed permutation $\pi \in \mathfrak{B}_n$ we have

$$\begin{aligned}\mathcal{E}(\pi) = \{\, f : [n] \to \pm X' \quad | \quad & x_0 \leq f(\pi(1)) \leq f(\pi(2)) \leq \cdots \leq f(\pi(n)), \\ & f(\pi(i)) = f(\pi(i+1)), \varepsilon(f(\pi(i))) > 0, \Rightarrow i \notin \mathrm{Des}(\pi), \\ & f(\pi(i)) = f(\pi(i+1)), \varepsilon(f(\pi(i))) < 0, \Rightarrow i \in \mathrm{Des}(\pi) \,\}\end{aligned}$$

As with type A enriched P-partitions, we will rephrase the classification above to look more like the case of ordinary P-partitions. Let $i \leq^+ j$ mean that $i < j$ in $\pm X'$ or $i = j$ and $\varepsilon(i) > 0$. Similarly define $i \leq^- j$ to mean that $i < j$ in $\pm X'$ or $i = j$ and $\varepsilon(i) < 0$. The set of all type B enriched π-partitions $f : \pi \to \pm X'$ is all solutions to

$$x_0 \leq^{\pm} f(\pi(1)) \leq^{\pm} f(\pi(2)) \leq^{\pm} \cdots \leq^{\pm} f(\pi(n)) \tag{29}$$

where $f(\pi(s)) \leq^- f(\pi(s+1))$ if $s \in \mathrm{Des}(\pi)$ and $f(\pi(s)) \leq^+ f(\pi(s+1))$ otherwise. Notice that since $\varepsilon(x_0) = 1$, then $x_0 \leq^- f(\pi(1))$ is the same as saying $x_0 < f(\pi(1))$, and $x_0 \leq^+ f(\pi(1))$ is the same as $x_0 \leq f(\pi(1))$.

While we have in some sense already said precisely what type B enriched order polynomials are, we need to give a few more properties of them. First of all, let $c_l(P)$ denote the number of type B enriched P-partitions f such that $\{\, |f(i)| : i = 1, 2, \ldots, n \,\} = [l]$ as sets, and let $c_l^0(P)$ denote the number of type B enriched P-partitions f such that $\{\, |f(i)| : i = 1, 2, \ldots, n \,\} = \{0\} \cup [l]$. Then we have the following formula for the type B enriched order polynomial:

$$\Omega'_P(k) = \sum_{l=1}^{n} \binom{k}{l} c_l(P) + \sum_{l=0}^{n-1} \binom{k}{l} c_l^0(P).$$

This formula quickly shows that the enriched order polynomial has degree n. Notice also that if $P = \pi$, a signed permutation with $\pi(1) < 0$, the second term vanishes since $c_l^0(\pi) = 0$ for all l. Notice also the similarity between this formula and that of the left order polynomial in the type A case.

We can derive the generating function for type B enriched order polynomials in much the same way as the type A case. It should be clear that type B enriched π-partitions depend only on the descent set of π. We will see that they depend precisely on the number of peaks and the sign of $\pi(1)$. We remark that while we are only concerned with the generating function for order polynomials of permutations, we can obtain the order polynomial generating function for any poset by summing the generating functions for its linear extensions. Let

$$\varsigma(\pi) = \frac{1 - \frac{\pi(1)}{|\pi(1)|}}{2}$$

so that $\varsigma(\pi) = 0$ if $\pi(1)$ is positive, $\varsigma(\pi) = 1$ if $\pi(1)$ is negative.

THEOREM 4.2.2. *We have the following generating function for enriched P-partitions:*

$$\begin{aligned} \sum_{k\geq 0} \Omega'_\pi(k)t^k &= \frac{(1+t)^n}{(1-t)^{n+1}} \cdot \left(\frac{2t}{1+t}\right)^{\varsigma(\pi)} \cdot \left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}(\pi)} \\ &= \left(\frac{1}{2}\right)^{\varsigma(\pi)} \cdot \frac{(1+t)^{n+\varsigma(\pi)}}{(1-t)^{n+1}} \cdot \left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}(\pi)+\varsigma(\pi)} \end{aligned} \tag{30}$$

Notice that this formula implies that $\Omega'_\pi(x)$ on *both* the number of peaks and the sign of $\pi(1)$. Notice also the similarity between this generating function and the generating functions for type A enriched order polynomials:

$$\text{(Interior peaks)} \quad \sum_{k\geq 0} \Omega'_{(A;\pi)}(k)t^k = \frac{1}{2}\frac{(1+t)^{n+1}}{(1-t)^{n+1}} \cdot \left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}(\pi)+1}$$

$$\text{(left peaks)} \quad \sum_{k\geq 0} \Omega^{(\ell)}_{(A;\pi)}(k)t^k = \frac{(1+t)^n}{(1-t)^{n+1}} \cdot \left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}^{(\ell)}(\pi)}$$

The proof that follows is understandably very similar to that of the type A case.

