First test: Cohomology

In [BLO2] it is shown that the classifying space $|\mathcal{L}|_p^\wedge$ of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ has cohomology

$$H^*(|\mathcal{L}|_p^\wedge) = H^*(\mathcal{F}) := \varprojlim_{\mathcal{O}(\mathcal{F})} H^*(B(-)).$$

Fix a Frobenius transfer triple (f, t, X) over an elementary abelian p-group S. If

$$\mathcal{F} := \mathcal{F}_{S,f}(X)$$

is saturated, then (by the classification)

$$H^*(\mathcal{F}) = H^*(BS)^W,$$

where $W := Aut_{\mathcal{F}}(S) \leq Aut(S)$ has order prime to p.

Necessary Condition 1 *If X is the classifying space of a p-local finite group over an elementary abelian group S, then*

$$H^*(X) = H^*(BS)^W$$

for some subgroup $W \leq Aut(S)$ of order prime to p.

Applying the cohomology functor $H^*(-;\mathbf{F}_p)$ to a Frobenius transfer triple (f,t,X) we get maps

$$H^*(X) \xrightarrow{f^*} H^*(BS) \xrightarrow{t^*} H^*(X)$$

with the following properties:

CohI $t^* \circ f^* = id$.

CohII t^* is $H^*(X)$-linear by the Frobenius reciprocity property.

CohIII t^* is a morphism of unstable modules over the Steenrod algebra.

CohIV f^* is a morphism of unstable algebras over the Steenrod algebra.

Hence $H^*(X)$ is a direct summand of $H^*(BS)$ as a $H^*(X)$-module and as a module over the Steenrod algebra.

Finiteness conditions:

Fact 1 *If S is a finite p-group, then $H^*(BS)$ is Noetherian.*

Lemma 1 *Let S be a finite p-group and (f, t, X) be a Frobenius transfer triple over S. Then $H^*(X)$ is Noetherian and in particular X is of $\mathbf{F}_p$-finite type.*

Proof: Uses CohI, CohII and an algebraic lemma of Dwyer-Wilkerson.

Lemma 2 *Let S be a finite p-group and (f, t, X) be a Frobenius transfer triple over S. Then X is of $\mathbf{Z}_{(p)}$-finite type.*

Proof: By the Universal coefficient theorem, it suffices to prove this for $\mathbf{F}_p$ and $\mathbf{Q}$-coefficients. The former is done above. By a transfer argument, $H^*(BS; \mathbf{Q}) = \mathbf{Q}$, proving the latter.

Notation:

$\mathcal{A}^* = mod\ p$ Steenrod Algebra.

$\mathcal{U}$ = category of unstable modules over $\mathcal{A}^*$.

$\mathcal{U}'$ = full subcategory of $\mathcal{U}$ with evenly graded objects.

$\mathcal{K}$ = category of unstable algebras over $\mathcal{A}^*$.

$\mathcal{K}'$ = full subcategory of $\mathcal{K}$ with evenly graded objects.

In all cases, morphisms are of degree zero.

Unstable condition:

$$P^k(x) = \begin{cases} x^p & \text{if } |x| = 2k, \\ 0 & \text{if } |x| < 2k. \end{cases}$$

In "Finite H-Spaces and Algebras", Adams and Wilkerson study the following category. $\mathcal{AW}$ = full subcategory of $\mathcal{K}'$ whose objects are integral domains.

They make precise the notions of "algebraic extension" and "algebraic closure" and prove the following:

Proposition 1 (Adams-Wilkerson) *Every object R^* in $\mathcal{AW}$ has an algebraic closure H^* in $\mathcal{AW}$. If R^* has finite transcendence degree, then so does H^*.*

Theorem 1 (Adams-Wilkerson) *The objects H^* in $\mathcal{AW}$, that are algebraically closed and of finite transcendence degree are precisely the polynomial algebras $\mathbf{F}_p[x_1, \ldots, x_n]$ on generators x_i of degree 2.*

The theorem we wish to apply is the following.

