We want to describe a $\mathbb{Z}$-basis for $A(G_1, G_2)$ for finite groups $G_1$ and $G_2$.

Recall that $A(G_1, G_2)$ is the group completion of the monoid of isomorphism classes of finite $(G_1, G_2)$-sets.

Similarly, for a finite group $G$, the *Burnside ring of* $G$ is the group completion $A(G)$ of the monoid of isomorphism classes of finite $G$-sets.

Now, a finite $(G_1, G_2)$-set $\Omega$ can be regarded as a left $(G_1 \times G_2)$ set by putting

$$(g_1, g_2).x := g_2 x g_1^{-1}$$

for $g_1 \in G_1, g_2 \in G_2$ and $x \in \Omega$. This gives us an imbedding of $\mathbb{Z}$-modules

$$A(G_1, G_2) \hookrightarrow A(G_1 \times G_2).$$

This allows us to approach $A(G_1, G_2)$ by studying the module $A(G_1 \times G_2)$.

**Remark.** *When* $G_1 = G_2$ *the module* $A(G_1, G_2)$ *has a ring structure, which one should note is very different from the ring structure on* $A(G_1 \times G_2)$.

Let $G$ be a finite group. The isomorphism type of a finite $G$-set $X$ is determined by the isomorphism types of its $G$-orbits. The following observations are left as exercises.

**Exercise.** *Each $G$-orbit of $X$ is isomorphic to a $G$-set of type $G/H$, where $H \leq G$.*

**Exercise.**

$$G/H_1 \cong G/H_2$$

*as $G$-sets if and only if $H_1$ and $H_2$ are conjugate in $G$.*

We conclude the following fact.

**Fact.** *A(G) is a free $\mathbb{Z}$-module with one basis element $[G/H]$ for each conjugacy class $[H]$ of subgroups $H \leq G$.*

We now study the image of $A(G_1, G_2)$ in $A(G_1 \times G_2)$.

It is easy to see that this image consists of exactly those $G_1 \times G_2$-bisets whose $G_2$-action (via the map $G_2 \to G_1 \times G_2$, $g \mapsto (1, g)$) is free.

Therefore a $G_1 \times G_2$-set is in the image of $A(G_1, G_2)$ if and only if each of its irreducible components is in the image. We deduce that it suffices to check which basis elements of $A(G_1 \times G_2)$ is in the image of $A(G_1, G_2)$. This is left as an exercise.

**Exercise.** *A $\mathbb{Z}$-basis for the image of $A(G_1, G_2)$ in $A(G_1 \times G_2)$ is given by the collection of basis elements of type $[G_1 \times G_2/\Delta_H^{\varphi}]$, where $H \leq G_1$ and $\varphi$ is a group homomorphism $H \to G_2$.*

Now let us interpret these results for bisets.

**Definition.** *Let $G_1$ and $G_2$ be finite groups. A* $(G_1, G_2)$-pair *is a pair $(H, \varphi)$ consisting of a subgroup $H \leq G_1$ and a homomorphism*

$$\varphi\colon H \to G_2.$$

*We say that two $(G_1, G_2)$-pairs $(H_1, \varphi_1)$ and $(H_2, \varphi_2)$ are* $(G_1, G_2)$-conjugate *if there exist elements $g \in G_1$ and $h \in G_2$ such that $c_g(H_1) = H_2$ and the following diagram commutes*

$$\begin{array}{ccc} H_1 & \xrightarrow{\varphi_1} & G_2 \\ \cong \big\downarrow c_g & & \big\downarrow c_h \\ H_2 & \xrightarrow{\varphi_2} & G_2. \end{array}$$

$(G_1, G_2)$-conjugacy is an equivalence relation on $(G_1, G_2)$-pairs and we write $[H, \psi]$ for the conjugacy class of a $(G_1, G_2)$-pair $(H, \psi)$.

For a $(G_1, G_2)$-pair $(H, \psi)$, we define a $(G_1, G_2)$-biset

$$G_2 \times_{(H,\psi)} G_1 = (G_2 \times G_1)/\sim,$$

where

$$(x, gy) \sim (x\psi(g), y)$$

for $x \in G_2$ , $y \in G_1$, $g \in H$.

