We have proved that a Frobenius transfer triple over an elementary abelian group S induces a p-local finite group over S.

Conversely we will now show that for a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ over *any* finite p-group S, the natural inclusion $\theta\colon BS \to |\mathcal{L}|^\wedge_p$ has a Frobenius transfer t, which makes $(\theta, t, |\mathcal{L}|^\wedge_p)$ a Frobenius transfer triple over S.

In the process of doing so we develop tools which allows us to prove properties of classifying spectra for saturated fusion systems, as defined in [BLO2], which are probably of greater interest than our original goal.

Given a saturated fusion system $\mathcal{F}$ over a finite p-group S, it is shown in [BLO2] how to associate a summand $\mathbb{B}\mathcal{F}$ of $\Sigma^{\infty}_{+}BS$ to $\mathcal{F}$. Furthermore, it is shown that if $\mathcal{F}$ has an associated linking system $\mathcal{L}$, then

$$\mathbb{B}\mathcal{F} \simeq \Sigma^{\infty}_{+}|\mathcal{L}|^{\wedge}_{p}.$$

Therefore Broto-Levi-Oliver refer to $\mathbb{B}\mathcal{F}$ as the *classifying spectrum* of $\mathcal{F}$.

We will enrich the classifying spectrum with a *structure map*

$$\sigma_{\mathcal{F}}\colon \Sigma^{\infty}_{+}BS \longrightarrow \mathbb{B}\mathcal{F}$$

and a *transfer*

$$t_{\mathcal{F}}\colon \mathbb{B}\mathcal{F} \longrightarrow \Sigma^{\infty}_{+}BS$$

such that

$$\sigma_{\mathcal{F}} \circ t_{\mathcal{F}} \simeq id_{\mathbb{B}\mathcal{F}}.$$

(Interlude:)
Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems over finite p-groups S_1 and S_2, respectively.
Definition. *A group homomorphism*

$$\gamma\colon S_1 \longrightarrow S_2$$

is $(\mathcal{F}_1, \mathcal{F}_2)$-fusion-preserving *if there exists a functor*

$$F_\gamma\colon \mathcal{F}_1 \longrightarrow \mathcal{F}_2$$

such that

$$F_\gamma(P) = \gamma(P)$$

for all $P \leq S$ *and*

$$\gamma|_Q \circ \varphi = F_\gamma(\varphi) \circ \gamma|_P$$

for all $\varphi \in$ *Hom*$_{\mathcal{F}_1}(P, Q)$.

We will prove that classifying spectra of saturated fusion systems have the following properties:

(A) $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ if and only if

$$\sigma_{\mathcal{F}} \circ \Sigma^{\infty}_{+} B\iota_Q \circ \Sigma^{\infty}_{+} B\varphi \simeq \sigma_{\mathcal{F}} \circ \Sigma^{\infty}_{+} B\iota_P.$$

(B) A $(\mathcal{F}_1, \mathcal{F}_2)$−fusion-preserving homomorphism $\gamma\colon S_1 \to S_2$ induces a map $\mathbb{B}\gamma^{\mathcal{F}_2}_{\mathcal{F}_1}$ making the following diagram commute up to homotopy:

$$\begin{array}{ccc} BS_1 & \xrightarrow{B\gamma} & BS_2 \\ \Big\downarrow{\scriptstyle \sigma_{\mathcal{F}_1}} & & \Big\downarrow{\scriptstyle \sigma_{\mathcal{F}_2}} \\ \mathbb{B}\mathcal{F}_1 & \xrightarrow{\mathbb{B}\gamma} & \mathbb{B}\mathcal{F}_2. \end{array}$$

This assignment is functorial and extends the group-theoretic case.

(C) Fusion-preserving monomorphisms admit transfers, which behave like transfers between groups in cohomology.