Theorem 2 (Adams-Wilkerson) *Let R^* be an algebra in $\mathcal{AW}$ of finite transcendence degree and let $H^* = \mathbf{F}_p[x_1, \ldots, x_n]$ be the algebraic closure in $\mathcal{AW}$. In order that R^* should admit an isomorphism*

$$R^* \cong (H^*)^W,$$

for some group W of automorphisms of H^*, the following two conditions are necessary and sufficient:

AW1 *The integral domain R^* is integrally closed in its field of fractions.*

AW2 *If $y \in R^{2dp}$ and $Q^r y = 0$ for each $r \geq 1$, then $y = x^p$ for some $x \in R^{2d}$.*

The second condition is really an “inseparably closed” condition. The operation Q^r is the Milnor primitive of dimension $2p^r - 2$ in $\mathcal{A}^*$. (Not to be confused with Q_r.)

Problem: When S elementary abelian,

$$H^*(BS) \cong E[y_1, \ldots, y_n] \otimes \mathbf{F}_p[x_1, \ldots, x_n],$$

where $|y_i| = 1$ and $|x_i| = 2$. Not evenly graded!

Solution: The forgetful functor $\theta\colon \mathcal{K}' \to \mathcal{K}$ has a right adjoint $\tilde{\theta}\colon \mathcal{K} \to \mathcal{K}'$.

By Lannes-Zarati:

$$\tilde{\theta} H^*(BS) \cong \mathbf{F}_p[x_1, \ldots, x_n].$$

By Goerss-Smith-Zarati, when we restrict ourselves to elements of $\mathcal{K}$, whose images in $\mathcal{U}$ are reduced injectives, we can move freely between $\mathcal{K}$ and $\mathcal{K}'$ via θ and $\tilde{\theta}$. Simply put, the reason is that morphisms between such elements are determined on the even graded part.

(An injective M in $\mathcal{U}$ is *reduced* if $Hom_{\mathcal{U}}(\Sigma N, M) = 0$ for every N in $\mathcal{U}$.)

Furthermore, Lannes has shown that the elements in $\mathcal{K}$, whose image in $\mathcal{U}$ are reduced injectives are precisely those that are isomorphic to direct summands of cohomology rings of elementary abelian groups. In particular, $H^*(BS)$ and $H^*(X)$ are reduced injectives.

Proposition 2 (Goerss-Smith-Zarati) *If K_1 and K_2 are two unstable Steenrod algebras, whose images in $\mathcal{U}$ are reduced $\mathcal{U}$-injectives, then K_1 is isomorphic to K_2 in $\mathcal{K}$ if and only if $\tilde{\theta}K_1$ is isomorphic to $\tilde{\theta}K_2$ in $\mathcal{K}'$.*

Lemma 3 *Let S be a finite elementary abelian p-group. Then there are isomorphisms*

$$\begin{array}{c} Aut(S) \\ {\scriptstyle H^*(-)}\Big\downarrow \cong \\ Aut_{\mathcal{K}}(H^*(BS)) \\ {\scriptstyle \tilde{\theta}}\Big\downarrow \cong \\ Aut_{\mathcal{K}'}(\tilde{\theta}H^*(BS)). \end{array}$$

Proof: The algebra automorphisms are determined on

$$H^2(BS) = H^2(\tilde{\theta}H^*(BS)) = S.$$

Proposition 3 *Let* (f, t, X) *be a Frobenius transfer triple over an elementary abelian* p*-group* S *and let* $W \leq Aut(S)$ *be the subgroup of automorphisms acting on* $H^*(X)$ *by the identity. Then*

$$H^*(X) = H^*(BS)^W$$

and W *has order prime to* p*.*

Proof: Put

$$R^* := \tilde{\theta} H^*(X)$$

and

$$H^* := \tilde{\theta} H^*(BS) = \mathbf{F}_p[x_1, \ldots, x_n].$$

Then H^* is the algebraic closure of R^* in $\mathcal{AW}$. (Recall that $R^* \to H^*$ is a finite extension). This allows us to apply Adams-Wilkerson.

