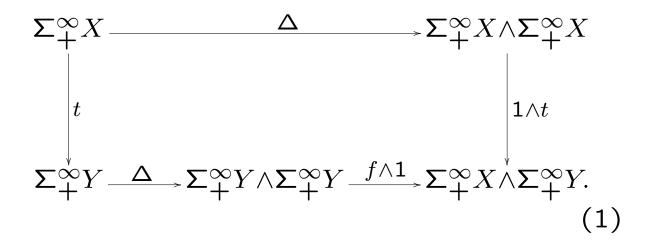
Definition 1 A map $f: Y \to X$ between connected spaces is called a homotopy monomorphism at p if its homotopy fibre F is $B\mathbf{Z}/p$ -local for every choice of basepoint.

In the special case where Y = BP is the classifying space of a finite *p*-group, we say that *f* is a *p*-subgroup inclusion.

Proposition 1 (Folklore) Let $f: Y \to X$ be a map between two *p*-complete spaces, with Noetherian cohomology rings. Then *f* is a homotopy monomorphism if and only if the induced map in cohomology makes $H^*(Y; \mathbf{F}_p)$ a finitely generated $H^*(X; \mathbf{F}_p)$ -module. **Definition 2** Let $f: Y \to X$ be a map of spaces. A Frobenius transfer of f is a stable map $t: \Sigma^{\infty}_{+}X \to \Sigma^{\infty}_{+}Y$ such that

$$\Sigma^{\infty}_{+} f \circ t \simeq id_{\Sigma^{\infty}_{+} X}$$

and the following diagram commutes up to homotopy



Definition 3 A Frobenius transfer triple over a finite p-group S is a triple (f, t, X), where

X is a connected, p-complete space with finite fundamental group

f is a subgroup inclusion $BS \to X$

t is a Frobenius transfer for f.

In general, given a map $f: BS \to X$, we get a fusion system $\mathcal{F}_{S,f}(X)$ over S by putting

 $Hom_{\mathcal{F}_{S,f}}(P,Q) = \{\varphi \in Inj(P,Q) \mid f|_{BP} \simeq f|_{BQ} \circ B\varphi\}$ for each $P,Q \leq S$.

For such a fusion system $\mathcal{F}_{S,f}$ we pose the following questions:

• Is $\mathcal{F}_{S,f}$ saturated?

- Does it have an associated centric linking system *L*? If so, is it unique?
- What is the relationship between $BS \xrightarrow{f} X$ and $BS \xrightarrow{\theta} |\mathcal{L}|_p^{\wedge}$? Are they equivalent as objects under BS.

In these lectures, these questions will be answered in a special case.

Theorem 1 Let S be a finite elementary abelian p-group. Let (f,t,X) be a Frobenius transfer triple over S and put $W := Aut_{\mathcal{F}_{S,f}(X)}(S)$. Then the following hold

- $\mathcal{F}_{S,f}(X)$ is equal to the saturated fusion system $\mathcal{F}_{S}(W \ltimes S)$
- $\mathcal{F}_{S,f}(X)$ has a unique associated centric linking system with classifying space $B(W \ltimes S)_p^{\wedge}$.
- There is a natural equivalence $B(W \ltimes S)_p^{\wedge} \xrightarrow{\simeq} X$ of objects under BS.

Conversely, we will show the following for p-local finite groups over *any* finite p-group S:

Theorem 2 Let $(S, \mathcal{F}, \mathcal{L})$ be a *p*-local finite group. Then the natural inclusion

$$\theta \colon BS \longrightarrow |\mathcal{L}|_p^{\wedge}.$$

has a Frobenius transfer

$$t: \Sigma^{\infty}_{+} |\mathcal{L}|_{p}^{\wedge} \to \Sigma^{\infty}_{+} BS$$

making (θ, t, X) a Frobenius transfer triple over S.

In the course of the proof of Theorem 2, we also obtain the following interesting results:

Theorem 3 (The next best thing) There is a functorial assignment

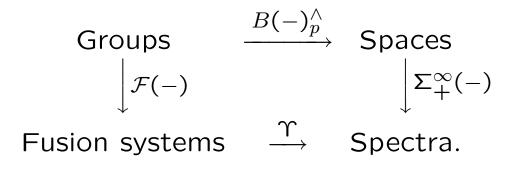
$$\Upsilon \colon (S, \mathcal{F}) \longmapsto \Sigma^{\infty}_{+} BS \xrightarrow{\sigma_{\mathcal{F}}} B\mathcal{F}$$

of a classifying spectrum to every saturated fusion system, such that:

 (S,\mathcal{F}) can be recovered from $(\sigma_{\mathcal{F}}, B\mathcal{F})$.

