Algebraic approach to p-local structure of a finite group:

Definition 1 (Puig?) *Let G be a finite group and S be a Sylow subgroup of G. The fusion system of G is the category $\mathcal{F}_S(G)$, whose objects are the subgroups of S,*

$$ob\mathcal{F} = \{P \leq S\},$$

and whose morphism sets are given by

$$Hom_{\mathcal{F}(G)}(P, Q) = Hom_G(P, Q)$$

for all p-subgroups $P, Q \leq S$.

Topological approach: $BG_p^\wedge$

By the Martino-Priddy conjecture, these two approaches are equivalent.

Theorem 1 (Martino-Priddy,Oliver) *Let G and G' be finite groups with p-Sylow subgroups S and S', respectively. Then*

$$\mathcal{F}_S(G) \cong \mathcal{F}_{S'}(G') \Leftrightarrow BG^{\wedge}_p \simeq BG'^{\wedge}_p.$$

Proof: "⇐:" Proved by Martino and Priddy ('96) using homotopy theoretic construction.

"⇒:" Proved by Oliver ('01/'02) by showing vanishing of obstructions to uniqueness of classifying spaces. Uses classification of finite simple groups.

Puig formalized fusion systems as follows:

Definition 2 (Puig) *Let S be a finite p-group. A fusion system over S is a category $\mathcal{F}$, whose objects are the subgroups of S,*

$$ob\mathcal{F} = \{P \leq S\},$$

and whose morphism sets satisfy the following conditions

(a) $Hom_S(P,Q) \subseteq Hom_{\mathcal{F}}(P,Q) \subseteq Inj(P,Q)$ *for all $P, Q \leq S$.*

(b) *Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.*

Terminology: Let $P \leq S$.

- Say P' is $\mathcal{F}$-*conjugate* to P if P' is isomorphic to P in $\mathcal{F}$.

- Say P is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for every $P' \leq S$ which is $\mathcal{F}$-conjugate to P.

- Say P is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for every $P' \leq S$ which is $\mathcal{F}$-conjugate to P.

- Say P is *centric* in $\mathcal{F}$ if $C_S P' \leq P'$for every P' which is $\mathcal{F}$-conjugate to P.

Definition 3 (Puig) *A fusion system $\mathcal{F}$ over a p-group S is saturated if the following two conditions hold:*

(I) *If $P \leq S$ is fully normalized in $\mathcal{F}$, then P is also fully centralized and $Aut_S(P)$ is a Sylow subgroup of $(Aut_{\mathcal{F}}(P))$.*

(II) *If $P \leq S$ and $\varphi \in Hom_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, then φ extends to $\bar{\varphi} \in Hom_{\mathcal{F}}(N_\varphi, S)$, where*

$$N_\varphi = \{g \in N_S(P) \mid \varphi \circ c_g \circ \varphi^{-1} \in Aut_S(\varphi P)\}.$$

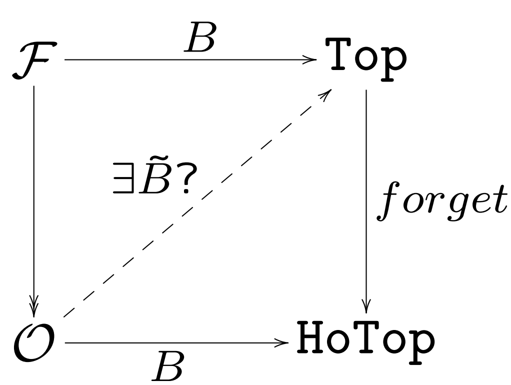
Condition (I) is a "prime-to-p condition". Condition (II) is a "maximum extension" condition.

We would like to form classifying spaces for arbitrary saturated fusion systems. Need to quotient out inner automorphisms of S before taking homotopy colimit.

Definition 4 (BLO) *The orbit category of a fusion system $\mathcal{F}$ over a p-group S is the category $\mathcal{O}(\mathcal{F})$ whose objects are the subgroups of S and whose morphisms are defined by*

$$Mor_{\mathcal{O}(\mathcal{F})}(P,Q) := Inn(Q)\backslash Hom_{\mathcal{F}}(P,Q).$$

We need a lifting $\tilde{B}\colon \mathcal{O}(\mathcal{F}) \to \texttt{Top}$ of the homotopy functor $B\colon \mathcal{O}(\mathcal{F}) \to \texttt{HoTop}$ in the following diagram:



Then $\underrightarrow{\text{Holim}}_{\mathcal{O}(\mathcal{F})}\ \tilde{B}$ is a classifying space for $\mathcal{F}$.

Definition 5 (BLO) *Let $\mathcal{F}$ be a fusion system over the p-group S. A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$, whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor*

$$\pi : \mathcal{L} \to \mathcal{F}^c,$$

and distinguished monomorphisms $P \xrightarrow{\delta_P} Aut_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions.

(A) *The functor π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, the center $Z(P)$ acts freely on $Mor_{\mathcal{L}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq Aut_{\mathcal{L}}(P)$), and π induces a bijection*

$$Mor_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\cong} Hom_{\mathcal{F}}(P,Q).$$

(B) *For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in Aut_{\mathcal{L}}(P)$ to $c_g \in Aut_{\mathcal{F}}(P)$.*

(C) *For each $f \in Mor_{\mathcal{L}}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:*

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \big\downarrow{\scriptstyle \delta_P(g)} & & \big\downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{f} & Q. \end{array}$$

Definition 6 (BLO) *A p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system over a finite p-group S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$.*

The classifying space of the p-local finite group is the p-completed geometric realization $|\mathcal{L}|_p^\wedge$.

A p-local finite group comes equipped with a natural inclusion

$$\theta\colon BS \longrightarrow |\mathcal{L}|_p^\wedge.$$

Let $\mathcal{F}$ be a saturated fusion system over S. Define a functor

$$\mathcal{Z} = \mathcal{Z}_{\mathcal{F}} : \mathcal{O}^c(\mathcal{F})^{op} \longrightarrow \mathtt{Ab},$$

by setting $\mathcal{Z}_{\mathcal{F}}(P) = Z(P)$ and

$$\mathcal{Z}_{\mathcal{F}}(P \xrightarrow{\varphi} Q) = (Z(Q) \xrightarrow{incl} Z(\varphi(P)) \xrightarrow{\varphi^{-1}} Z(P)).$$

Proposition 1 (BLO) *There is an obstruction class* $\eta(\mathcal{F}) \in \varprojlim^3_{\mathcal{O}^c(\mathcal{F})} (\mathcal{Z})$ *to the existence of a centric linking system associated to* $\mathcal{F}$*.*

The group $\varprojlim^2_{\mathcal{O}^c(\mathcal{F})} (\mathcal{Z})$ *acts freely and transitively on the set of isomorphism classes of centric linking systems associated to* $\mathcal{F}$ *(if any exist).*

Exercise: Let S be a finite abelian p-group.

1. Classify all saturated fusion systems over S

2. Classify all p-local finite groups over S

Part 1. should be accessible to anyone with a basic knowledge of group theory.
Part 2. requires some, albeit simple, methods from group cohomology.