Proof of AW1:

-Take an integral $x \in Fr(R^*)$.

-Write $x = a/b$, with $a, b \in R^*$.

-x is also integral over H^*.

-Since H^* is integrally closed, we get $x \in H^*$.

-Now have $a = bx$ in H^*.

-Apply t^* and use R^*-linearity (CohII):

$$a = t^*(a) = t^*(bx) = bt^*(x).$$

- Since H^* is an integral domain, this implies

$$x = a/b = t^*(x) \in R^*.$$

Proof of AW2:

-Take $y \in R^{2dp}$ such that $Q^r y = 0$ for all $r \geq 1$.

-$H^* = H^{\{id\}}$ satisfies AW2.

-Therefore there is an $x \in H^{2d}$ such that $x^p = y$.

-Apply t^* and preservation of Steenrod operations (CohIII):

$$y = t^*(y) = t^*(x^p) = t^*(P^d x) = P^d t^*(x) = (t^* x)^p.$$

-Since $t^*(x) \in R^{2d}$ we are done.

Have now shown that $R^* = (H^*)^W$ for some subgroup W of $Aut(S)$. Since R^* is a direct summand of H^* as unstable modules over the Steenrod algebra, a result of Lannes (which appears in the paper by Goerss-Smith-Zarati) shows that W must have order prime to p.

The right adjoint $\tilde{\theta}$ preserves inverse limits and in particular rings of invariants. We can therefore carry our results back over to $\mathcal{K}$ and get

$$H^*(X) \cong H^*(BS)^W.$$

Finally, it is easy to see that W may be replaced by the group of automorphisms of S acting on $H^*(X)$ by the identity. (A posteriori this step is unnecessary).

Lannes's theorem allows us to carry the cohomological information over to homotopy:

Theorem 3 (Lannes) *Let X be a connected space and V a finite elementary abelian p-group. Suppose that X is nilpotent, that $\pi_1 X$ is finite and that $H^*(X)$ is of finite type. Then the natural map*

$$[BV, X] \to Hom_{\mathcal{K}}(H^*(X), H^*(BV))$$

is a bijection.

The case where $H^*(X) = U(M)$ for a module M is due to Miller.

Lannes's theorem applies to Frobenius transfer triples: Already know that $\pi_1(X)$ is finite and that $H^*(X)$ is of finite type. Since X is also p-complete, $\pi_1(X)$ is a finite p-group and therefore X is nilpotent.

Remark: The reasoning in the last sentence is false, although the statement may still be true. In the worst case, we will assume that X is nilpotent as part of the definition of a Frobenius transfer triple.

Lemma 4 *Let (f, t, X) be a Frobenius transfer triple over a finite elementary abelian p-group S and let W be the subgroup of $Aut(S)$ acting on $H^*(X)$ by the identity. Then*

$$Aut_{\mathcal{F}_{f,S}(X)}(S) = W$$

.

Proof: The map f induces the inclusion

$$H^*(BS)^W \hookrightarrow H^*(BS)$$

in cohomology. Now,

$$\begin{aligned} \varphi &\in Aut_{\mathcal{F}_{f,S}(X)}(S) \\ &\Updownarrow \\ f \circ B\varphi &\simeq f \\ &\Updownarrow \\ B\varphi^* \circ f^* &= f^* \\ &\Updownarrow \\ \varphi &\in W. \end{aligned}$$

Proposition 4 *Let* (f, t, X) *be a Frobenius transfer triple over a finite elementary abelian group* S *and put* $W := Aut_{\mathcal{F}_{f,S}}(S)$*. Then* W *has order prime to* p *and* $\mathcal{F}_{f,S}$ *is equal to the fusion system* $\mathcal{F}_S(W \ltimes S)$*. In particular* $\mathcal{F}_{f,S}$ *is saturated.*

Proof: The maps f and $\theta\colon BS \to B(W \ltimes S)^\wedge_p$ both induce the inclusion

$$H^*(X) = H^*(BS)^W \hookrightarrow H^*(BS)$$

in cohomology.