- $\sigma_{\mathcal{F}}$ admits a "transfer" $t_{\mathcal{F}}: B\mathcal{F} \xrightarrow{\sigma_{\mathcal{F}}} \Sigma^{\infty}_{+}BS$, such that $\sigma_{\mathcal{F}} \circ t_{\mathcal{F}} \simeq id_{B\mathcal{F}}$.
- $(\sigma_{\mathcal{F}}, B\mathcal{F})$ agrees with $(\Sigma_{+}^{\infty}\theta, \Sigma_{+}^{\infty}|\mathcal{L}|_{p}^{\wedge})$ in the case of *p*-local finite groups.
- There is a theory of transfers for "injective" morphisms between saturated fusion systems.

The functor Υ also agrees with the *p*-completed stable classifying spaces of finite groups. That is, the following diagram of functors commutes:

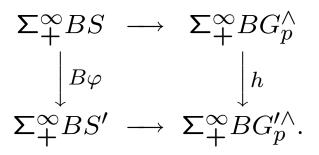


In view of Oliver's solution of the Martino-Priddy conjecture, we get the following Corollary.

Corollary 1 Let G and G' be finite groups with Sylow subgroups S and S', respectively. Then the following are equivalent

(i) $BG_p^{\wedge} \simeq BG_p^{\prime \wedge}$

(ii) There is an isomorphism $\varphi : S \to S'$ and a stable equivalence $h: \Sigma^{\infty}_{+}BG^{\wedge}_{p} \to \Sigma^{\infty}_{+}BG'^{\wedge}_{p}$ making the following diagram commute



Using the transfer theory, we also obtain the following corollary

Corollary 2 Let \mathcal{F} and \mathcal{F}' be saturated fusion systems over a finite p-group S. Then $\mathcal{F} \subset \mathcal{F}'$ if and only if $B\mathcal{F}'$ is a stable summand of $B\mathcal{F}$ (as objects under $\Sigma^{\infty}_{+}BS$).

Let S be a finite abelian p-group and \mathcal{F} be a fusion system over S. Then we get the following simplifications:

• Every $P \leq S$ is both fully centralized and fully normalized, since $C_S(P) = N_S(P) = S$.

•
$$Aut_S(P) = \{id\}$$
 for all $P \leq S$.

• For
$$P \leq S$$
 and $\varphi \in Hom_{\mathcal{F}}(P, S)$, we have

$$N_{\varphi} = \{g \in N_{S}(P) \mid \varphi \circ c_{g} \circ \varphi^{-1} \in Aut_{S}(\varphi P)\}$$

$$= \{g \in S \mid \varphi \circ id \circ \varphi^{-1} \in \{id\}\}$$

$$= S.$$

• Since $C_S(P) = S$ for every $P \leq S$, the only \mathcal{F} -centric subgroup is S itself.

Lemma 1 Let \mathcal{F} be a fusion system over a finite abelian p-group S. Then \mathcal{F} is saturated if and only if the following two conditions are satisfied:

 (\mathbf{I}_{ab}) $|Aut_{\mathcal{F}}(S)|$ is prime to p.

(II_{*ab*}) Every $\varphi \in Hom_{\mathcal{F}}(P,Q)$ is the restriction of some $\tilde{\varphi} \in Aut_{\mathcal{F}}(S)$. Therefore, the only saturated fusion systems over S are the ones coming from semi-direct products $W \ltimes S$, where $W \leq Aut(S)$ has order prime to p.

These have a canonical classifying space $B(W \ltimes S)_p^{\wedge}$. The obstructions to existence and uniqueness of classifying spaces reduces to group cohomology

 $H^*(W;S),$

which vanishes by a transfer argument. Hence the classifying space is unique.

Proposition 2 If S is an abelian finite p-group, then the assignment

 $W \mapsto (S, \mathcal{F}(W \ltimes S), \mathcal{L}_S^c(W \ltimes S))$

gives a bijective correspondence between subgroups $W \leq Aut(S)$ of order prime to p and p-local finite groups over S. In particular, there are no exotic p-local finite groups over S. Outline of proof of Theorem 1:

1. Use a theorem of Adams and Wilkerson to show that

$$H^*(X) = H^*(BS)^W = H^*(B(W \ltimes S)_p^{\wedge}),$$

for some $W \leq Aut(S)$ of order prime to p.

2. Use Lannes's theorem to deduce that

$$\mathcal{F}_{S,f}(X) = \mathcal{F}_S(W \ltimes S)$$

3. By classification of *p*-local finite groups, we have a unique classifying space $B(W \ltimes S)_p^{\wedge}$ for $\mathcal{F}_{S,f}(X)$. We use Wojtkowiak's obstruction theory to produce an equivalence

$$B(W \ltimes S)_p^{\wedge} \xrightarrow{\simeq} X$$

of objects under BS.