We reach our original goal by the following:

(D) When $\mathcal{F}$ has an associated centric linking system $\mathcal{L}$, the natural inclusion

$$\theta\colon BS \to |\mathcal{L}|_p^\wedge$$

suspends to the structure map

$$\sigma_{\mathcal{F}} : \Sigma_+^\infty BS \to \mathbb{B}\mathcal{F}$$

and $t_{\mathcal{F}}$ is a Frobenius transfer for θ.

Finally, we will also give an explicit basis for the $\mathbf{Z}_p^\wedge$-modules $[\mathbb{B}\mathcal{F}_1, \mathbb{B}\mathcal{F}_2]$.

Preliminaries on the Segal conjecture

We need to have a good understanding of the group $\{BS_{1+}, BS_{2+}\}$ of homotopy classes of stable maps $\Sigma_+^\infty BS_1 \to \Sigma_+^\infty BS_2$ for finite p-groups.

The Segal conjecture relates stable maps $\Sigma_+^\infty BG_1 \to \Sigma_+^\infty BG_2$ to (G_1, G_2)-bisets for finite groups.

$\mathcal{M}or(G_1, G_2)$:= Set of isomorphism classes of finite sets with right G_1-action and free left G_2-action such that the actions commute.

$\mathcal{M}or(G_1, G_2)$ is a monoid under disjoint union.

$A(G_1, G_2)$:= Grothendieck group completion of $\mathcal{M}or(G_1, G_2)$.

We describe a "natural" homomorphism

$$\alpha\colon A(G_1, G_2) \longrightarrow \{BG_{1+}, BG_{2+}\}.$$

Take a representative Ω of an element in $\mathcal{M}or(G_1, G_2)$ and put

$$\Lambda := \Omega/G_2.$$

The projection map

$$EG_1 \times_{G_1} \Lambda \to BG_1$$

is a finite covering (the fibre is Λ) and we get a transfer

$$\tau\colon \Sigma_+^\infty BG_1 \to \Sigma_+^\infty BG_2.$$

We also have a principal G_2-fibre sequence

$$G_2 \to EG_1 \times_{G_1} \Omega \to EG_1 \times_{G_1} \Lambda$$

since the G_2-action on Ω was free. Let

$$\xi\colon EG_1 \times_{G_1} \Lambda \to BG_2$$

be the classifying map.

Finally,

$$\alpha(\Omega) := \Sigma_+^\infty \xi \circ \tau.$$

There is a pairing

$$A(G_2, G_3) \times A(G_1, G_2) \to A(G_1, G_3)$$

$$(\Lambda, \Omega) \mapsto \Lambda \times_{G_2} \Omega.$$

This makes $A(G, G)$ a ring called the *double Burnside ring of* G.

The morphism α sends this pairing to the composition pairing of stable maps:

$$\alpha(\Lambda \times_{G_2} \Omega) = \alpha(\Lambda) \circ \alpha(\Omega).$$

We could regard α as a functor from the category whose objects are finite groups and whose morphism sets are $A(G_1, G_2)$ to the category whose objects are finite groups and whose morphism sets are $\{BG_{1+}, BG_{2+}\}$.

Put $A(G) := A(G, 1)$.

There is a pairing

$$A(G) \times A(G, G') \to A(G, G'),$$

$$(X, \Omega) \to X \times \Omega,$$

where the actions are given by

$$(x, y).g = (x.g, y.g)\ ,\ g'.(x, y) = (x, g'.y).$$

In particular, $A(G)$ is a ring and $A(G, G')$ is a module over $A(G)$. $A(G)$ is called the *Burnside ring of* G.

$A(G)$ has an augmentation

$$A(G) \to \mathbb{Z}, \quad \Omega \mapsto |\Omega|.$$

We let $I(G)$ denote the augmentation ideal (i.e. the kernel of the augmentation).

Lewis-May-McClure have shown that the following theorem is a consequence of the Segal conjecture, which was proved by Carlsson.