Let V and V' be two subgroups in S. Then

$$\begin{array}{c}
\varphi \in Hom_{\mathcal{F}_{f,S}(X)}(V,V') \\
\Updownarrow \\
f \circ B\iota_{V'} \circ B\varphi \simeq f \circ B\iota_V \\
\Updownarrow \\
B\varphi^* \circ B\iota_{V'}^* \circ f^* = B\iota_V^* \circ f^* \\
\Updownarrow \\
B\varphi^* \circ B\iota_{V'}^* \circ \theta^* = B\iota_V^* \circ \theta^* \\
\Updownarrow \\
\theta \circ B\iota_{V'} \circ B\varphi \simeq \theta \circ B\iota_V \\
\Updownarrow \\
\varphi \in Hom_{\mathcal{F}_{\theta,S}(B(W \ltimes S)^\wedge_p)}(V,V') \\
\Updownarrow \\
\varphi \in Hom_{\mathcal{F}_{\theta,S}(W \ltimes S)}(V,V').
\end{array}$$

We have now shown that (f, t, X) induces a saturated fusion system $\mathcal{F}_S(W \ltimes S)$, which has a classifying space $B(W \ltimes S)^\wedge_p$. It only remains to show that X is equivalent to $B(W \ltimes S)^\wedge_p$, as objects under BS. This can be achieved by applying Wojtkowiak's obstruction theory, presented here in a special case.

Theorem 4 (Wojtkowiak) *Let S be a finite abelian p-group and W a group of order prime to p, that acts on S. For any nilpotent p-local space X of $\mathbf{Z}_{(p)}$-finite type with trivial W-action, the natural map*

$$[B(W \ltimes S), X] \xrightarrow{-\circ Bi_S} [BS, X]^W$$

is a bijection.

Proposition 5 *Let (f, t, X), S and W be as before. Let θ be the p-completed inclusion*

$$\theta\colon BS \to B(W \ltimes S)^{\wedge}_{p}.$$

There is an equivalence

$$h\colon (\theta, B(W \ltimes S)^{\wedge}_{p}) \to (f, X)$$

of spaces under BS.

Proof: By Wojtkowiak there is a bijection

$$[B(W \ltimes S), X] \xrightarrow{-\circ B\iota} [BS, X]^{W},$$

where $B\iota$ is the inclusion $BS \hookrightarrow B(W \ltimes S)$ (before completion). Since $f \in [BS, X]^{W}$ by definition of W, we get a map h fitting into the following diagram.

$$\begin{array}{ccc} & BS & \\ {\scriptstyle B\iota}\swarrow & & \searrow{\scriptstyle f} \\ B(W \ltimes S) & \overset{\exists h}{\dashrightarrow} & X. \end{array}$$

Applying the cohomology functor, we get

$$\begin{array}{ccc} & H^*(BS) & \\ \nearrow & & \nwarrow \\ H^*(BS)^W & \xleftarrow{\;h^*\;} & H^*(BS)^W. \end{array}$$

We conclude that h^* must be an isomorphism. Upon p-completion, we now get a homotopy equivalence

$$h_p^\wedge \colon B(W \ltimes S)_p^\wedge \xrightarrow{\simeq} X_p^\wedge = X$$

fitting into the diagram

$$\begin{array}{ccc} & BS & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle \theta} \\ B(W \ltimes S)_p^\wedge & \xrightarrow[\simeq]{h_p^\wedge} & X. \end{array}$$