The group actions are the obvious right $G_1$-action and free left $G_2$-action given by

$$(x, y).g_1 = (x, yg_1)$$

and

$$g_2.(x, y) = (g_2 x, y)$$

for $x, g_2 \in G_2$ and $y, g_1 \in G_1$.

We again proceed by a sequence of observations, the proofs of which are left as exercises.

**Exercise.** *Let* $(H,\psi)$ *and* $(H',\psi')$ *be* $(G_1,G_2)$*-pairs. Then*

$$G_2 \times_{(H,\psi)} G_1 \cong G_2 \times_{(H',\psi')} G_1$$

*as* $(G_1,G_2)$*-bisets if and only if* $(H,\psi)$ *and* $(H',\psi')$ *are* $(G_1,G_2)$*-conjugate.*

In view of this we will abuse notation and also denote the isomorphism class of the $(G_1,G_2)$-biset $G_2 \times_{(H,\psi)} G_1$ by $[H,\psi]$.

**Exercise.**

$$G_1 \times G_2/\Delta_H^{\psi} \cong G_1 \times G_2/\Delta_{H'}^{\psi'}$$

*as* $(G_1 \times G_2)$*-sets if and only if* $(H,\psi)$ *and* $(H',\psi')$ *are* $(G_1,G_2)$*-conjugate.*

**Exercise.** *The image of* $[G_1 \times G_2/\Delta_H^\psi]$ *in* $A(G_1, G_2)$ *is* $[H, \psi]$*.*

We conclude:

**Fact.** $A(G_1, G_2)$ *is a free* $\mathbb{Z}$*-module with one basis element* $[H, \psi]$ *for every conjugacy class* $[H, \psi]$ *of* $(G_1, G_2)$*-pairs.*

Recall that there is a $\mathbb{Z}$-module morphism

$$\alpha\colon A(G_1, G_2) \longrightarrow \{BG_{1+}, BG_{2+}\}.$$

**Exercise.** *$\alpha$ sends the basis element $[H, \psi]$ to the homotopy class of the map*

$$\Sigma_+^\infty B\psi \circ tr_H\colon \Sigma_+^\infty BG_1 \to \Sigma_+^\infty BG_2,$$

*where*

$$tr_H\colon \Sigma_+^\infty BG_1 \to \Sigma_+^\infty BH$$

*is the transfer of the inclusion $H \leq G_1$.*

Recall also that by the Segal conjecture, for finite $p$-groups $S_1$ and $S_2$, $\alpha$ induces an isomorphism

$$A(S_1, S_2)^\wedge_p \xrightarrow{\cong} \{BS_{1+}, BS_{2+}\}^\wedge_p,$$

where $\{BS_{1+}, BS_{2+}\}^\wedge_p$ denotes the group of homotopy classes of maps between $p$-completed classifying spectra

$$(\Sigma^\infty_+ BS_1)^\wedge_p \to (\Sigma^\infty_+ BS_2)^\wedge_p.$$

In view of this and the last exercise, we will further abuse notation and let $[P, \psi]$ also denote the homotopy class of the map

$$(\Sigma^\infty_+ B\psi \circ tr_P)^\wedge_p\colon (\Sigma^\infty_+ BS_1)^\wedge_p \to (\Sigma^\infty_+ BS_2)^\wedge_p,$$

*when $S_1$ and $S_2$ are finite $p$-groups.*

We now get the following version of the Segal conjecture.

**Theorem (Segal conjecture).** *Let $S_1$ and $S_2$ be finite $p$-groups. Then $\{BS_{1+}, BS_{2+}\}_p^\wedge$ is a free $\mathbf{Z}_p^\wedge$-module with one basis element $[P, \psi]$ for every conjugacy class $[S_1, S_2]$ of $(S_1, S_2)$-pairs.*

It is more common to get rid of the sphere spectra corresponding to the added basepoints before describing the group of homotopy classes of stable maps. (Recall that $\Sigma^\infty_+ BS \simeq \Sigma^\infty BS \vee S^0$.) Although we will not use this version of the Segal conjecture, we present it below.

**Theorem (Segal conjecture).** *Let $S_1$ and $S_2$ be finite $p$-groups. Then $\{BS_{1+}, BS_{2+}\}$ is a free $\mathbf{Z}_p^\wedge$-module with one basis element $[P, \psi]$ for every conjugacy class $[S_1, S_2]$ of $(S_1, S_2)$-pairs, where $\psi$ is nontrivial.*