Theorem (Segal conjecture). α *induces an isomorphism*

$$A(G_1, G_2)^{\wedge}_I \longrightarrow \{BG_{1+}, BG_{2+}\},$$

where

$$A(G_1, G_2)^{\wedge}_I = \varprojlim_n \left(A(G_1, G_2)/I^n\right)$$

is the completion with respect to the ideal $I = I(G_1)$*.*

When G_1 is a p-group this takes a simple form

Theorem. *Let* S *be a finite* p*-group and* G *be a finite group. Let* $\tilde{A}(S, G)$ *be the kernel of the morphism*

$$A(S, G) \to A(S),\ \Omega \to \Omega/G.$$

Then α *induces an isomorphism*

$$\tilde{A}(S, G)^{\wedge}_p := \mathbf{Z}^{\wedge}_p \otimes \tilde{A}(S, G) \longrightarrow \{BS, BG\}.$$

Historical note: Let G be a finite group.

Atiyah (ca. 1960): There is an isomorphism

$$R(G)^\wedge_I \longrightarrow KU(BG),$$

where $R(G)$ is the complex representation ring and I is the kernel of the augmentation

$$R(G) \to \mathbf{Z}, \;\; V \mapsto \dim(V).$$

Segal conjectured that the analogous result holds for stable cohomotopy.

Segal conjecture (weak form): The map

$$A(G)^\wedge_I \longrightarrow \pi^0_S(BG_+) := \{BG_+, S^0\}$$

is an isomorphism.

Lin: Proved conjecture for $G = \mathbf{Z}/2$.
Gunawardena: $G = \mathbf{Z}/p$, p odd prime.
Ravenel: General finite cyclic groups.
Carlsson: Elementary abelian 2-groups.
Adams-Gunawardena-Miller:
Odd elementary abelian groups.
May-McClure: Reduce question to finite p-groups.
Carlsson: Uses A-G-M result and induction to prove p-group case.

The proofs are actually of a stronger form of the conjecture. This was in fact necessary, since a statement involving only $\pi_S^0(BG_+)$ (and not higher cohomotopy groups) does not lend itself to induction.

Segal introduced equivariant stable cohomotopy groups

$$\pi_G^{-*}(S^0) = \bigotimes_K \pi_*^S(BW(K)_+),$$

where the sum is taken over conjugacy classes of subgroups of G and $W(K) = N_G(K)/K$.

Note that $\pi_G^*(S^0)$ is a an $A(G)$-module and $\pi_G^0(S^0) \cong A(G)$.

Segal conjecture (strong form): The map

$$\pi_G^*(S^0)^{\wedge}_{I(G)} \longrightarrow \pi_S^*(BG_+)$$

is an isomorphism.

We will study $A(S_1, S_2)$ in detail for finite p-groups. Therefore we apply an unusual form of the Segal conjecture, which maintains more of the structure of $A(S_1, S_2)$.

Theorem (Segal conjecture). *If S is a finite p-group and G any finite group, then α induces an isomorphism*

$$A(S, G)^{\wedge}_p \xrightarrow{\cong} \{BS_+, BG_+\}^{\wedge}_p,$$

where $\{BS_+, BG_+\}^{\wedge}_p$ denotes the group of homotopy classes of stable maps $\Sigma^\infty_+ BS^{\wedge}_p \to \Sigma^\infty_+ BG^{\wedge}_p$.

Note that $\Sigma^\infty_+ BG \simeq \Sigma^\infty BG \vee \mathbb{S}^0$. We will only be interested in the case where G is a p-group, in which case $\Sigma^\infty BG$ is p-complete and the effect of p-completing $\Sigma^\infty_+ BG$ is exactly to p-complete the $\mathbb{S}^0$-term.

It has been shown (Lewis-May-McClure) that in this case the difference between $\{BS_+, BG_+\}$ and $\{BS_+, BG_+\}^{\wedge}_p$ is that the $\mathbb{Z}$-term corresponding to $\{S^0, S^0\}$ in the former becomes a $\mathbf{Z}^{\wedge}_p$-term in the latter.