PROOF. Fix any permutation $\pi \in \mathfrak{B}_n$. We have the following formula for the generating function of ordinary order polynomials of type B (see, e.g., Reiner [**Rei93**]):

$$\sum_{k\geq 0} \Omega_\pi(k) t^k = \frac{t^{\operatorname{des}(\pi)}}{(1-t)^{n+1}}$$

From [**Pet**], we see that

$$\Omega'_\pi(k) = 2^{(\operatorname{pk}(\pi)+\varsigma(\pi))} \cdot \sum_{\substack{D\subset\{0\}\cup[n-1] \\ \operatorname{Pk}(\pi)\subset D\Delta(D+1) \\ \text{and } 0\in D \text{ if } \pi(1)<0}} \Omega_D(k),$$

where $\Omega_D(k)$ denotes the ordinary type B order polynomial of any signed permutation with descent set D. Putting these two facts together, we get:

$$\sum_{k\geq 0} \Omega'_\pi(k) t^k = \frac{2^{\operatorname{pk}(\pi)+\varsigma(\pi)}}{(1-t)^{n+1}} \cdot \sum_{\substack{D\subset\{0\}\cup[n-1] \\ \operatorname{Pk}(\pi)\subset D\Delta(D+1) \\ \text{and } 0\in D \text{ if } \pi(1)<0}} t^{|D|}$$

To obtain the generating function for the sets D by size, we proceed in two cases. If we don't require that 0 is in D, that is, if $\pi(1)$ is positive, then we get $(2t)^{\operatorname{pk}(\pi)}(1+t)^{n-2\operatorname{pk}(\pi)}$ exactly as in the type A case. If $\pi(1) < 0$, we have that 0 is always in D (and hence $|D| > 0$), while for any $j \in \operatorname{Pk}(\pi)$, j must be greater than 1 and exactly one of j or $j-1$ will be in D. There are $n - 2\operatorname{pk}(\pi) - 1$ remaining elements of $\{0\} \cup [n-1]$, and they can be included in D or not:

$$\begin{aligned} \sum_{\substack{D\subset\{0\}\cup[n-1] \\ \operatorname{Pk}(\pi)\subset D\Delta(D+1) \\ \text{and } 0\in D \text{ if } \pi(1)<0}} t^{|D|} &= t\,\underbrace{(t+t)(t+t)\cdots(t+t)}_{\operatorname{pk}(\pi)}\underbrace{(1+t)(1+t)\cdots(1+t)}_{n-2\operatorname{pk}(\pi)-1} \\ &= t(2t)^{\operatorname{pk}(\pi)}(1+t)^{n-2\operatorname{pk}(\pi)-1} \end{aligned}$$

Taking the two cases together, we can write

$$\sum_{\substack{D\subset\{0\}\cup[n-1]\\ \mathrm{Pk}(\pi)\subset D\triangle(D+1)\\ \text{and } 0\in D \text{ if } \pi(1)<0}} t^{|D|} = t^{\varsigma(\pi)}(2t)^{\mathrm{pk}(\pi)}(1+t)^{n-2\,\mathrm{pk}(\pi)-\varsigma(\pi)}$$

Finally, we get

$$\sum_{k\geq 0}\Omega'_\pi(k)t^k = \frac{(1+t)^n}{(1-t)^{n+1}}\cdot\left(\frac{2t}{1+t}\right)^{\varsigma(\pi)}\cdot\left(\frac{4t}{(1+t)^2}\right)^{\mathrm{pk}(\pi)}$$

as desired. □

4.3. The peak algebra of type B

We now move on to find orthogonal idempotents for the Eulerian peak algebra of the hyperoctahedral group. Let

$$\rho(x) = \sum_{\pi\in\mathfrak{B}_n}\Omega'_\pi((x-1)/4)\pi = \sum_{i=0}^{\lfloor n/2\rfloor}\Omega'_{i,+}((x-1)/4)E_i^+ + \Omega'_{i,-}((x-1)/4)E_i^-$$

where $\Omega'_{i,+}(x)$ is the order polynomial for any permutation π with i peaks and $\pi(1) > 0$, $\Omega'_{i,-}(x)$ is defined similarly for π such that $\pi(1) < 0$.

Theorem 4.3.1. *As polynomials in x and y with coefficients in the group algebra of the hyperoctahedral group, we have*

$$\rho(x)\rho(y) = \rho(xy). \tag{31}$$

We can let e'_i, $i = 0, 1, 2, \ldots, n$ be the coefficient of x^i in $\rho((x-1)/4) = \sum_{i=0}^{n} e'_i x^i$. Then we get a set of $n+1$ orthogonal idempotents since Theorem 4.3.1 gives $(e'_i)^2 = e'_i$ and $e'_i e'_j = 0$ if $i \neq j$. Therefore the Eulerian peak algebra of type B is a commutative subalgebra of dimension $n+1$.

Proof. This proof is nearly identical to the proofs of the analogous Theorem 3.3.1. By equating the coefficient of π on both sides of equation (31) it suffices to prove that for any permutation $\pi \in \mathfrak{B}_\mathfrak{n}$ and positive integers k, l, we have

$$\Omega'_B(\pi; 4kl + k + l) = \sum_{\sigma\tau=\pi} \Omega'_B(\sigma; k)\Omega'_B(\tau; l).$$

We will interpret $\Omega'_B(\pi; 4kl + k + l)$ as counting maps $f : \pi \to \pm[l]' \times \pm[k]'$, where we take the up-down order on $\pm[l]' \times \pm[k]'$. We count up the columns that have positive exponent and down columns with negative exponent. Notice that we can restrict our attention to all the points greater than or equal to $(0, 0)$, since everything else is determined by the symmetry property of type B enriched P-partitions: $f(-i) = -f(i)$. We consider $\Omega'_B(\pi; 4kl + k + l)$ to be the number of solutions to

$$(0, 0) \le (i_1, j_1) \le (i_2, j_2) \le \cdots \le (i_n, j_n) \le (l, k) \tag{32}$$

where $(i_s, j_s) \le^- (i_{s+1}, j_{s+1})$ if $s \in \mathrm{Des}(\pi)$ and $(i_s, j_s) \le^+ (i_{s+1}, j_{s+1})$ otherwise. For example, if $\pi = (-3, 1, -2)$, we will count the number of points $((i_1, j_1), (i_2, j_2), (i_3, j_3))$ such that

$$(0, 0) \le^- (i_1, j_1) \le^+ (i_2, j_2) \le^- (i_3, j_3) \le (l, k).$$

Here $(i, j) \le^+ (i', j')$ means $i < i'$, or if $i = i'$ with $\varepsilon(i) > 0$ and $j \le^+ j'$, or if $i = i'$ with $\varepsilon(i) < 0$ and $j \ge^- j'$. Similarly, $(i, j) \le^- (i', j')$ if $i < i'$, or if $i = i'$ with $\varepsilon(i) > 0$ and $j \le^- j'$, or if $i = i'$ with $\varepsilon(i) < 0$ and $j \ge^+ j'$.

Just as with the type A case, we will want to group the solutions to (32) into cases that we will count using enriched order polynomials. Here there are are 2^n cases, indexed by subsets of $[0, n-1]$. The grouping depends on π and proceeds as follows. Let $F = ((i_1, j_1), \ldots, (i_n, j_n))$ be any solution to (32), and fix $\pi(0) = i_0 = j_0 = 0$. For any $s = 0, 1, 2, \ldots, n-1$, if $\pi(s) < \pi(s+1)$, then $(i_s, j_s) \le^+ (i_{s+1}, j_{s+1})$, which falls

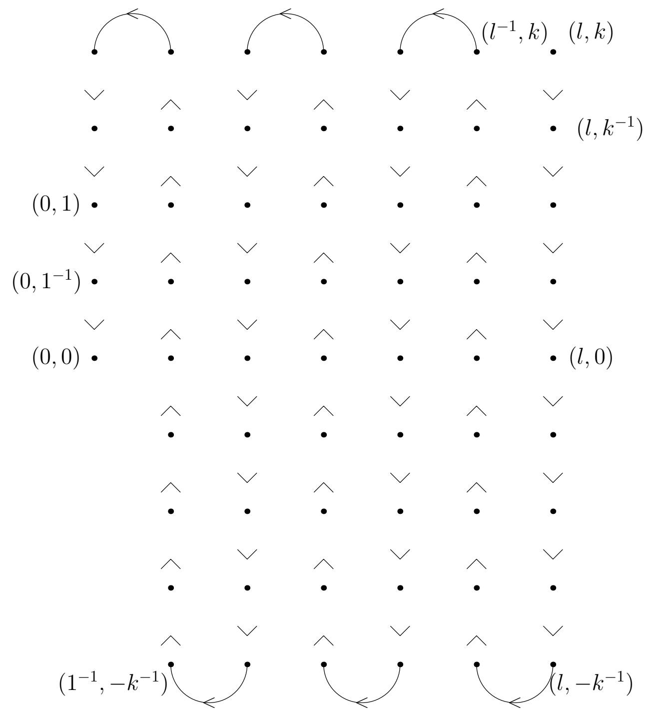


FIGURE 4.2. The up-down order on $\pm[l]' \times \pm[k]'$ with points greater than or equal to $(0, 0)$.

into one of two mutually exclusive cases:

$$i_s \leq^+ i_{s+1} \text{ and } j_s \leq^+ j_{s+1} \text{ or,} \tag{33}$$

$$i_s \leq^- i_{s+1} \text{ and } j_s \geq^- j_{s+1}. \tag{34}$$

If $\pi(s) > \pi(s+1)$, then $(i_s, j_s) \leq^- (i_{s+1}, j_{s+1})$, which we split into cases:

$$i_s \leq^+ i_{s+1} \text{ and } j_s \leq^- j_{s+1} \text{ or,} \tag{35}$$

$$i_s \leq^- i_{s+1} \text{ and } j_s \geq^+ j_{s+1}. \tag{36}$$

We define I_F to be the set of all s such that either (34) or (36) holds for F. Notice that in both cases, $i_s \leq^- i_{s+1}$. Now for any $I \subset [0, n-1]$, let S_I be the set of all solutions F to (32) satisfying $I_F = I$.

For any particular $I \subset [0, n-1]$, form the poset P_I of the elements $0, \pm 1, \pm 2, \ldots, \pm n$ by $\pi(s) <_{P_I} \pi(s+1)$ if $s \notin I$, $\pi(s) >_{P_I} \pi(s+1)$ if $s \in I$, where we extend all our relations by the symmetry property of type B posets. We form a zig-zag poset of n elements labeled consecutively by $0, \pi(1), \pi(2), \ldots, \pi(n)$ with downward zigs corresponding to the elements of I. So if $\pi = (-3, 1, -2)$ and $I = \{0, 2\}$, then our type B poset P_I is

$$2 >_{P_I} -1 <_{P_I} 3 >_{P_I} 0 >_{P_I} -3 <_{P_I} 1 >_{P_I} -2.$$

For any solution F in S_I, let $f : [n] \to \pm[k]'$ be defined by $f(\pi(s)) = j_s$. We will show that f is an enriched P_I-partition. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (33) tells us that $f(\pi(s)) = j_s \leq^+ j_{s+1} = f(\pi(s+1))$. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (35) tells us that $f(\pi(s)) = j_s \leq^- j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (34) gives us that $f(\pi(s)) = j_s \geq^- j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (36) gives us that $f(\pi(s)) = j_s \geq^+ j_{s+1} = f(\pi(s+1))$. In other words, we have verified that f is a P_I-partition. So for any particular solution in S_I, the n-tuple $(j_1, \ldots, j_n)$ can be thought of as an enriched P_I-partition.

Conversely, any enriched P_I-partition f gives a solution in S_I since if $j_s = f(\pi(s))$, then

$$((i_1, j_1), \ldots, (i_n, j_n)) \in S_I$$

if and only if $0 \leq i_1 \leq \cdots \leq i_n \leq l$ and $i_s \leq^- i_{s+1}$ for all $s \in I$, $i_s \leq^+ i_{s+1}$ for $s \notin I$. We can therefore turn our attention to counting enriched P_I-partitions, and the remainder of the argument follows the proof of Theorem 2.1.2. □

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